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SOME CHARACTERIZATIONS OF NONLINEAR FIRST ORDER DIFFERENTIAL EQUATIONS ON UNBOUNDED INTERVALS

BAPURAO C. DHAGE

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Abstract. In this article, we discuss the existence as well as global attractivity and ultimate positivity of solutions for a certain nonlinear first order ordinary differential equation on the unbounded intervals of real line via a classical fixed point theorem of Schauder (Smart [12, page 15]). Our hypotheses and claims have also been explained with the help of two realizations under some natural conditions.

1. Introduction

Let $\mathbb{R}$ denote the real line and let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Let $\mathscr{CRB}(\mathbb{R}_+)$ denote the class of functions $a : \mathbb{R}_+ \to \mathbb{R} - \{0\}$ satisfying the following conditions:

(i) a is continuous, and
(ii) $\lim_{t\to\infty} a(t) = \pm\infty$.

There do exist functions belonging to the above class of functions $\mathscr{CRB}(\mathbb{R}_+)$. In fact, if $a_1(t) = t+1$, $a_2(t) = e^t$, then $a_1, a_2 \in \mathscr{CRB}(\mathbb{R}_+)$. Again, the class of continuous and strictly monotone functions $a : \mathbb{R}_+ \to \mathbb{R} - \{0\}$ satisfy the above criteria. Note that if $a \in \mathscr{CRB}(\mathbb{R}_+)$, then the reciprocal function $\overline{a} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and $\lim_{t\to\infty} \overline{a}(t) = 0$.

Given a function $a \in \mathscr{CRB}(\mathbb{R}_+)$, consider the initial value problem of the following nonlinear first order ordinary differential equation (in short ODE),

$$\left.\begin{aligned} \frac{d}{dt}\big[a(t)x(t)\big] &= f(t, x(t)) \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

By a *solution* to the ODE (1.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ such that the function $t \mapsto a(t)x(t)$ is absolutely continuous and satisfies the equations in (1.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of real-valued functions defined and continuous on $\mathbb{R}_+$.

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The ODE (1.1) is a quadratic perturbation of second type for the well-known nonlinear first order ordinary differential equation (see Dhage and Lakshmikantham [9] and the references therein),

$$\left.\begin{aligned} x'(t) &= f(t,x(t)) \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.2}$$

which has been extensively studied in the literature for different characterizations of the solutions. Therefore, the ODE (1.1) is new to the literature and includes a good number of known nonlinear differential equations. Let $k : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $\lim_{t\to\infty} e^{K(t)} = \infty$, where $K(t) = \int_0^t k(s)\,ds$. Indeed, the differential equation,

$$\left.\begin{aligned} x'(t) + k(t)x(t) &= f(t,x(t)) \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.3}$$

is of the type ODE (1.1) on unbounded interval $\mathbb{R}_+$. The ODE (1.2) again includes as a special case the well-known Bernoulli's equation,

$$\left.\begin{aligned} x'(t) + k(t)x(t) &= q(t)x^n(t) \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.4}$$

where n is a nonnegative real number and the function $q : \mathbb{R}_+ \to \mathbb{R}$ is continuous. Some special cases of (1.3) have also been treated in Burton and Furumochi [4] for asymptotic stability of solutions. Thus, our ODE (1.1) is of general interest and therefore, the global attractivity and ultimate positivity results proved in this paper include the existence as well as attractivity results for the above mentioned ODEs (1.2)-(1.3) on $\mathbb{R}_+$.

2. Auxiliary Results

We seek the solutions of the ODE (1.1) in the space $BC(\mathbb{R}_+,\mathbb{R})$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Define a standard supremum norm $\|\cdot\|$ in $BC(\mathbb{R}_+,\mathbb{R})$ by

$$\|x\| = \sup_{t\in\mathbb{R}_+} |x(t)|.$$

Clearly, $BC(\mathbb{R}_+,\mathbb{R})$ becomes a Banach space with respect to above norm defined in it. By $L^1(\mathbb{R}_+,\mathbb{R})$ we denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ and the norm $\|\cdot\|_{L^1}$ in $L^1(\mathbb{R}_+,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^\infty |x(s)|\,ds.$$

We employ the Schauder's fixed point theorem for establishing the main result of this paper. Before stating this result, we give a useful definition.

DEFINITION 2.1. An operator Q on a Banach space X into itself is called compact if for any bounded subset S of X, $Q(S)$ is a relatively compact subset of X. If Q is continuous and compact, then it is called completely continuous on X.

The details of completely continuous nonlinear operators appear in Granas and Dugundji [10]. Our key result which is used as a tool in the analysis of the problem (1.1) is

THEOREM 2.1. (Smart [12, page 15]) *Let S be a non-empty, closed, convex and bounded subset of the Banach space X and let $Q : S \to S$ be a continuous and compact operator. Then the operator equation $Qx = x$ has a solution.*

In order to introduce further concepts used in the subsequent part of this paper, let $Q : BC(\mathbb{R}_+,\mathbb{R}) \to BC(\mathbb{R}_+,\mathbb{R})$ be an operator and consider the following operator equation in $BC(\mathbb{R}_+,\mathbb{R})$,

$$Qx(t) = x(t) \tag{2.1}$$

for all $t \in \mathbb{R}_+$. Below we give different characterizations of the solutions for the operator equation (2.1) in the space $BC(\mathbb{R}_+,\mathbb{R})$.

DEFINITION 2.2. We say that solutions of the equation (2.1) are *locally attractive* if there exists a closed ball $\overline{\mathscr{B}_r}(x_0)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of the equation (3.2) belonging to $\overline{\mathscr{B}_r}(x_0)$ we have that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \tag{2.2}$$

In the case when the limit (2.2) is uniform with respect to the set $\overline{\mathscr{B}_r}(x_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leqslant \varepsilon \tag{2.3}$$

for all $x,y \in \overline{\mathscr{B}_r}(x_0)$ being solutions of (2.1) and for $t \geqslant T$, we will say that solutions of equation (2.1) are *uniformly locally attractive* on $\mathbb{R}_+$.

DEFINITION 2.3. A solution $x = x(t)$ of equation (2.1) is said to be *globally attractive* if (2.2) holds for each solution $y = y(t)$ of (2.1) in $BC(\mathbb{R}_+,\mathbb{R})$. In other words, we may say that solutions of the equation (2.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (2.1) in $BC(\mathbb{R}_+,\mathbb{R})$, the condition (2.2) is satisfied. In the case when the condition (2.2) is satisfied uniformly with respect to the space $BC(\mathbb{R}_+,\mathbb{R})$, i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (2.2) is satisfied for all $x,y \in BC(\mathbb{R}_+,\mathbb{R})$ being the solutions of (2.1) and for $t \geqslant T$, we will say that solutions of the equation (2.1) are *uniformly globally attractive* on $\mathbb{R}_+$.

REMARK 2.1. Let us mention that the concept of global attractivity of solutions is recently introduced in Hu and Yan [11] while the concepts of uniform local and global attractivity (in the above sense) were introduced in Banas and Rzepka [1].

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (2.1) in the space $BC(\mathbb{R}_+,\mathbb{R})$.

DEFINITION 2.4. (Dhage [7]) A solution x of the equation (2.1) is called *locally ultimately positive* if there exists a closed ball $\overline{\mathscr{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that $x \in \overline{\mathscr{B}}_r(0)$ and

$$\lim_{t\to\infty}\big[|x(t)| - x(t)\big] = 0. \tag{2.4}$$

In the case when the limit (2.4) is uniform with respect to the solution set of the operator equation (2.1) in $BC(\mathbb{R}_+,\mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\big||x(t)| - x(t)\big| \leqslant \varepsilon \tag{2.5}$$

for all x being the solutions of (2.1) in $BC(\mathbb{R}_+,\mathbb{R})$ and for $t \geqslant T$, we will say that solutions of equation (2.1) are *uniformly locally ultimately positive* on $\mathbb{R}_+$.

DEFINITION 2.5. (Dhage [8]) A solution $x \in BC(\mathbb{R}_+,\mathbb{R})$ of the equation (2.1) is called *globally ultimately positive* if (2.4) is satisfied. In the case when the limit (2.4) is uniform with respect to the solution set of the operator equation (2.1) in $BC(\mathbb{R}_+,\mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (2.5) is satisfied for all x being the solutions of (2.1) in $BC(\mathbb{R}_+,\mathbb{R})$ and for $t \geqslant T$, we will say that solutions of equation (2.1) are *uniformly globally ultimately positive* on $\mathbb{R}_+$.

REMARK 2.2. We note that the global attractivity implies the local attractivity and uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation (2.1) on $\mathbb{R}_+$. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (2.1) on unbounded intervals. However, the converse of the above two statements may not be true.

3. Attractivity and positivity results

In this section, we prove the global attractivity and positivity results for the ODE (1.1) on $\mathbb{R}_+$ under some suitable conditions. First we prove the global attractivity and ultimate positivity results for the ODE (1.1) on $\mathbb{R}_+$.

We need the following definitions in the sequel.

DEFINITION 3.1. By a *solution* of the equation (1.1) we mean a function $x \in BC(\mathbb{R}_+,\mathbb{R}) \cap AC(\mathbb{R}_+,\mathbb{R})$ that satisfies the equations in (1.1), where $AC(\mathbb{R}_+,\mathbb{R})$ is the space of absolutely continuous real-valued functions on right half real axis $\mathbb{R}_+$.

DEFINITION 3.2. A function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is called *Carathéodory* if:

(i) $t \mapsto f(t,x)$ is measurable for all $x \in \mathbb{R}$, and

(ii) $x \mapsto f(t,x)$ is continuous for all $t \in \mathbb{R}_+$.

The class of Carathéodory functions f on $\mathbb{R}_+ \times \mathbb{R}$ is denoted by $\mathscr{C}(\mathbb{R}_+ \times \mathbb{R},\mathbb{R})$.

We need the following hypotheses in what follows.

(H_1) The function f is Carathéodory.

(H_2) There exists a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x)| \leqslant h(t) \text{ a.e. } t \in \mathbb{R}_+$$

for all $x \in \mathbb{R}$. Moreover, we assume that $\lim_{t\to\infty} |\overline{a}(t)| \int_0^t h(s)\,ds = 0$.

(H_3) $a(0) \geqslant 0$ and $x_0 \geqslant 0$.

REMARK 3.1. If the hypothesis (H_2) holds true and $a \in \mathscr{C}\mathscr{R}\mathscr{B}(\mathbb{R}_+)$, then $\overline{a} \in BC(\mathbb{R}_+,\mathbb{R})$ and the function $w : \mathbb{R}_+ \to \mathbb{R}$ defined by the expression

$$w(t) = |\overline{a}(t)| \int_0^t h(s)\,ds$$

is continuous and the number $W = \sup_{t\geqslant 0} w(t)$ exists.

THEOREM 3.1. *Assume that the hypotheses* (H_1)-(H_2) *hold. Then the ODE* (1.1) *has a solution and solutions are uniformly globally attractive on* $\mathbb{R}_+$.

Proof. The ODE (1.1) is equivalent to the nonlinear integral equation

$$x(t) = a(0)x_0\overline{a}(t) + \overline{a}(t) \int_0^t f(s,x(s))\,ds \tag{3.1}$$

for $t \in J$. Set $X = BC(\mathbb{R}_+,\mathbb{R})$. Define an operator Q on X by

$$Qx(t) = a(0)x_0\overline{a}(t) + \overline{a}(t) \int_0^t f(s,x(s))\,ds, \tag{3.2}$$

for all $t \in \mathbb{R}_+$. We show that Q defines a mapping $Q : X \to X$. Let $x \in X$ be arbitrary. Obviously, Qx is a continuous function on $\mathbb{R}_+$. We show that Qx is bounded on $\mathbb{R}_+$. Thus, if $t \in \mathbb{R}_+$, then we obtain:

$$\begin{aligned}
|Qx(t)| &\leqslant |a(0)|\,|x_0|\,|\overline{a}(t)| + |\overline{a}(t)| \int_0^t |f(s,x(s))|\,ds \\
&\leqslant |a(0)|\,|x_0|\,\|\overline{a}\| + |\overline{a}(t)| \int_0^t h(s)\,ds.
\end{aligned}$$

Since

$$\lim_{t\to\infty} |\overline{a}(t)| \int_0^t h(s)\,ds = 0,$$

and the function $w : \mathbb{R}_+ \to \mathbb{R}$ defined by $w(t) = |\overline{a}(t)| \int_0^t h(s)\,ds$ is continuous, there is a constant $W > 0$ such that

$$W = \sup_{t\geqslant 0} w(t) = \sup_{t\geqslant 0} |\overline{a}(t)| \int_0^t h(s)\,ds.$$

Therefore,

$$|Qx(t)| \leqslant |a(0)|\,|x_0|\|\overline{a}\| + W$$

for all $t \in \mathbb{R}_+$. As a result, we have that

$$\|Qx\| \leqslant |a(0)|\,|x_0|\,\|\overline{a}\| + W \tag{3.3}$$

for all $x \in X$ and therefore, Q maps X into X itself. Define a closed ball $\overline{\mathscr{B}}_r(0)$ centered at origin of radius r, where $r = |a(0)|\,|x_0|\,\|\overline{a}\| + W$. Clearly Q defines a mapping $Q : X \to \overline{\mathscr{B}}_r(0)$ and in particular $Q : \overline{\mathscr{B}}_r(0) \to \overline{\mathscr{B}}_r(0)$. We show that Q satisfies all the conditions of Theorem 2.1. First, we show that Q is continuous on $\overline{\mathscr{B}}_r(0)$. To do this, let us fix arbitrarily $\varepsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{\mathscr{B}}_r(0)$ converging to a point $x \in \overline{\mathscr{B}}_r(0)$. Then we get:

$$\begin{aligned}
|(Qx_n)(t) - (Qx)(t)| &\leqslant |\overline{a}(t)| \int_0^t |f(s,x_n(s)) - f(s,x(s))|ds \\
&\leqslant |\overline{a}(t)| \int_0^t [|f(s,x_n(s))| + |f(s,x(s))|]ds \\
&\leqslant 2|\overline{a}(t)| \int_0^t h(s)\,ds \\
&= 2w(t)
\end{aligned} \tag{3.4}$$

Hence, in virtue of hypothesis (H_2), we infer that there exists a $T > 0$ such that $w(t) \leqslant \varepsilon$ for $t \geqslant T$. Thus, for $t \geqslant T$, from the estimate (3.4) we derive that

$$|(Qx_n)(t) - (Qx)(t)| \leqslant 2\varepsilon \quad \text{as} \quad n \to \infty.$$

Furthermore, let us assume that $t \in [0,T]$. Then, by Lebesgue dominated convergence theorem, we obtain the estimate:

$$\begin{aligned}
\lim_{n\to\infty} Qx_n(t) &= \lim_{n\to\infty} \left[a(0)x_0\overline{a}(t) + \overline{a}(t) \int_0^t f(s,x_n(s))\,ds \right] \\
&= a(0)x_0\overline{a}(t) + \overline{a}(t) \int_0^t \left[\lim_{n\to\infty} f(s,x_n(s)) \right] ds \\
&= Qx(t)
\end{aligned} \tag{3.5}$$

for all $t \in [0,T]$. Thus, $Qx_n \to Qx$ as $n \to \infty$ uniformly on $\mathbb{R}_+$ and hence Q is a continuous operator on $\overline{\mathscr{B}}_r(0)$ into $\overline{\mathscr{B}}_r(0)$.

Next, we show that B is compact on $\overline{\mathscr{B}}_r(0)$. To finish, it is enough to show that every sequence $\{Qx_n\}$ in $Q(\overline{\mathscr{B}}_r(0))$ has a Cauchy subsequence. Now by (H_2),

$$\begin{aligned}
|Qx_n(t)| &\leqslant |a(0)|\,|x_0||\overline{a}(t)| + |\overline{a}(t)| \int_0^t |f(s,x_n(s))|\,ds \\
&\leqslant |a(0)|\,|x_0|\,\|\overline{a}\| + w(t) \\
&\leqslant |a(0)|\,|x_0|\,\|\overline{a}\| + W
\end{aligned} \tag{3.6}$$

for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|Qx_n\| \leqslant |a(0)|\,|x_0|\,\|\overline{a}\| + W$ for all $n \in \mathbb{N}$. This shows that $\{Qx_n\}$ is a uniformly bounded sequence in $Q(\overline{\mathscr{B}_r}(0))$.

Next, we show that $\{Qx_n\}$ is also a equicontinuous sequence in $Q(\overline{\mathscr{B}_r}(0))$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a constant $T_1 > 0$ such that $|w(t)| < \varepsilon/8$ for all $t \geqslant T_1$. Similarly, since $\lim\limits_{t\to\infty} \overline{a}(t) = 0$, for above $\varepsilon > 0$, there is a $T_2 > 0$ such that

$$|\overline{a}(t)| < \frac{\varepsilon}{8|a(0)|\,|x_0|} \quad \text{for all } t \geqslant T_2.$$

Thus, if $T = \max\{T_1, T_2\}$, then $|w(t)| < \varepsilon/8$ and

$$|\overline{a}(t)| < \frac{\varepsilon}{8|a(0)|\,|x_0|} \quad \text{for all } t \geqslant T.$$

Let $t, \tau \in \mathbb{R}_+$ be arbitrary. If $t, \tau \in [0, T]$, then we have

$$\begin{aligned}
|Qx_n(t) - Qx_n(\tau)| &\leqslant |a(0)|\,|x_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left|\overline{a}(t)\int_0^t f(s,x_n(s))\,ds - \overline{a}(\tau)\int_0^\tau f(s,x_n(s))\,ds\right| \\
&\leqslant |a(0)|\,|x_0|\,|\overline{a}(t) - \overline{a}(\tau)| + \left|\overline{a}(t)\int_0^t f(s,x_n(s))\,ds - \overline{a}(\tau)\int_0^t f(s,x_n(s))\,ds\right| \\
&\quad + \left|\overline{a}(\tau)\int_0^t f(s,x_n(s))\,ds - \overline{a}(\tau)\int_0^\tau f(s,x_n(s))\,ds\right| \\
&\leqslant |a(0)|\,|x_0|\,|\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)|\left|\int_0^t f(s,x_n(s))\,ds\right| \\
&\quad + |\overline{a}(\tau)|\left|\int_\tau^t f(s,x_n(s))\,ds\right| \\
&\leqslant |a(0)|\,|x_0|\,|\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)|\int_0^T h(s)\,ds + \|\overline{a}\|\left|\int_\tau^t h(s)\,ds\right| \\
&\leqslant |a(0)|\,|x_0|\,|\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)|\int_0^T h(s)\,ds + \|\overline{a}\|\,|p(t) - p(\tau)| \\
&\leqslant \big[|a(0)|\,|x_0| + \|h\|_{L^1}\big]\,|\overline{a}(t) - \overline{a}(\tau)| + \|\overline{a}\|\,|p(t) - p(\tau)|
\end{aligned}$$

where $p(t) = \displaystyle\int_0^t h(s)\,ds$ and $\|h\|_{L^1} = \displaystyle\int_0^\infty h(s)\,ds$.

By the uniform continuity of the function $\overline{a}$ and p on $[0, T]$, for above ε we have the numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending only on ε such that

$$|t - \tau| < \delta_1 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\varepsilon}{8\big[|a(0)|\,|x_0| + \|h\|_{L^1}\big]}$$

and

$$|t - \tau| < \delta_2 \Longrightarrow |p(t) - p(\tau)| < \frac{\varepsilon}{8\|\overline{a}\|}$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Then

$$|t - \tau| < \delta_3 \Longrightarrow |Qx_n(t) - Qx_n(\tau)| < \frac{\varepsilon}{4}$$

for all $n \in \mathbb{N}$.

Again, if $t, \tau > T$, then we have a $\delta_4 > 0$ depending only on ε such that

$$\begin{aligned}
|Qx_n(t) - Qx_n(\tau)| &\leqslant |a(0)|\,|x_0|\,|a(t) - a(\tau)| \\
&\quad + \left|\overline{a}(t)\int_0^t f(s,x_n(s))\,ds - \overline{a}(t)\int_0^\tau f(s,x_n(s))\,ds\right| \\
&\leqslant |a(0)|\,|x_0|\,|\overline{a}(t)| + |a(0)|\,|x_0|\,|\overline{a}(\tau)| + w(t) + w(\tau) \\
&< \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$ whenever $t - \tau| < \delta_4$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|Qx_n(t) - Qx_n(\tau)| \leqslant |Qx_n(t) - Qx_n(T)| + |Qx_n(T) - Qx_n(\tau)|.$$

Take $\delta = \min\{\delta_3, \delta_4\} > 0$ depending only on ε. Therefore, from the above obtained estimates, it follows that

$$|Qx_n(t) - Qx_n(T)| < \frac{\varepsilon}{2} \quad \text{and} \quad |Qx_n(T) - Qx_n(\tau)| < \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta$. As a result, $|Qx_n(t) - Qx_n(\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$ and for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta$. This shows that $\{Qx_n\}$ is a equicontinuous sequence in X. Now an application of Arzelà-Ascoli theorem yields that $\{Qx_n\}$ has a uniformly convergent subsequence on the compact subset $[0, T]$ of $\mathbb{R}$. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{Qx_n\}$ is Cauchy in X. Now $|Qx_n(t) - Qx(t)| \to 0$ as $n \to \infty$ for all $t \in [0, T]$. Then for given $\varepsilon > 0$ there exits an $n_0 \in \mathbb{N}$ such that

$$\sup_{0 \leqslant p \leqslant T} \overline{a}(p)\int_0^p |f(s,x_m(s)) - f(s,x_n(s))|\,ds < \frac{\varepsilon}{2}$$

for all $m, n \geqslant n_0$. Therefore, if $m, n \geqslant n_0$, then we have

$$\begin{aligned}
\|Qx_m - Qx_n\| &= \sup_{0 \leqslant t < \infty}\left|\overline{a}(t)\int_0^t |f(s,x_m(s)) - f(s,x_n(s))|\,ds\right| \\
&\leqslant \sup_{0 \leqslant p \leqslant T}\left|\overline{a}(p)\int_0^p |f(s,x_m(s)) - f(s,x_n(s))|\,ds\right| \\
&\quad + \sup_{p \geqslant T}\overline{a}(p)\int_0^p \big[|f(s,x_m(s))| + |f(s,x_n(s))|\big]\,ds \\
&< \varepsilon.
\end{aligned}$$

This shows that $\{Qx_n\} \subset Q(\overline{\mathscr{B}_r(0)}) \subset X$ is Cauchy. Since X is complete, $\{Qx_n\}$ converges to a point in X. As $Q(\overline{\mathscr{B}_r(0)})$ is closed $\{Qx_n\}$ converges to a point in $Q(\overline{\mathscr{B}_r(0)})$. Hence $Q(\overline{\mathscr{B}_r(0)})$ is relatively compact and consequently Q is a continuous and compact operator on $\overline{\mathscr{B}_r}(0)$ into itself. Now an application of Theorem 2.1 to the operator Q on $\overline{\mathscr{B}_r}(0)$ yields that Q has a fixed point in $\overline{\mathscr{B}_r}(0)$ which further implies that the ODE (1.1) has a solution on $\mathbb{R}_+$.

Finally, we show that the solutions are uniformly attractive on $\mathbb{R}_+$. Let $x, y \in \overline{\mathscr{B}}_r(0)$ be any two solutions the ODE (1.1) on $\mathbb{R}_+$. Then,

$$\begin{aligned} |x(t) - y(t)| &\leqslant \left|\overline{a}(t) \int_0^t f(s,x(s))\,ds - \overline{a}(t) \int_0^t f(s,y(s))\,ds\right| \\ &\leqslant 2|\overline{a}(t)| \int_0^t |f(s,x(s))|\,ds \\ &= 2w(t) \end{aligned} \tag{3.7}$$

for all $t \in \mathbb{R}_+$. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T > 0$ such that $w(t) < \frac{\varepsilon}{2}$ for all $t \geqslant T$. Therefore, $|x(t) - y(t)| \leqslant \varepsilon$ for all $t \geqslant T$, and so all the solutions of the ODE (1.1) are uniformly globally attractive on $\mathbb{R}_+$.

THEOREM 3.2. *Assume that the hypotheses* (H_1)-(H_3) *hold. Then the ODE* (1.1) *has a solution and solutions are uniformly globally attractive and ultimately positive on* $\mathbb{R}_+$.

Proof. By Theorem 3.1, the ODE (1.1) has a solution in $\overline{\mathscr{B}}_r(0)$, where $r = \|\overline{a}\|\,|a(0)|\,|x_0| + W$ and the solutions are uniformly globally attractive on $\mathbb{R}_+$. We know that for any $x, y \in \mathbb{R}$, one has the inequality,

$$|x| + |y| \geqslant |x+y| \geqslant x+y,$$

and therefore,

$$\big||x+y| - (x+y)\big| \leqslant \big|\,|x| + |y| - (x+y)\big| \leqslant \big||x| - x\big| + \big||y| - y\big| \tag{3.8}$$

for all $x, y \in \mathbb{R}$. Now, for any solution $x \in \overline{\mathscr{B}}_r(0)$, one has

$$\begin{aligned} \big||x(t)| - x(t)\big| &= \Bigg|\left|a(0)x_0\overline{a}(t) + \overline{a}(t)\int_0^t f(s,x(s))\,ds\right| \\ &\qquad - \left(a(0)x_0\overline{a}(t) + \overline{a}(t)\int_0^t f(s,x(s))\,ds\right)\Bigg| \\ &\leqslant \big||a(0)|\,|x_0| - a(0)x_0\big|\,|\overline{a}(t)| + |\overline{a}(t)| \int_0^t |f(s,x(s))|\,ds \\ &\qquad + |\overline{a}(t)| \int_0^t |f(s,x(s))|\,ds \\ &\leqslant 2w(t). \end{aligned}$$

Since $\lim\limits_{t\to\infty} w(t) = 0$, there is a real number $T > 0$ such that $\big||x(t)| - x(t)\big| \leqslant \varepsilon$ for all $t \geqslant T$. Hence solutions of the ODE (1.1) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$. This completes the proof.

REMARK 3.2. Note that the attractivity results proved in Theorems 3.1 and 3.2 also yield that the solutions of the ODE (1.1) are asymptotically attractive towards a line $x(t) = 0 \ \forall t \in \mathbb{R}_+$. To see this, let x be any solution of the (1.1) existing on $\mathbb{R}_+$. Then by hypothesis (H_2),

$$\begin{aligned} |x(t) &\leqslant |a(0)x_0\overline{a}(t)| + \left|\overline{a}(t)\int_0^t f(s,x(s))\,ds\right| \\ &\leqslant |a(0)|\,|x_0|\,|\overline{a}(t)| + |\overline{a}(t)|\int_0^t |f(s,x(s))|\,ds \\ &\leqslant |a(0)|\,|x_0|\,|\overline{a}(t)| + w(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$. Taking the limit superior as $t \to \infty$ in the above inequality yields,

$$\limsup_{t\to\infty} |x(t)| \leqslant |a(0)|\,|x_0| \limsup_{t\to\infty} |\overline{a}(t)| + \limsup_{t\to\infty} w(t) = 0.$$

As a result, $\lim_{t\to\infty} |x(t)| = 0$. This shows that every solution x to the ODE (1.1) under hypothesis (H_2) is asymptotic to the line $x(t) = 0$ on $\mathbb{R}_+$.

EXAMPLE 3.1. Let us consider the following ODE,

$$\left.\begin{aligned} \frac{d}{dt}\left[e^t x(t)\right] &= e^{-t}\frac{|x(t)|}{1+x^2(t)} \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= 1, \end{aligned}\right\} \tag{3.9}$$

where, $e^{-t} \in L^1(\mathbb{R}_+,\mathbb{R})$ and $\lim\limits_{t\to\infty} e^{-t}\int_0^t e^{-s}\,ds = 0$.

Here, $a(t) = e^t$ which is positive and increasing on $\mathbb{R}_+$ and so $a \in \mathscr{C}\mathscr{R}\mathscr{B}(\mathbb{R}_+)$ and

$$\|\overline{a}\| = \sup_{t\geqslant 0} \overline{a}(t) = \sup_{t\geqslant 0} e^{-t} \leqslant 1.$$

Here, $f(t,x) = \dfrac{e^{-t}|x|}{1+x^2}$ for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Clearly, the function f satisfies the hypothesis (H_2) with growth function $h(t) = e^{-t}$ on $\mathbb{R}_+$ so that

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} e^{-t}\int_0^t e^{-s}\,ds = 0.$$

Now, we apply Theorem 3.1 to ODE (3.9) and conclude that it has a solution and solutions are uniformly globally attractive on $\mathbb{R}_+$. As $x_0 = 1 \geqslant 0$, the hypothesis (H_3) of Theorem 3.2 is satisfied. Hence, solutions of the ODE (3.9) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$.

EXAMPLE 3.2. Let us consider the following ODE,

$$\left.\begin{aligned} x'(t) + \frac{1}{t+1}x(t) &= \frac{\log(1+|x(t)|)}{(t+1)^{3/2}(1+|x(t)|)} \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= 1. \end{aligned}\right\} \tag{3.10}$$

The above differential equation (3.10) can be written as

$$\left.\begin{aligned} \frac{d}{dt}\big[(t+1)x(t)\big] &= \frac{\log(1+|x(t)|)}{(t+1)^{1/2}(1+|x(t)|)} \text{ a.e. } t \in \mathbb{R}_+ \\ x(0) &= 1 \in \mathbb{R}. \end{aligned}\right\}$$

Here, $a(t) = t+1$ which is positive and increasing on $\mathbb{R}_+$ and so $a \in \mathscr{C}\mathscr{R}\mathscr{B}(\mathbb{R}_+)$ and

$$\|\overline{a}\| = \sup_{t\geqslant 0} \overline{a}(t) = \sup_{t\geqslant 0} \frac{1}{t+1} \leqslant 1.$$

Here,

$$f(t,x) = \frac{\log(1+|x|)}{(t+1)^{1/2}(1+|x|)}$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Clearly, the function f satisfies the hypothesis (H_2) with growth function $h(t) = (t+1)^{-1/2}$ on $\mathbb{R}_+$ so that

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \frac{1}{t+1} \int_0^t (s+1)^{-1/2}\, ds = 0.$$

Now we apply Theorem 3.1 to ODE (3.10) to conclude that it has a solution and solutions are uniformly globally attractive on $\mathbb{R}_+$. As $x_0 = 1 \geqslant 0$ and $a(0) = 1$, the hypothesis (H_3) of Theorem 3.2 is satisfied. Hence, solutions of the ODE (3.10) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$.

4. The Conclusion

The study of characterizations of solutions given in Remark 3.2 is initiated by Burton and Furumochi [4] for nonlinear differential equations and by Banas and Rzepka [1] for nonlinear integral equations under the titles stability and asymptotic stability of solutions respectively. The former studies make use of functional analytic techniques while the later studies make use of measure theoretic techniques in formulating the main stability results for the differential and integral equations in question. Later, similar studies have been exploited for different types of perturbed nonlinear integral equations. See for example, Banas and Dhage [2], Burton and Zhang [3] and Dhage [5, 6, 7, 8]. However, we do not find much literature on nonlinear differential equations in this direction. Therefore, it seems that the ideas of this paper will open a new vistas for the research work in the theory of nonlinear differential equations. There are some advantages and disadvantages of the above mentioned two approaches one over the other. In the present work, we have used the fixed point theoretic technique of Schauder [12, page 15] for proving the attractivity and positivity of solutions for the initial value problems of first order ordinary differential equations only. However, these characterizations can also be proved for different types of nonlinear differential and integral equations by employing the suitable fixed point theorems needed for the situations. The choice of the fixed point theorems depends upon the prevailing situations and the circumstances of the nonlinearities involved in the problems. The clever selection of the fixed point theorems yields very powerful existence results as well as different characterizations of the nonlinear differential equations. In a forthcoming paper, it is planed to discuss other characterizations such as global asymptotic and monotonic attractivity of the solutions for nonlinear differential equation (1.1) via classical fixed point theory under some natural conditions.

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Bapurao C. Dhage
Kasubai, Gurukul Colony
Ahmedpur - 413515, Distr. Latur, Maharashtra
India
e-mail: bcdhage@yahoo.co.in