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OSCILLATIONS OF EVEN ORDER LINEAR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we devote to investigation of even order impulsive delay differential equations, new oscillation criteria for every solution of equations are established.

1. Introduction

In this paper we study the oscillatory behavior of even order linear impulsive delay differential equations of the form

$$\begin{cases} x^{(n)}(t)+p(t)x(t-\tau)=0, & t \geqslant t_0,\ t \neq t_k, \\ x^{(i)}(t_k^+)=a_k^{(i)}x^{(i)}(t_k), & i=0,1,\cdots n-1,\ k=1,2\cdots \\ x^{(i)}(t_0^+)=x_0^{(i)}, \end{cases} \tag{1}$$

where $x^{(0)}(t)=x(t),\tau>0$, n is even, $0 \leqslant t_0<t_1<t_2<\cdots<t_k<\cdots$ such that $t_k \to \infty$ as $k \to \infty$, $p(t)$ is positive and contionuous in $[t_0,+\infty)$ and for $k=1,2,\ldots,a_k^{(i)}$ are positive numbers. We also adopt the definitions that

$$x^{(i)}(t_k)=\lim_{h\to -0}\frac{x^{(i-1)}(t_k+h)-x^{(i-1)}(t_k)}{h}$$

and

$$x^{(i)}(t_k^+)=\lim_{h\to +0}\frac{x^{(i-1)}(t_k+h)-x^{(i-1)}(t_k^+)}{h}.$$

Let $\varphi(t):[t_0-\tau,t_0]\to R$.Both φ and φ' have at most finite number of the first class discontinuous points on $[t_0-\tau,t_0]$. By a solution $x=x(t)$, we mean a real function on $[t_0-\tau,\infty)$ such that $x^{(i)}(t_0^+)=x_0^{(i)}$ for $i=0,1,\cdots,n-1$ and for $t\in[t_0-\tau,t_0],x(t)=\varphi(t)$,and $x(t)$ satisfies $x^{(n)}(t)+p(t)x(t-\tau)=0$ at each point $t\in[t_0-\tau,\infty)$ with the possible exception of the points $t\neq t_k,t_k+\tau$ and $x^{(i)}(t_k^+)=$

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$a_k^{(i)}x^{(i)}(t_k)$ for any t_k. A solution of (1) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

Impulsive differential equations are mathematical apparatus for simulation of process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, etc. So there have been quite a few results on properties of their solutions in recent years [8]-[13]. For example, in [5], K. Gopalsamy and B. G. Zhang investigated oscillation of first order delay differential equations with impulses. [14] generalized the results of [5]. In [13] J. R. Yan established Oscillation criteria for nonlinear several delays impulsive differential equations. The oscillatory property of second order ordinary differential equations with impulses had been studied in [4]-[9]. In [3], L. Berezansky and E. Braverman obtained explicit conditions of oscillation and nonoscillation for sufficiently general class of second order impulsive linear delay differential equation. These results were base on the coresponding equations without impulses in [2]. In [10], even order nonlinear differential equations with impulses were studied. But papers devoted to study of the oscillations of even order impulsive delay differential equations are quite rare.

In this paper, appling some known Lemmas and some new Lemmas, we investigate the oscillatory property of (1). We will also provide examples to show that although even order delay differential equations without impulses may have nonoscillatory solutions,adding impulses may lead to oscillatory solutions. That is, impulses may change the oscillatory behavior of an equation. Finally, we also point out that our results may be generalized to even order several delays differential equations with impulses

$$\begin{cases} x^{(n)}(t)+\sum\limits_{j=1}^{m} p_j(t)x(t-\tau_j)=0, \ t\geqslant t_0,\ t\neq t_k, \\ x^{(i)}(t_k^+)=a_k^{(i)}x^{(i)}(t_k), i=0,1,\cdots n-1,\ k=1,2\cdots \end{cases}$$

where n is even, $\tau_j>0$.

For background material on oscillation of high order differential equations without impulses, we may see the references [7]-[12].

2. Main results

We will establish oscillatory results based on combinations of the following conditions:

(i) $\lim\limits_{t\to+\infty}\int_{t_0}^{t}\prod\limits_{t_0<t_k<s}\frac{a_k^{(i)}}{a_k^{(i-1)}}ds=+\infty, i=1,2,\ldots,n-1$;

(ii) there exists a positive integer k_0 such that for $k\geqslant k_0$ and for natural number any $l\in\{1,3,\cdots,n-1\}$

$$a_k^{(j)}\geqslant c_k^{(l)}(j=0,1,\cdots,l-1),$$

where $c_k^{(l)}=\max\{a_k^{(l)},a_k^{(l+1)},\cdots,a_k^{(n-1)}\}$;

(iii) $\lim\limits_{t\to+\infty}\sup\prod\limits_{t-2\tau<t_k<t}a_k<+\infty$, where $a_k=\max\{a_k^{(n-1)},1\}$;

(iv) $c=\lim\limits_{t\to+\infty}\inf\int_{t-\tau}^{t}(s-\tau)^{n-1}p(s)ds>0$.

The main results of the paper are as follows.

THEOREM 1. *Assume that the conditions* $(i),(ii)$ *hold. If*

$$\lim_{t\to+\infty}\sup\int_{t}^{t+\tau}\prod_{s-\tau\leqslant t_k<s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds>(n-1)!,\tag{2}$$

then every solution of (1) is oscillatory.

THEOREM 2. *Assume that the conditions* $(i),(ii),(iii)$ *and* (iv) *hold. If*

$$\lim_{t\to+\infty}\inf\int_{t-\tau}^{t}\prod_{s-\tau\leqslant t_k<s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds>\frac{(n-1)!}{e},\tag{3}$$

then every solution of (1) is oscillatory.

To prove Theorem 1 and 2, we need the following Lemmas.

LEMMA 1. (Lakshmikantham et al. [8]) *Assume that:*
(A_0) $m\in PC^1(R_+,R)$ *and* $m(t)$ *is left-continuous at* t_k, $k=1,2,\ldots,$
(A_1) $k=1,2,\ldots,$ $t\geqslant t_0$,

$$m'(t)\leqslant p(t)m(t)+q(t),\ t\neq t_k,\tag{4}$$
$$m(t_k^+)\leqslant d_km(t_k)+b_k,\tag{5}$$

where $p,q\in PC^1(R_+,R), d_k\geqslant 0$ *and* b_k *are real constants. Then for* $t\geqslant t_0$

$$\begin{aligned}m(t)\leqslant m(t_0)\prod_{t_0\leqslant t_k<t}d_k exp(\int_{t_0}^{t}p(s)ds)+\sum_{t_0<t_k<t}(\prod_{t_0<t_j<t}d_j exp(\int_{t_k}^{t}p(s)ds))b_k\\ +\int_{t_0}^{t}\prod_{s<t_k<t}d_k exp(\int_{s}^{t}p(\sigma)d\sigma)q(s)ds.\end{aligned}\tag{6}$$

REMARK 1. If inequalities (4) and (5) are reversed, then as an conclution, inequality (6) is also reversed.

LEMMA 2. [10] *Let* $x(t)$ *be a solution of* (1). *Suppose there exists some* $T\geqslant t_0$ *such that* $x(t)>0$ *for* $t\geqslant T$. *If condition* (i) *holds, then there exist* $T^{'}\geqslant T$ *and an integer* $l\in\{1,3,\cdots,n-1\}$ *such that for* $t\geqslant T^{'}$:

$$x^{(i)}(t)>0,\ \ i=0,1,\cdots,l,$$
$$(-1)^{i-1}x^{(i)}(t)>0,\ \ i=l+1,l+2,\cdots,n-1.$$

LEMMA 3. *Let* $x(t)$ *be a solution of* (1). *Suppose there exists some* $T\geqslant t_0$ *such that* $x(t)>0$ *for* $t\geqslant T$. *If conditions* $(i),(ii)$ *hold, then for any* $0<\theta<1$, *there exists some* $T^{'}\geqslant T$ *such that for* $t\geqslant T^{'}$, $x(t)\geqslant\frac{\theta}{(n-1)!}t^{n-1}x^{(n-1)}(t)$.

Proof. Without loss of generality, we may assume that $k_0 = 1$. Since $x(t) > 0 (t \geqslant T)$, by Lemma 2 there exists some $T_1 \geqslant T$ such that for $t \geqslant T_1$,

$$x^{'}(t) > 0 \cdots x^{(l)}(t) > 0,\ (-1)^{i-1}x^{(i)}(t) > 0,\ i = l+1, \cdots, n-1.$$

In order to finish our proof, we consider two cases as follows.

Case $1:\ l = 1$. Let $u(t) = -\frac{x^{(n-2)}(t)}{x^{(n-1)}(t)}$. Then $u(t) > 0,\ t \geqslant T_1$,

$$u^{'}(t) = \frac{-(x^{(n-1)}(t))^2 + x^{(n-2)}(t)x^{(n)}(t)}{(x^{(n-1)}(t))^2} = \frac{x^{(n)}(t)}{x^{(n-1)}(t)}u(t) - 1 < -1 \tag{7}$$

and $u(t_k^+) = \frac{a_k^{(n-2)}}{a_k^{(n-1)}}u(t_k), t_k \geqslant T_1$. By Lemma 1, we get

$$u(s) < u(t)\prod_{t \leqslant t_k < s}\frac{a_k^{(n-2)}}{a_k^{(n-1)}} - \int_t^s \prod_{u \leqslant t_k < s}\frac{a_k^{(n-2)}}{a_k^{(n-1)}}du, s > t \geqslant T_1. \tag{8}$$

Since $u(s) > 0$, we have

$$u(t) > \int_t^s \prod_{t \leqslant t_k < u}\frac{a_k^{(n-1)}}{a_k^{(n-2)}}du. \tag{9}$$

So

$$x^{(n-2)}(t) < -x^{(n-1)}(t)\int_t^s \prod_{t \leqslant t_k < u}\frac{a_k^{(n-1)}}{a_k^{(n-2)}}du. \tag{10}$$

It follows from (1) that

$$x^{(n)}(t) = -p(t)x(t-\tau) < 0,\ t \geqslant T_1 + \tau,$$
$$x^{(n-1)}(t_k^+) = a_k^{(n-1)}x^{(n-1)}(t_k),\ t_k \geqslant T_1 + \tau.$$

Set $v(t) = x^{(n-1)}(t)$. Then

$$v^{'}(t) < 0,\ t \geqslant T_1 + \tau,\ \ v(t_k^+) = a_k^{(n-1)}v(t_k), t_k \geqslant T_1 + \tau.$$

Applying Lemma 1, we obtain

$$v(u) < v(t)\prod_{t \leqslant t_k < u}a_k^{(n-1)}, u > t \geqslant T_1 + \tau.$$

That is

$$x^{(n-1)}(u) < x^{(n-1)}(t)\prod_{t \leqslant t_k < u}a_k^{(n-1)}, u > t \geqslant T_1 + \tau. \tag{11}$$

From (10) and (11), we have

$$x^{(n-2)}(t) < -\int_t^s \prod_{t \leqslant t_k < u}\frac{1}{a_k^{(n-2)}}x^{(n-1)}(u)du. \tag{12}$$

Let $m(t) = x^{(n-3)}(t)$. Then:

$$m(t_k^+) = x^{(n-3)}(t_k^+) = a_k^{(n-3)} x^{(n-3)}(t_k) = a_k^{(n-3)} m(t_k),$$
$$m'(t) < -\int_t^s \prod_{t \leqslant t_k < u} \frac{1}{a_k^{(n-2)}} x^{(n-1)}(u)du.$$

By Lemma 1, we have

$$m(s) < m(t) \prod_{t \leqslant t_k < s} a_k^{(n-3)} - \int_t^s \prod_{v \leqslant t_k < s} a_k^{(n-3)} \Big(\int_v^s \prod_{v \leqslant t_k < u} \frac{1}{a_k^{(n-2)}} x^{(n-1)}(u)du \Big) dv. \quad (13)$$

From condition (ii) and $m(s) > 0$, we get

$$x^{(n-3)}(t) > \int_t^s \prod_{t \leqslant t_k < u} \frac{1}{c_k^{(n-3)}} (u-t) x^{(n-1)}(u)du. \quad (14)$$

By repeating the above argument repeatedly, we may show that

$$x'(t) \geqslant \int_t^s \prod_{t \leqslant t_k < u} \frac{1}{c_k^{(1)}} \frac{(u-t)^{n-3}}{(n-3)!} x^{(n-1)}(u)du, \ s > t \geqslant T_1 + \tau. \quad (15)$$

Since $x(t_k^+) = a_k^{(0)} x(t_k)$, $t_k \geqslant T_1 + \tau$, by Lemma 1 and noting that

$$x^{(n-1)}(t) < \prod_{u \leqslant t_k < s} a_k^{(n-1)} x^{(n-1)}(u), \ t > u,$$

and condition (ii), we have

$$\begin{aligned} x(t) &> x(T_1^+) \prod_{T_1 < t_k < t} a_k^{(0)} + \int_{T_1}^t \prod_{v \leqslant t_k < t} a_k^{(0)} \Big(\int_v^s \prod_{v \leqslant t_k < u} \frac{1}{c_k^{(1)}} \frac{(u-v)^{n-3}}{(n-3)!} x^{(n-1)}(u)du \Big) dv \\ &> \int_{T_1}^t \prod_{v \leqslant t_k < t} a_k^{(0)} \Big(\int_v^s \prod_{v \leqslant t_k < u} \frac{1}{c_k^{(1)}} \frac{(u-v)^{n-3}}{(n-3)!} x^{(n-1)}(u)du \Big) dv \\ &> \int_{T_1}^t \prod_{v \leqslant t_k < v} a_k^{(0)} \Big(\int_v^t \prod_{v \leqslant t_k < u} \frac{1}{c_k^{(1)}} \frac{(u-v)^{n-3}}{(n-3)!} x^{(n-1)}(u)du \Big) dv \\ &\geqslant x^{(n-1)}(t) \int_{T_1}^t \prod_{v \leqslant t_k < t} a_k^{(0)} \Big(\int_v^t \prod_{v \leqslant t_k < u} \frac{1}{c_k^{(1)}} \prod_{u \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} \frac{(u-v)^{n-3}}{(n-3)!} du \Big) dv \\ &\geqslant x^{(n-1)}(t) \int_{T_1}^t \Big(\int_v^t \frac{(u-v)^{n-3}}{(n-3)!} du \Big) dv \geqslant x^{(n-1)}(t) \frac{(t-T_1)^{n-1}}{(n-1)!}. \end{aligned} \quad (16)$$

Thus, for any $0 < \theta < 1$, there exists some $T' \geqslant T_1 + \tau$ such that for $t \geqslant T'$,

$$x(t) \geqslant \frac{\theta}{(n-1)!} t^{n-1} x^{(n-1)}(t).$$

Case $2:\ 2<l\leqslant n-1$. From the proof of Lemma 3, we have:

$$x^{(l-1)}(t)>x^{(n-1)}(t)\frac{(t-T_1)^{n-l}}{(n-l)!},\ \ t\geqslant T_1+\tau,$$
$$x^{(l-2)}(t_k^+)=a_k^{(l-2)}x^{(l-2)}(t_k),\ \ t_k\geqslant T_1+\tau.$$

Let $w(t)=x^{(l-2)}(t)$. Then

$$w^{'}(t)>x^{(n-1)}(t)\frac{(t-T_1)^{n-l}}{(n-l)!}\quad\text{and}\quad w(t_k^+)=a_k^{(l-2)}w(t_k),\ \ t_k\geqslant T_1+\tau. \tag{17}$$

By Lemma 1 and noting that

$$x^{(n-1)}(t)<\prod_{u\leqslant t_k<t}a_k^{(n-1)}x^{(n-1)}(u),\ \ t>u\geqslant T_1+\tau$$

and condition (ii), we get

$$\begin{aligned}
w(t)&>w(T_1^+)\prod_{T_1<t_k<t}a_k^{(l-2)}+\int_{T_1}^{t}\prod_{u\leqslant t_k<t}a_k^{(l-2)}x^{(n-1)}(u)\frac{(u-T_1)^{n-l}}{(n-l)!}du\\
&>\int_{T_1}^{t}\prod_{u\leqslant t_k<t}a_k^{(l-2)}x^{(n-1)}(u)\frac{(u-T_1)^{n-l}}{(n-l)!}du\\
&>x^{(n-1)}(t)\int_{T_1}^{t}\prod_{u\leqslant t_k<t}\frac{a_k^{(l-2)}}{a_k^{(n-1)}}\frac{(u-T_1)^{n-l}}{(n-l)!}du\\
&>x^{(n-1)}(t)\int_{T_1}^{t}\frac{(u-T_1)^{n-l}}{(n-l)!}du=x^{(n-1)}(t)\frac{(t-T_1)^{n-l+1}}{(n-l+1)!}.
\end{aligned} \tag{18}$$

That is

$$x^{(l-2)}(t)>x^{(n-1)}(t)\frac{(t-T_1)^{n-l+1}}{(n-l+1)!}. \tag{19}$$

By repeating the above argument,we have

$$x(t)>x^{(n-1)}(t)\frac{(t-T_1)^{n-1}}{(n-1)!}. \tag{20}$$

Thus, for any $0<\theta<1$, there exists some $T^{'}\geqslant T_1+\tau\geqslant T$ such that for $t\geqslant T^{'}$, $x(t)\geqslant\frac{\theta}{(n-1)!}t^{n-1}x^{(n-1)}(t)$.

LEMMA 4. *Let* $x(t)$ *be an eventually positive solution of* (1). *If conditions* (i), (ii), (iii) *and* (iv) *hold, then for any sufficiently large* t, *there exists a* $t^*\in[t-\tau,t]$ *such that*

$$\frac{x^{(n-1)}(t^*-\tau)}{x^{(n-1)}(t^*)}<+\infty.$$

Proof. Without loss of generality, we may assume that $k_0 = 1$. Since $x(t)$ is a eventually positive solution of (1). We may assume that $x(t) > 0(t \geqslant T_1 \geqslant t_0)$. By Lemma 2, there exist a $T_2 \geqslant T_1$ and an integer $l \in \{1, 2, \ldots, n-1\}$ such that for $t \geqslant T_2$,

$$x^{'}(t) > 0 \cdots x^{(l)}(t) > 0, \quad (-1)^{(i-1)} x^{(i)}(t) > 0, \;\; i = l+1, \cdots, n-1.$$

From Lemma 3, there exists θ, $0 < \theta < 1$ and $T_3 \geqslant T_1$ such that for $t \geqslant T_3$,

$$x(t) \geqslant \frac{\theta}{(n-1)!} t^{n-1} x^{(n-1)}(t).$$

Since condition (iv) holds, so choosing $0 < L \leqslant c$, then exists $T^{'}$ such that for $t \geqslant T^{'}$

$$\int_{t-\tau}^{t} (s-\tau)^{n-1} p(s) ds > L. \tag{21}$$

Fixing a $t \geqslant T^{'}$,then there exists some $t^* \in [t-\tau, t]$ such that

$$\int_{t-\tau}^{t^*} (s-\tau)^{n-1} p(s) ds > \frac{L}{2} \;\text{ and }\; \int_{t^*}^{t} (s-\tau)^{n-1} p(s) ds > \frac{L}{2}. \tag{22}$$

We suppose that T is sufficiently large and (22) holds for $t \geqslant T$ and

$$x^{(n-1)}(t-\tau) > 0 \;\; x(t-\tau) \geqslant \frac{\theta}{(n-1)!} (t-\tau)^{n-1} x^{(n-1)}(t-\tau). \tag{23}$$

In view of (1) and (23), we have

$$x^{(n)}(t) + \frac{\theta}{(n-1)!} (t-\tau)^{n-1} p(t) x^{(n-1)}(t-\tau) \leqslant 0, t \geqslant T \tag{24}$$

Integrating the both sides of (24) on $[t-\tau, t^*]$, we have

$$\begin{aligned} x^{(n-1)}(t^*) - x^{(n-1)}((t-\tau)^+) - \sum_{t-\tau < t_k < t^*} (a_k^{(n-1)} - 1) x^{(n-1)}(t_k) \\ + \frac{\theta}{(n-1)!} \int_{t-\tau}^{t^*} (s-\tau)^{n-1} p(s) x^{(n-1)}(s-\tau) ds \leqslant 0. \end{aligned} \tag{25}$$

From (25), it follows that

$$\begin{aligned} x^{(n-1)}((t-\tau)^+) + \sum_{t-\tau < t_k < t^*} (a_k^{(n-1)} - 1) x^{(n-1)}(t_k) \\ \geqslant \frac{\theta}{(n-1)!} \int_{t-\tau}^{t^*} (s-\tau)^{n-1} p(s) x^{(n-1)}(s-\tau) ds. \end{aligned} \tag{26}$$

Since

$$x^{(n)}(t) < 0, \;\; t \geqslant T \quad \text{and} \quad x^{(n-1)}(t_k^+) = a_k^{(n-1)} x^{(n-1)}(t_k), \;\; t_k > T,$$

by Lemma 1, it is easy to see that

$$x^{(n-1)}(s-\tau) > \prod_{s-\tau\leqslant t_k<t^*-\tau} \frac{1}{a_k^{(n-1)}} x^{(n-1)}(t^*-\tau), t-\tau \leqslant s \leqslant t^*. \tag{27}$$

We claim

$$x^{(n-1)}((t-\tau)^+) + \sum_{t-\tau<t_k<t^*} (a_k^{(n-1)}-1)x^{(n-1)}(t_k) \leqslant \prod_{t-\tau<t_k<t^*} a_k x^{(n-1)}((t-\tau)^+). \tag{28}$$

Indeed, we assume that $t_{p+1} < t_{p+2} < \cdots < t_{p+m}$ are m impulsive points on $(t-\tau, t^*)$. Note that for $t_k > t-\tau$,

$$x^{(n-1)}(t_k) < \prod_{t-\tau<t_j<t_k} a_k^{(n-1)} x^{(n-1)}((t-\tau)^+) \leqslant \prod_{t-\tau<t_j<t_k} a_k^{(n-1)} x^{(n-1)}((t-\tau)^+).$$

Thus

$$\begin{aligned}
&x^{(n-1)}((t-\tau)^+) + \sum_{t-\tau<t_k<t^*} (a_k^{(n-1)}-1)x^{(n-1)}(t_k) \\
&\quad \leqslant x^{(n-1)}((t-\tau)^+) + [(a_{p+1}^{(n-1)}-1)x^{(n-1)}(t_{p+1}) + (a_{p+2}^{(n-1)}-1)x^{(n-1)}(t_{p+2}) + \cdots \\
&\qquad + (a_{p+m}^{(n-1)}-1)x^{(n-1)}(t_{p+m})] \\
&\quad = x^{(n-1)}((t-\tau)^+) + (a_{p+1}^{(n-1)}x^{(n-1)}(t_{p+1}) + a_{p+2}^{(n-1)}x^{(n-1)}(t_{p+2}) + \cdots \\
&\qquad + a_{p+m}^{(n-1)}x^{(n-1)}(t_{p+m})) - (x^{(n-1)}(t_{p+1}) + x^{(n-1)}(t_{p+2}) + \cdots + x^{(n-1)}(t_{p+m})) \\
&\quad \leqslant x^{(n-1)}((t-\tau)^+) + \sum_{t-\tau<t_k<t^*} (a_k-1) \prod_{t-\tau<t_j<t_k} a_k x^{(n-1)}((t-\tau)^+) \\
&\quad = x^{(n-1)}((t-\tau)^+)[1 + \sum_{t-\tau<t_k<t^*} (a_k-1) \prod_{t-\tau<t_j<t_k} a_k] \\
&\quad = x^{(n-1)}((t-\tau)^+)[1 + (a_{p+1}-1) + (a_{p+2}-1)a_{p+1} \\
&\qquad + \cdots + (a_{p+m}-1)a_{p+1}a_{p+2}\cdots a_{p+m-1}] \\
&\quad = x^{(n-1)}((t-\tau)^+)a_{p+1}a_{p+2}\cdots a_{p+m} = \prod_{t-\tau<t_k<t^*} a_k x^{(n-1)}((t-\tau)^+).
\end{aligned} \tag{29}$$

From (22), (25), (27) and (28), we have

$$\begin{aligned}
\prod_{t-\tau<t_k<t^*} a_k x^{(n-1)}((t-\tau)^+) &\geqslant \frac{\theta}{(n-1)!}\int_{t-\tau}^{t^*} (s-\tau)^{n-1} p(s) x^{(n-1)}(s-\tau) ds \\
&> \frac{\theta}{(n-1)!} x^{(n-1)}(t^*-\tau) \int_{t-\tau}^{t^*} \prod_{s-\tau\leqslant t_k<t^*-\tau} \frac{1}{a_k^{(n-1)}} (s-\tau)^{n-1} p(s) ds \\
&\geqslant \frac{L\theta}{2(n-1)!} \prod_{t-2\tau<t_k<t^*-\tau} \frac{1}{a_k} x^{(n-1)}(t^*-\tau), t-\tau < s < t^*.
\end{aligned} \tag{30}$$

So

$$x^{(n-1)}(t^*-\tau) < \frac{2(n-1)!}{L\theta} \prod_{t-\tau<t_k<t^*} a_k \prod_{t^-2\tau<t_k<t^*-\tau} a_k x^{(n-1)}((t-\tau)^+). \quad (31)$$

Similar to (31), we get

$$x^{(n-1)}((t-\tau)^+) < \frac{2(n-1)!}{L\theta} \prod_{t^*\leqslant t_k<t} a_k \prod_{t^*-\tau\leqslant t_k<t-\tau} a_k x^{(n-1)}(t^*). \quad (32)$$

So

$$x^{(n-1)}(t^*-\tau) < (\frac{2(n-1)!}{L\theta})^2 \prod_{t-\tau<t_k<t} a_k \prod_{t-2\tau<t_k<t-\tau} a_k x^{(n-1)}(t^*) \quad (33)$$

Thus, from condition (iii), we have $\frac{x^{(n-1)}(t^*-\tau)}{x^{(n-1)}(t^*)} < +\infty$.

REMARK 2. $x(t)$ is a eventually negative solution of (1),we also have conclusions similar to Lemmas 2-4.

LEMMA 5. [1] *Definite a sequence* $\{h^n(1)\}_{n=1}^{\infty}$ *satisfing*

$$h^1(1) = h(1),\ h^2(1) = e^{\theta L h^1(1)}, \cdots,\ h^n(1) = e^{\theta L h^{n-1}(1)},$$

where $h(s) = e^{\theta L s}$. *If* $\theta L > \frac{1}{e}$, *then sequence* $\{h^n(1)\}_{n=1}^{\infty}$ *is monotonically increasing and tends to* $+\infty$.

We now turn to the proof of theorems 1 and 2.

Proof of Theorem 1. Without loss of generality, we may assume that $k_0 = 1$. If (1) has a nonoscillation $x(t)$, we may assume that $x(t) > 0 (t \geqslant T \geqslant t_0)$. By Lemma 2, there exist $T_1 \geqslant T$ and an integer $l \in \{1, 3, \cdots, n-1\}$ such that for $t \geqslant T_1$,

$$x'(t) > 0 \cdots x^{(l)}(t) > 0,\ \ (-1)^{(i-1)} x^{(i)}(t) > 0,\ i = l+1, \cdots, n-1.$$

Let

$$c_1 = \lim_{t\to+\infty} \sup \frac{1}{(n-1)!} \int_t^{t+\tau} \prod_{s-\tau\leqslant t_k<s} \frac{1}{a_k^{(n-1)}} (s-\tau)^{n-1} p(s) ds.$$

From (2) we have $c_1 > 1$. Taking $L,\ 1 < L \leqslant c_1$, by Lemma 3 there exist a constant $\theta,\ 2/(L+1) < \theta < 1$ and $t^* \geqslant T_1$ such that for $t \geqslant t^* + \tau$,

$$x(t-\tau) \geqslant \frac{\theta}{(n-1)!} (t-\tau)^{n-1} x^{(n-1)}(t-\tau). \quad (34)$$

From (1) and (34), we have

$$x^{(n)}(t) + \frac{\theta}{(n-1)!} (t-\tau)^{n-1} p(t) x^{(n-1)}(t-\tau) \leqslant 0,\ t \geqslant t^* + \tau,\ t \neq t_k,\ t_k \geqslant t^* + \tau. \quad (35)$$

Set $v(t)=x^{(n-1)}(t)$. From above formula, it follows that

$$\begin{cases} v^{'}(t)\leqslant -\frac{\theta}{(n-1)!}(t-\tau)^{n-1}p(t)x^{(n-1)}(t-\tau),\ t\geqslant t^*+\tau,\ t\neq t_k,\\ v(t_k^+)=a_k^{(n-1)}v(t_k),\ t_k\geqslant t^*+\tau. \end{cases} \tag{36}$$

Applying Lemma 1, we obtain

$$\begin{aligned} v(t+\tau)\leqslant v(t^+)\prod_{t<t_k<t+\tau}a_k^{(n-1)}\\ -\frac{\theta}{(n-1)!}\int_t^{t+\tau}\prod_{s\leqslant t_k<t+\tau}a_k^{(n-1)}(s-\tau)^{n-1}p(s)x^{(n-1)}(s-\tau)ds. \end{aligned}$$

That is

$$\begin{aligned} x^{(n-1)}(t+\tau)\leqslant x^{(n-1)}(t^+)\prod_{t<t_k<t+\tau}a_k^{(n-1)}\\ -\frac{\theta}{(n-1)!}\int_t^{t+\tau}\prod_{s\leqslant t_k<t+\tau}a_k^{(n-1)}(s-\tau)^{n-1}p(s)x^{(n-1)}(s-\tau)ds. \end{aligned} \tag{37}$$

Since

$$x^{(n)}(t)<0,\ t\geqslant t^*+\tau \text{ and } x^{(n-1)}(t_k^+)=a_k^{(n-1)}x^{(n-1)}(t_k),\ t_k>t^*+\tau,$$

by Lemma 1, we have:

$$x^{(n-1)}(s-\tau)\geqslant x^{(n-1)}(t^+)\prod_{s-\tau\leqslant t_k\leqslant t}\frac{1}{a_k^{(n-1)}}. \tag{38}$$

From the inequalites (37) and (38), it follows that

$$\begin{aligned} x^{(n-1)}(t+\tau)\leqslant x^{(n-1)}(t^+)\prod_{t<t_k<t+\tau}a_k^{(n-1)}\Big[1-\\ \frac{\theta}{(n-1)!}\int_t^{t+\tau}\prod_{s-\tau\leqslant t_k<s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds\Big],\ t\geqslant t^*+\tau. \end{aligned} \tag{39}$$

In view of (2). Then for $\varepsilon=\frac{L-1}{2}>0$, there exists $T_\lambda>t^*+\tau$ such that

$$\frac{1}{(n-1)!}\int_{T_\lambda}^{T_\lambda+\tau}\prod_{s-\tau\leqslant t_k<s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds>L-\varepsilon=\frac{L+1}{2}. \tag{40}$$

From (39), (40) and $\frac{2}{L+1}<\theta<1$, we have

$$x^{(n-1)}(T_\lambda+\tau)\leqslant x^{(n-1)}(T_\lambda^+)\Big[1-$$

$$\frac{\theta}{(n-1)!}\int_{T_\lambda}^{T_\lambda+\tau}\prod_{s-\tau\leqslant t_k<s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds\Big]<0. \quad (41)$$

This is a contrary to the fact that $x^{(n-1)}(t)>0,\ t\geqslant T_1$. Thus every solution of (1) is oscillatory.

Proof of Theorem 2. Without loss of generality, we may assume that $k_0=1$. If (1) has a nonoscillation $x(t)$, we may assume that $x(t)>0(t\geqslant T\geqslant t_0)$. By Lemma 2, there exist $T_1\geqslant T$ and an integer $l\in\{1,3,\cdots,n-1\}$ such that for $t\geqslant T_1$,

$$x^{'}(t)>0\ \cdots x^{(l)}(t)>0,\ \ (-1)^{(i-1)}x^{(i)}(t)>0,\ i=l+1,\cdots,n-1.$$

Let

$$c_2=\liminf_{t\to+\infty}\frac{1}{(n-1)!}\int_{t-\tau}^{t}\prod_{s-\tau\leqslant t_k\leqslant s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds,$$

(3) yields $c_2>1/e$. Taking L such that $1/e<L\leqslant c$, by Lemma 3 there exist a constant θ, $2/(Le+1)<\theta<1$ and $t^*\geqslant T_1$ such that for $t\geqslant t^*+\tau$,

$$x(t-\tau)\geqslant\frac{\theta}{(n-1)!}(t-\tau)^{n-1}x^{(n-1)}(t-\tau). \quad (42)$$

From Lemma 4, for any sufficiently large t, there exists $T_2\in[t-\tau,t]$ such that

$$\frac{x^{(n-1)}(T_2-\tau)}{x^{(n-1)}(T_2)}<+\infty \quad (43)$$

From (3), it follows that for sufficiently large t

$$\frac{1}{(n-1)!}\int_{t-\tau}^{t}\prod_{s-\tau\leqslant t_k<s}\frac{1}{a_k^{(n-1)}}(s-\tau)^{n-1}p(s)ds>L. \quad (44)$$

We suppose that (42), (43) and (44) hold for $t\geqslant t^*+\tau$. On the other hand, noting that

$$x^{(n-1)}(t-\tau)\geqslant\prod_{t-\tau\leqslant t_k<t}\frac{1}{a_k^{(n-1)}}x^{(n-1)}(t). \quad (45)$$

From (1), (42) and (45), we have

$$x^{(n)}(t)+\frac{\theta}{(n-1)!}\prod_{t-\tau\leqslant t_k<t}\frac{1}{a_k^{(n-1)}}(t-\tau)^{n-1}p(t)x^{(n-1)}(t)\leqslant 0,\ t\geqslant t^*+\tau. \quad (46)$$

Set $v(t)=x^{(n-1)}(t)$, then $v(t_k^+)=a_k^{(n-1)}v(t_k)$ and

$$v^{'}(t)\leqslant-\frac{\theta}{(n-1)!}\prod_{t-\tau\leqslant t_k<t}\frac{1}{a_k^{(n-1)}}(t-\tau)^{n-1}p(t)v(t),\ t\geqslant t^*+\tau,\ t\neq t_k. \quad (47)$$

Applying Lemma 1, we obtain

$$x^{(n-1)}(t) \leqslant x^{(n-1)}(t-\tau) \prod_{t-\tau \leqslant t_k < t} a_k^{(n-1)} \exp\Big[- \frac{\theta}{(n-1)!} \int_{t-\tau}^{t} \prod_{s-\tau \leqslant t_k < s} \frac{1}{a_k^{(n-1)}} (s-\tau)^{n-1} p(s) ds \Big]. \quad (48)$$

That is

$$\frac{x^{(n-1)}(t-\tau)}{x^{(n-1)}(t)} \geqslant \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} \exp\Big[\frac{\theta}{(n-1)!} \int_{t-\tau}^{t} \prod_{s-\tau \leqslant t_k < s} \frac{1}{a_k^{(n-1)}} (s-\tau)^{n-1} p(s) ds \Big]$$
$$\geqslant \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} e^{\theta L} = \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} h^1(1) \quad (49)$$

where $h(s) = e^{\theta L s}$, $\theta L > 1/e$. Thus, from (1), (42), (45) and (49), we have

$$x^{(n)}(t) + \frac{\theta}{(n-1)!} h^1(1) \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} (t-\tau)^{n-1} p(t) x^{(n-1)}(t) \leqslant 0,\ t \geqslant t^* + \tau. \quad (50)$$

Applying the above method, by Lemma 1 we have

$$\frac{x^{(n-1)}(t-\tau)}{x^{(n-1)}(t)} \geqslant \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} e^{\theta L h^1(1)} = \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} h^2(1),\ t \geqslant t^* + 2\tau. \quad (51)$$

By repeating the above method repeatedly, we get

$$\frac{x^{(n-1)}(t-\tau)}{x^{(n-1)}(t)} \geqslant \prod_{t-\tau \leqslant t_k < t} \frac{1}{a_k^{(n-1)}} h^n(1),\ t \geqslant t^* + n\tau, n = 1, 2, \ldots. \quad (52)$$

From Lemma 5, we have $h^n(1) \to \infty$. That is $\frac{x^{(n-1)}(t-\tau)}{x^{(n-1)}(t)} \to \infty$. This is a contrary to (43). Thus every solution of (1) is oscillatory.

REMARK 3. It is easy to see that the above theorems 1 and 2 can be generalized to the equations of the form

$$\begin{cases} x^{(n)}(t) + \sum_{j=1}^{m} p_j(t) x(t-\tau_j) = 0, & t \geqslant t_0,\ t \neq t_k, \\ x^{(i)}(t_k^+) = a_k^{(i)} x^{(i)}(t_k),\ i = 0, 1, \cdots n-1,\ k = 1, 2 \cdots \end{cases}$$

where n is even, $\tau_j > 0$. Indeed, if we replace the conditions $(iii), (iv)$, (2) and (3) with $(iii'), (iv')$, $(2')$ and $(3')$ as follows:
(iii') $\lim_{t \to +\infty} \sup \prod_{t-2\tau < t_k < t} a_k < +\infty$, where $a_k = \max\{a_k^{(n-1)}, 1\}, \tau = \max\{\tau_1, \ldots, \tau_m\}$,

(iv') $c = \liminf_{t\to+\infty} \int_{t-\tau'}^{t} \sum_{j=1}^{m} (s-\tau_j)^{n-1} p_j(s)ds > 0$, where $\tau' = \min\{\tau_1, \dots, \tau_m\}$

$(2')$

$$\limsup_{t\to+\infty} \int_t^{t+\tau} \sum_{j=1}^{m} \prod_{s-\tau_j \leqslant t_k < s} \frac{1}{a_k^{(n-1)}} (s-\tau_j)^{n-1} p_j(s)ds > (n-1)!,$$

$(3')$

$$\liminf_{t\to+\infty} \int_{t-\tau'}^{t} \sum_{j=1}^{m} \prod_{s-\tau_j \leqslant t_k < s} \frac{1}{a_k^{(n-1)}} (s-\tau')^{n-1} p_j(s)ds > \frac{(n-1)!}{e}.$$

Then, we will obtain the similar Theorems of the above equations.

EXAMPLE 1. Consider the Equation

$$\begin{cases} x^{(n)}(t) + \frac{1\times 3\times\cdots\times(2n-3)}{2^n} t^{-n+\frac{1}{2}} (t-\frac{1}{3})^{-\frac{1}{2}} x(t-\frac{1}{3}) = 0, \ t \geqslant \frac{1}{2}, \ t \neq k, \\ x^{(i)}(k^+) = \frac{k^{i-2}}{(k+1)^i} (x^{(i)}(k)), \ i = 0,1,2\cdots,n-1, \ k = 1,2,\cdots \end{cases} \tag{53}$$

where n is even, $t_k = k$,

$$a_k^{(i)} = \frac{k^{i-2}}{(k+1)^i}, \ \ p(t) = \frac{1\times 3\times\cdots\times(2n-3)}{2^n} t^{-n+\frac{1}{2}} (t-\frac{1}{3})^{-\frac{1}{2}} \text{ and } \ \tau = \frac{1}{3}.$$

It is easy to see that conditions (i) and (ii) satisfy:

$$\begin{aligned} &\limsup_{t\to+\infty} \int_t^{t+\tau} \prod_{s-\tau \leqslant t_k < s} \frac{1}{a_k^{(n-1)}} (s-\tau)^{n-1} p(s)ds \\ &\qquad\qquad = \limsup_{k\to+\infty} \frac{1}{a_k^{(n-1)}} \int_{t_k}^{t_k+\tau} (s-\tau)^{n-1} p(s)ds = +\infty > (n-1)!. \end{aligned}$$

By Theorem 1, every solution of (53) is oscillatory. But the delay differential equation

$$x^{(n)}(t) + \frac{1\times 3\times\cdots\times(2n-3)}{2^n} t^{-n+\frac{1}{2}} (t-\frac{1}{3})^{-\frac{1}{2}} x(t-\frac{1}{3}) = 0, \ t \geqslant \frac{1}{2}$$

has a nonnegative solution $x = \sqrt{t}$. This example shows that impulses play an important role in the oscillatory behavior of equations under perturbing impulses.

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