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ON DEGENERATE NON–UNIFORMLY ELLIPTIC PROBLEMS

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Abstract. We are interested in the degenerate problem: $b(v)-\mathrm{div}A(v,\nabla g(v))=f$ in Ω with the boundary condition $v=a$, where $a:\partial\Omega\to\mathbb{R}$ is measurable such that $g(a)=0$. We suppose that the vector field A satisfies the Leray-Lions conditions, that b,g are continuous, nondecreasing with $\lim_{r\to\pm\infty}|b+g|(r)<+\infty$, that g hat a flat region $[A_1,A_2]$ and is strictly increasing on $\mathbb{R}\setminus[A_1,A_2]$ for some $A_1\leqslant 0\leqslant A_2$. Using monotonicity methods, we prove the existence and uniqueness of a renormalized entropy solution (with possibly infinite values).

1. Introduction

In this paper, we study a class of partially degenerate elliptic problems of the type:

$$P_{b,g}(f,a)\begin{cases} b(v)-\ \mathrm{div}\,A(v,\nabla g(v))=f & \text{in } \Omega,\\ v=a & \text{on } \Gamma:=\partial\Omega,\end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with regular boundary if $N>1$, $f\in L^1(\Omega)$, $a:\Gamma\to\mathbb{R}$ is measurable with $g(a)=0$ a.e. on Γ, $b,g:\mathbb{R}\to\mathbb{R}$ are nondecreasing, continuous such that $b(0)=g(0)=0$ and $\lim_{r\to\pm\infty}|b+g|(r)<+\infty$. The vector field $A:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is continuous and satisfies the following conditions:

- the growth condition: there exists $p>1$ such that

$$|A(r,\xi)-A(r,0)|\leqslant C(|r|)|\xi|^{p-1}\quad \text{for all } (r,\xi)\in\mathbb{R}\times\mathbb{R}^N, \tag{1.2}$$

$$r\mapsto|A(r,0)| \text{ is bounded,} \tag{1.3}$$

where $C:\mathbb{R}^+\to\mathbb{R}^+$ is non-decreasing such that

$$\int_{-\infty}^{+\infty}(C(|r|))^{\frac{p}{p-1}}d\,g(r)<\infty, \tag{1.4}$$

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- the coerciveness condition:

$$(A(r,\xi)-A(r,0))\cdot\xi \geqslant \lambda|\xi|^p \quad \text{for all } r\in\mathbb{R},\ \xi\in\mathbb{R}^N \tag{1.5}$$

  for some $\lambda>0$.

The function g is locally Lipschitz and has a flat region on which it keeps a constant value i.e. there exists $A_1 \leqslant A_2 \in \mathbb{R}$ such that g is constant in $[A_1,A_2]$ and strictly increasing in $\mathbb{R}\setminus[A_1,A_2]$. For simplicity, we assume $A_1\leqslant 0\leqslant A_2$ and that $g\equiv 0$ on $[A_1,A_2]$. Conditions (1.2) and (1.5) are rather classical in the theory of elliptic problems and assure the boundedness and the coerciveness of the operator $v\to -\operatorname{div} A(v,\nabla v)$. Remark that in our framework, the problem $P_{b,g}(f,a)$ is not uniformly elliptic and the classical theory of Leray-Lions is not available even in the variational setting. A model example is the problem $b(v)-\operatorname{div}\phi(v)+\Delta g(v)=f$ where $\phi:\mathbb{R}\to\mathbb{R}^N$ is a continuous vector field and $b,g:\mathbb{R}\to\mathbb{R}$ are nondecreasing, continuous such that $b(0)=g(0)=0$. In the case where b and g verify the range condition $R(b+g)=\mathbb{R}$, existence and uniqueness results of a renormalized entropy solution with a.e. finite values are already proved in [3] for the corresponding evolution problem. Here, we also cover the case where the operator $v\to -\operatorname{div} A(v,\nabla g(v))$ strongly degenerates when $|v|\to\infty$. This means that the solutions (at least those obtained by approximation methods) may reach the values $+\infty$ or $-\infty$.

Let us consider the simple case where $b\equiv 0, \phi\equiv 0$ and g is increasing with $\lim_{r\to\pm\infty} g(r)<\infty$: For $f\in L^2(\Omega)$, we try to solve the problem

$$-\Delta g(v)=f \quad \text{in } \Omega,\ \ g(v)=0 \ \text{ on } \Gamma, \tag{1.6}$$

or equivalently to find v such that $g(v)\in W_0^{1,2}(\Omega)$ and $g(v)$ is equal to the unique solution of $-\Delta u=f$ in Ω, $u=0$ on Γ. As u is in general not in the range of g, it is clear that even in the variational setting, the problem has not usually a weak solution with a.e. finite values.

On the other hand, as g is only assumed to be nondecreasing (which means that the diffusion term can partially degenerate), it is well known that the problem is ill-posed in the variational setting. Indeed, the weak solution in the usual distributional sense is not suitable in order to assure uniqueness results (see [13], [18] and [19]). Furthermore, the condition on the boundary can not be assumed pointwise but has to be understood as an entropy condition on the boundary (see [4], [13], [22], and the bibliography therein). Remark that by our assumptions on g, it follows that $|a|\leqslant d:=\max(-A_1,A_2)$ a.e. on Γ and that $\overline{A}(a)\in L^\infty(\Gamma)$, where $\overline{A}:\mathbb{R}\times\partial\Omega\to\mathbb{R}$ is defined by

$$\overline{A}(s):=\sup\{|A(r,0)\cdot\vec{\eta}(x)|,\ r\in[-s^-,s^+]\}.$$

Here, we denote by $\vec{\eta}(x)$ the unit outer normal to $\partial\Omega$ in x.

In order to prove the "partial" uniqueness result, we assume that A verifies the additional condition

$$(A(r,\xi)-A(s,\eta))\cdot(\xi-\eta)\geqslant (B(g(r))-B(g(s)))(1+|\xi|^p+|\eta|^p), \tag{1.7}$$

for all $r,s \in \mathbb{R}$, $\xi,\eta \in \mathbb{R}^N$, for some function $B : \mathbb{R} \to \mathbb{R}$ which is locally Lipschitz continuous on $\mathbb{R}$.

Hypothesis (1.7) implies in particular that

$$(A(r,\xi) - A(r,\eta)) \cdot (\xi - \eta) \geqslant 0 \text{ for all } r \in \mathbb{R}.$$

An other difficulty is related to the nonregularity of our data f which is only supposed to be in $L^1(\Omega)$. To overcome this problem, we use the notion of renormalized entropy solution introduced in [8] which only involves the truncations of the solution in the entropy inequalities.

More results on non uniformly coercive problems can be found in among other manuscripts [10], [11] and the bibliography therein. The reader interested in degenerate diffusion problems is referred to [5], [13], [20], [23].

The outline of the paper is as follows: In the following section, after a short introduction of our notations, we give our concept of renormalized entropy solution with a few comments and we present the main results. Section 3 is devoted to the proof of the partial uniqueness result. Finally, in Section 4, we establish the existence result.

2. Notations, definitions and main results

In this section, we give our definition of renormalized entropy solution with a few comments, then we present our main results. Let us first set some notations. We denote by $\mathscr{M}(\overline{\Omega})$ the set of Radon measures on $\overline{\Omega}$ and by $M(\Gamma)$ the set of measurable on Γ with values in $\mathbb{R}$. For any measurable function $v : \Omega \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, for any $s \in \overline{\mathbb{R}}$, we denote by $\chi_{\{v<s\}}$ (resp, $\chi_{\{v>s\}}$, $\chi_{\{v=s\}}$) the characteristic function of the set $\{x \in \Omega; v(x) < s\}$ (resp, $\{x \in \Omega; v(x) > s\}$, $\{x \in \Omega; v(x) = s\}$). For any $k \geqslant 0$, the functions T_k, h_k and H_k are defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k,r))$, $h_k(r) = 1 - |T_{k+1}(r) - T_k(r)|$ and $H_k(r) := \min(\frac{r^+}{k}, 1)$. The operators sign^+ and H_0 are defined by: $\operatorname{sign}^+(r) = 0$ if $r < 0$, $= [0,1]$ if $r = 0$ and $= 1$ if $r > 0$ and

$$H_0(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

Moreover, for $r,k \in \mathbb{R}$, we set $r \vee k = \max(r,k)$, $r \wedge k = \min(r,k)$, $r \vee +\infty = +\infty$, $r \wedge +\infty = r = r \vee -\infty$, $r \wedge -\infty = -\infty$.

By $T^1, T^{1,2}$ and T^2, we denote the truncation functions defined successively by

$$T^1(r) = r \wedge A_1, \;\; T^{1,2}(r) = A_1 \vee r \wedge A_2 \quad \text{and } T^2(r) = r \vee A_2$$

and for $k,l \in \mathbb{R}$, for a.e. $x \in \Gamma$, we define

$$\omega^+(x,k,l) := \max_{k \leqslant r,s \leqslant l \vee k} |(A(r,0) - A(s,0)) \cdot \vec{\eta}(x)|,$$

$$\omega^-(x,k,l) := \max_{l \wedge k \leqslant r,s \leqslant k} |(A(r,0) - A(s,0)) \cdot \vec{\eta}(x)|.$$

Finally, we denote

$$E := \{r \in R(g)/(g^{-1})_0 \text{ is discontinuous at } r\}. \tag{2.1}$$

Throughout the paper we suppose that conditions (1.2)-(1.7) are satisfied.

DEFINITION 2.1. Let $f \in L^1(\Omega)$ and $a \in M(\Gamma)$ with $g(a) = 0$ a. e. on Γ. A measurable function $v : \Omega \to \overline{\mathbb{R}}$ is said to be a renormalized entropy solution of the problem $P_{b,g}(f,a)$ if

$$g(T_k v) = T_{g(k)} g(v) \in W_0^{1,p}(\Omega), \ \forall k > 0, \tag{2.2}$$

$$A(v, Dg(v))\chi_{\{|v|<\infty\}} \in (L^{p'}(\Omega))^N \tag{2.3}$$

and there exist two families $(\mu_l)_l$ and $(\nu_l)_l$ of bounded measures on $\overline{\Omega}$ such that:

$$\mu_l, \nu_l \in (W^{-1,p'}(\Omega) + L^1(\Omega) + L^1(\Gamma)) \cap \mathscr{M}(\overline{\Omega}), \tag{2.4}$$

$$\mu_l(\{v \leqslant l\}) = 0, \ \nu_l(\{v \geqslant l\}) = 0, \tag{2.5}$$

$$\lim_{l\to\infty} \int_\Omega \xi \, d\mu_l(v) = 0 \text{ for all } \xi \in \mathscr{D}^+(\mathbb{R}^N) \text{ with } \operatorname{supp}(\nabla\xi) \subset \{v < +\infty\}, \tag{2.6}$$

$$\lim_{l\to\infty} \int_\Omega \xi \, d\nu_l(v) = 0 \text{ for all } \xi \in \mathscr{D}^+(\mathbb{R}^N) \text{ with } \operatorname{supp}(\nabla\xi) \subset \{v > -\infty\}, \tag{2.7}$$

and the following inequalities are satisfied: for all $l \geqslant k \in \mathbb{R}$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $\operatorname{sign}^+(g(a\wedge l) - g(k))\xi = 0$ a.e. on Γ,

$$\begin{aligned} -\int_\Omega b(v\wedge l)\chi_{\{v\wedge l>k\}}\xi - \int_\Omega \chi_{\{v\wedge l>k\}}(A(v\wedge l, \nabla g(v\wedge l)) - A(k,0))\cdot\nabla\xi \\ + \int_\Omega \chi_{\{v\wedge l>k\}} f\xi \geqslant -\int_\Gamma \omega^+(x,k,a\wedge l)\xi + \int_\Omega \xi \, d\mu_l(v) \end{aligned} \tag{2.8}$$

and for all $l \leqslant k \in \mathbb{R}$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $\operatorname{sign}^+(g(k) - g(a\vee l))\xi = 0$ a.e. on Γ,

$$\begin{aligned} \int_\Omega b(v\vee l)\chi_{\{k>v\vee l\}}\xi + \int_\Omega \chi_{\{k>v\vee l\}}(A(v\vee l, \nabla g(v\vee l)) - A(k,0))\cdot\nabla\xi \\ - \int_\Omega \chi_{\{k>v\vee l\}} f\xi \geqslant -\int_\Gamma \omega^-(x,k,a\vee l)\xi + \int_\Omega \xi \, d\nu_l(v). \end{aligned} \tag{2.9}$$

Here, we set $-\infty \wedge l = -\infty$, $+\infty \vee l = +\infty$, $+\infty \wedge l = -\infty \vee l = l$ and we extend the truncation function T_k to $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ by setting $T_k(+\infty) = k$ and $T_k(-\infty) = -k$.

REMARK 2.2. 1. As in [9], we denote by $\chi_{\{|v|<\infty\}} A(v, Dg(v))$ (in (2.3)), the measurable field on Ω satisfying

$$\chi_{\{|v|<k\}}\chi_{\{|v|<\infty\}} A(v, Dg(v)) = A(v, \nabla g(T_k v)) \text{ for all } k > 0.$$

It makes sense thanks to condition (2.2) on v.

2. A simple computation shows that if v is a renormalized solution of $P_{b,g}(f,a)$ then $-v$ is a renormalized entropy solution of

$$\tilde{P}_{\tilde{b},\tilde{g}}(-f,-a)\begin{cases}\tilde{b}(v)-\operatorname{div}\tilde{A}(v,\nabla\tilde{g}(v))=-f & \text{in } \Omega\\ v=-a & \text{on } \Gamma:=\partial\Omega,\end{cases}$$

with $\tilde{g}(r)=-g(-r)$, $\tilde{b}(r)=-b(-r)$ and $\tilde{A}(r,\xi)=-A(-r,-\xi)$.

3. Condition (2.3) is not satisfied if we not assume (1.4). Indeed, in the case where $b\equiv 0$, $g(r)=r$ and $A(r,\xi)=\xi$, it is well known that the entropy solution of the problem $-\Delta u=f$, $f\in L^1(\Omega)$ is not usually in $W^{1,p}(\Omega)$ (see [9]).

4. As g is increasing on $\mathbb{R}\setminus[A_1,A_2]$, it follows from the definition that for all $k>0$ and $L>d:=\max(|A_1|,|A_2|)$, the function $T_{k+L}v-T_Lv\in W_0^{1,p}(\Omega)$.

5. Taking $\xi=\varphi\in\mathscr{D}^+(\Omega)$ with $\nabla\varphi=0$ a.e. on $v=+\infty$ as test function in (2.8), letting $l\to+\infty$, taking into account (2.6), we find

$$\int_\Omega \chi_{\{k<v<\infty\}}\big[b(v)\varphi+(A(v,Dg(v))-A(k,0))\cdot\nabla\varphi-f\varphi\big]$$
$$\leqslant\int_{\{v=+\infty\}}f\varphi-b(+\infty)\int_{\{v=+\infty\}}\varphi. \quad (2.10)$$

Similarly, taking $\xi=\varphi\in\mathscr{D}^+(\Omega)$ with $\nabla\varphi=0$ a.e. on $v=-\infty$ as test function in (2.9), letting $l\to-\infty$, taking into account (2.7), we find

$$-\int_\Omega \chi_{\{k>v>-\infty\}}\big[b(v)\varphi+(A(v,Dg(v))-A(k,0))\cdot\nabla\varphi-f\varphi\big]$$
$$\leqslant-\int_{\{v=-\infty\}}f\varphi+b(-\infty)\int_{\{v=-\infty\}}\varphi. \quad (2.11)$$

6. In particular, it follows from (2.10) that

$$\int_\Omega \chi_{\{k\leqslant v<\infty\}}\big[b(v)\varphi+(A(v,Dg(v))-A(k,0))\cdot\nabla\varphi-f\varphi\big]$$
$$\leqslant\int_{\{v=+\infty\}\cup\{v=k\}}f\varphi-b(+\infty)\int_{\{v=+\infty\}}\varphi+b(k)\int_{\{v=k\}}\varphi. \quad (2.12)$$

Similarly, it follows from (2.11) that

$$-\int_\Omega \chi_{\{k\geqslant v>-\infty\}}\big[b(v)\varphi+(A(v,Dg(v))-A(k,0))\cdot\nabla\varphi-f\varphi\big]$$
$$\leqslant-\int_{\{v=-\infty\}\cup\{v=k\}}f\varphi+b(-\infty)\int_{\{v=-\infty\}}\varphi-b(k)\int_{\{v=k\}}\varphi. \quad (2.13)$$

In the framework of example (1.6), with the same hypothesis described in the introduction, letting $k\to-\infty$ in (2.12) and $k\to+\infty$ in (2.13), combining the two inequalities, we find

$$\int_\Omega \chi_{\{|v|<\infty\}}\big[b(v)\varphi+(A(v,Dg(v))-A(k,0))\cdot\nabla\varphi-f\varphi\big]$$

$$= \int_{\{v=+\infty\}} f\varphi - \int_{\{v=-\infty\}} f\varphi - b(+\infty)\int_{\{v=+\infty\}} \varphi + b(-\infty)\int_{\{v=-\infty\}} \varphi, \quad (2.14)$$

which is the definition of renormalized solution proposed in [10].

7. Now, under the same assumption as in (6), taking $\xi = \varphi(1-h_p(v^+))$ with $\varphi \in \mathscr{D}^+(\Omega)$ and $\nabla\varphi = 0$ a.e. on $v = +\infty$ as test function in (2.12), letting $p \to +\infty$, we find the energie estimate

$$\int_{\{v=+\infty\}} f\varphi - b(+\infty)\int_{\{v=+\infty\}} \varphi = \lim_{p\to+\infty}\int_{\{p\leqslant v\leqslant p+1\}} A(T_{p+1}v, \nabla g(T_{p+1}v))\cdot\nabla(T_{p+1}v)\varphi. \quad (2.15)$$

Similarly, taking $\xi = \varphi(1-h_p(-v^-))$ with $\nabla\varphi = 0$ a.e. on $v = -\infty$ as test function in (2.13), letting $p \to +\infty$, we find

$$-\int_{\{v=-\infty\}} f\varphi + b(-\infty)\int_{\{v=-\infty\}} \varphi = \lim_{p\to+\infty}\int_{\{-p-1\leqslant v\leqslant -p\}} A(T_{p+1}v, \nabla g(T_{p+1}v))\cdot\nabla(T_{p+1}v)\varphi. \quad (2.16)$$

Here, we denote $r^+ = r\vee 0$ and $r^- = (-r)\vee 0$.

8. We have considered in this paper the case where $g+b$ is bounded on $\mathbb{R}$ but we can also deal with functions b, g such that $\lim_{r\to+\infty}(b+g)(r) = +\infty$ and $\lim_{r\to-\infty}(b+g)(r) \neq -\infty$ or $\lim_{r\to+\infty}(b+g)(r) \neq +\infty$ and $\lim_{r\to-\infty}(b+g)(r) = -\infty$. In the first case, (as shown in [2]) $v < +\infty$ a.e. on Ω and (2.6) holds true for every test function $\xi \in \mathscr{D}^+(\mathbb{R}^N)$ without any extra condition on supp(ξ). Similarly, in the second case, $v > -\infty$ a.e. on Ω and (2.7) holds true for every test function $\xi \in \mathscr{D}^+(\mathbb{R}^N)$.

9. We can also consider the case where the function g has a finite number of flat regions not necessarily arround 0.

The main results are the following.

THEOREM 2.3. *For all $f \in L^1(\Omega)$ and $a \in M(\Gamma)$ with $g(a) = 0$ a.e. on Γ, there exists $v : \Omega \to \overline{\mathbb{R}}$ such that v is a renormalized entropy solution of $P_{b,g}(f,a)$.*

We also prove the following comparison result.

THEOREM 2.4. *Let $(a_1, f_1) \in C(\Gamma)\times L^1(\Omega)$, $(a_2, f_2) \in M(\Gamma)\times L^1(\Omega)$ with a_2 satisfying $g(a_1) = g(a_2) = 0$ a.e. on Γ. Let v_1 be a renormalized entropy solution of the problem $P_{b,g}(a_1, f_1)$, v_2 be a renormalized entropy solution of the problem $P_{b,g}(a_2, f_2)$. Then, for any $l < 0$, there exists $\kappa \in L^\infty(\Omega)$ with $\kappa \in sign^+(T_l v_1 - T_l v_2)$ a.e. in Ω such that, for any $\xi \in \mathscr{D}^+(\mathbb{R}^N)$,*

$$\int_\Omega (b(T_l v_1) - b(T_l v_2))^+\xi$$

$$+\int_\Omega (A(T_l v_1, \nabla g(T_l v_1)) - A(T_l v_2, \nabla g(T_l v_2))) \cdot \nabla \xi \chi_{\{T_l v_1 > T_l v_2\}}$$
$$\leqslant \int_\Omega \kappa (f_1 - f_2)\xi - 2\int_\Omega \xi \, d(\mu_l(v_1) + \nu_{-l}(v_2)) + \int_\Gamma \omega^-(x, T_l a_1, T_l a_2)\xi. \tag{2.17}$$

As a consequence, we deduce the following "partial uniqueness" result.

THEOREM 2.5. *Let $f \in L^1(\Omega)$ and $a : \Gamma \to \mathbb{R}$ with $g(a) = 0$ a.e. on Γ. Let v_i, $i = 1,2$ be two renormalized entropy solutions of $P_{b,g}(f,a)$ with $\{v_1 = +\infty\} = \{v_2 = +\infty\}$ and $\{v_1 = -\infty\} = \{v_2 = -\infty\}$. Then, $b(v_1) = b(v_2)$ a.e. in Ω.*

Theorem 2.5 makes sense only in the case where b is not completely degenerate. If b is strictly increasing, (under the assumptions of Theorem 2.5), it follows that $v_1 = v_2$ a.e. on Ω. In the case where $b \equiv 0$, we need some extra conditions on the field A in order to deduce a partial uniqueness result in $g(v)$.

3. Proofs of the comparison and uniqueness results

The following Lemma plays a crucial role in the proof of the comparison result.

LEMMA 3.1. *Let $f \in L^1(\Omega)$, $a \in M(\Gamma)$ with $g(a) = 0$ a.e. on Γ and v be a renormalized entropy solution of $P_{b,g}(a,f)$. Then, for every $l > 0$, there exists a positive constant C depending only on $\| f \|_{L^1(\Omega)}$ and l such that for any $\delta > 0$,*

$$\int_{\Omega \cap \{0 \leqslant g(T_l v) \leqslant \delta\}} |\nabla g(T_l v)|^p \leqslant \delta C, \tag{3.1}$$

$$\int_{\Omega \cap \{0 \leqslant -g(T_l v) \leqslant \delta\}} |\nabla g(T_l v)|^p \leqslant \delta C. \tag{3.2}$$

Proof. We use $\xi := T_\delta(g(T_l v)^+ \in W_0^{1,p}(\Omega)$ as test function in (2.8). Then, by the Green-Gauss formula and the growth condition (1.2), we obtain (3.1). The second estimation (3.2) can be proved in a similar way.

LEMMA 3.2. *Let $(a,f) \in M(\Gamma) \times L^1(\Omega)$ with $g(a) = 0$ a.e. on Γ and v be a renormalized entropy solution of $P_{b,g}(a,f)$. Then*

$$\int_\Omega \chi_{\{v \wedge l > k\}} \{-b(v \wedge l)\xi + f\xi - (A(v \wedge l, \nabla g(v \wedge l)) - A(k,0)) \cdot \nabla \xi\}$$
$$\geqslant \lim_{\delta \to 0} \int_\Omega (A(v, \nabla g(v \wedge l)) - A(v \wedge l, 0)) \cdot \nabla g(v \wedge l) H'_\delta(g(v \wedge l) - g(k))\xi$$
$$+ \int_\Omega \xi \, d\mu_l(v) \tag{3.3}$$

for any $(k,\xi) \in \mathbb{R} \times \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$ such that $g(k) \notin E$ and $(g(a \wedge l) - g(k))^+ \xi = 0$ a.e. on Γ.

Moreover,

$$\begin{aligned}\int_\Omega \chi_{\{v\vee l<k\}}\{b(v\vee l)\xi - f\xi + (A(v\vee l,\nabla g(v\vee l)) - A(k,0))\cdot\nabla\xi\}\\ \geqslant \lim_{\delta\to 0}\int_\Omega (A(v\vee l,\nabla g(v\vee l)) - A(v\vee l,0))\cdot\nabla g(v\vee l)H'_\delta(g(v\vee l)-g(k))\xi\\ +\int_\Omega \xi\, d\nu_l(v) \qquad (3.4)\end{aligned}$$

for any $(k,\xi)\in\mathbb{R}\times\mathscr{D}(\mathbb{R}^N)$, $\xi\geqslant 0$ *such that* $g(k)\notin E$ *and* $(g(k)-g(a\vee l))^+\xi = 0$ *a.e. on* Γ.

Proof. We choose $H_\delta(g(v\wedge l)-g(k))^+\xi$ as test function in (2.8). By the divergence theorem, proceeding as in the proof of Lemma 1 in [13], letting $\delta\to 0$, we get inequality (3.3). Inequality (3.4) can be proved in a similar way.

PROOF OF THEOREM 2.4. The main idea of our proof is to compare locally v_1 and v_2 on a sufficiently small ball $\mathscr{B}(x,r)$ such that $\mathscr{B}(x,r)\cap\Gamma\neq\emptyset$ and $\max_{\Gamma\cap\mathscr{B}(x,r)} a - \min_{\Gamma\cap\mathscr{B}(x,r)} a\leqslant\varepsilon$. As usual we use Kruzhkov's technique of doubling variables (cf. [18], [19]): We choose two variables x and y and consider v_1 as a function of $y\in\Omega$ and v_2 as a function of $x\in\Omega$. For arbitrary $\alpha>0$, let $(B_i^\alpha)_{i=0\dots m_\alpha}$ be a covering of $\overline{\Omega}$ satisfying $B_0^\alpha\cap\partial\Omega=\emptyset$, and such that, for each $i\geqslant 1$, B_i^α is a ball of diameter $\leqslant\alpha$, contained in some larger ball $\tilde{B}_i^\alpha$ with $\tilde{B}_i^\alpha\cap\partial\Omega$ is part of the graph of a Lipschitz function. Let $(\phi_i^\alpha)_{i=0\dots m_\alpha}$ denote a partition of unity subordinate to the covering $(B_i^\alpha)_i$. Now, let $i\in\{1,\dots,m_\alpha\}$ be fixed in the following. For simplicity, we omit the dependence on α and i and simply set $\phi=\phi_i^\alpha$, $B=B_i^\alpha$. As in [13], we choose a sequence of mollifiers $(\rho_n)_n$ in $\mathbb{R}^N$ such that $x\mapsto\rho_n(x-y)\in\mathscr{D}^+(\Omega)$, for all $y\in B$, $\sigma_n(x)=\int_\Omega\rho_n(x-y)\,dy$ is an increasing sequence for all $x\in B$, and $\sigma_n(x)=1$ for all $x\in B$ with $d(x,\mathbb{R}^N\backslash\Omega)>c/n$ for some $c=c(i,\alpha)$ depending on $B=B_i^\alpha$. Define the test function

$$\zeta_n(x,y)=\xi(x)\phi(x)\rho_n(x-y)$$

with $\xi\in\mathscr{D}^+(\mathbb{R}^N)$ with $\mathrm{supp}(\xi)\subset\{|v_2|<\infty\}$. Note that, for n sufficiently large,

$$\begin{aligned} y\mapsto\zeta_n(x,y)\in\mathscr{D}(\mathbb{R}^N)\ \text{ for any } x\in\Omega,\\ x\mapsto\zeta_n(x,y)\in\mathscr{D}(\Omega)\ \text{ for any } y\in\Omega.\end{aligned}$$

Moreover, the function

$$\begin{aligned}\hat{\zeta}_n(x)=\int_\Omega\zeta_n(x,y)\,dy = \xi(x)\phi(x)\int_\Omega\rho_n(x-y)\,dy\\ =\xi(x)\phi(x)\sigma_n(x) \qquad (3.5)\end{aligned}$$

satisfies $\hat{\zeta}_n\in\mathscr{D}(\Omega)$, $0\leqslant\hat{\zeta}_n\leqslant\xi$, $\forall n$.

Let $\Omega_1:=\{y\in\Omega/v_1(y)\in E\}$ and $\Omega_2:=\{x\in\Omega/v_2(x)\in E\}$. Then, $\nabla_y g(T_l v_1)=0$ a. e. in Ω_1 and $\nabla_x g(T_l v_2)=0$ a. e. in Ω_2 for all $l>0$. Moreover, $H_0(T_l v_1 - T_l v_2) = H_0(g(T_l v_1)-g(T_l v_2))$ a.e in $(\Omega\setminus\Omega_1)\times\Omega\cup\Omega\times(\Omega\setminus\Omega_2)$.

First inequality. From now on, we denote by $\tilde{A}:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ the vector field defined by:

$$\tilde{A}(r,\xi)=A(r,\xi)-A(r,0). \tag{3.6}$$

Let $k_i^\alpha := \max\limits_{\{\mathscr{B}(x,r)\cap\Gamma\}} a_1$ and $l>0$ such that $k_i^\alpha\in(-l,l)$. We first prove the following inequality:

$$\begin{aligned}\int_\Omega \xi\phi d(\mu_l+\nu_{-l}) &\leqslant -\int_\Omega (b(T_lv_1\vee k_i^\alpha)-b(T_lv_2\vee k_i^\alpha))^+\xi\phi\\ &\quad -\int_\Omega \chi_{\{T_lv_1\vee k_i^\alpha>T_lv_2\vee k_i^\alpha\}}\Big(A\big(T_lv_1\vee k_i^\alpha,\nabla g(T_lv_1\vee k_i^\alpha)\big)\\ &\qquad\qquad -A\big(T_lv_2\vee k_i^\alpha,\nabla g(T_lv_2\vee k_i^\alpha)\big)\Big)\cdot\nabla_x(\xi\phi)\\ &\quad +\int_\Omega \kappa_1\chi_{\{T_lv_1>k_i^\alpha\}}(f_1-\chi_{\{T_lv_2\geqslant k_i^\alpha\}}f_2)\xi\phi+\lim_{n\to\infty}<\mathscr{M}_{k_i^\alpha,-l}(v_2),\xi\phi\sigma_n>,\end{aligned} \tag{3.7}$$

where $\kappa_1\in L^\infty(\Omega)$, $\kappa_1\in\ \mathrm{sign}^+(T_lv_1-T_lv_2\vee k_i^\alpha)$ and $\mathscr{M}_{k_i^\alpha}$ is a distribution which will be defined later (see (3.18)).

As v_1 is a renormalized entropy solution of $P_{b,g}(a_1,f_1)$, choosing $k=v_2\wedge l\vee k_i^\alpha$ and $\xi(y)=\zeta_n(x,y)$ in (2.8) and (3.3), integrating (2.8) in x over Ω_2 and (3.3) over $\Omega\setminus\Omega_2$, combining the two inequalities, we find

$$\begin{aligned}&\lim_{\delta\to0}\int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}}\tilde{A}(v_1\wedge l,\nabla_yg(v_1\wedge l))\cdot\nabla_yg(v_1\wedge l)H'_\delta(g(v_1\wedge l)-g(v_2\wedge l\vee k_i^\alpha))\zeta_n\\ &\qquad+\int_{\Omega\times\Omega}\zeta_n\,d\mu_l(v_1)\\ &=\lim_{\delta\to0}\int_{\Omega\times\{\Omega\setminus\Omega_2\}}\tilde{A}(v_1\wedge l,\nabla_yg(v_1\wedge l))\cdot\nabla_yg(v_1\wedge l)H'_\delta(g(v_1\wedge l)-g(v_2\wedge l\vee k_i^\alpha))\zeta_n\\ &\qquad+\int_{\Omega\times\Omega}\zeta_n\,d\mu_l(v_1)\\ &\leqslant-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l>v_2\vee k_i^\alpha\}}b(v_1\wedge l)\zeta_n+\int_\Omega\chi_{\{v_1\wedge l>v_2\wedge l\vee k_i^\alpha\}}f_1\zeta_n\\ &\qquad-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l>v_2\vee k_i^\alpha\}}(A(v_1\wedge l\vee k_i^\alpha,0)-A(v_2\wedge l\vee k_i^\alpha,0))\cdot\nabla_y\zeta_n\\ &\qquad-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\wedge l\vee k_i^\alpha\}}\tilde{A}(v_1\wedge l\vee k_i^\alpha,\nabla_yg(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_y\zeta_n.\end{aligned} \tag{3.8}$$

Now, as $:x\mapsto\zeta_n(x,y)H_\delta(g(v_1\wedge l\vee k_i^\alpha)-g(v_2\wedge l\vee k_i^\alpha))\in W_0^{1,p}(\Omega)$ for a.e. $y\in\Omega$, we have

$$\int_{\Omega\times\Omega}\tilde{A}(v_1\wedge l,\nabla_yg(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_x(H_\delta(g(v_1\wedge l)-g(v_2\vee k_i^\alpha))\zeta_n)=0. \tag{3.9}$$

Therefore, going to the limit on δ, we get

$$\lim_{\delta\to0}\int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}}\tilde{A}(v_1\wedge l\vee k_i^\alpha,\nabla_yg(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_xg(v_2\wedge l\vee k_i^\alpha)$$

$$\begin{aligned}
&H'_\delta(g(v_1\wedge l\vee k_i^\alpha)-g(v_2\wedge l\vee k_i^\alpha))\zeta_n\\
&=\int_{\Omega\times\Omega}\chi_{\{g(v_1\wedge l\vee k_i^\alpha)>g(v_2\wedge l\vee k_i^\alpha)\}}\tilde A(v_1\wedge l\vee k_i^\alpha,\nabla_y g(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_x\zeta_n\\
&=\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\wedge l\vee k_i^\alpha\}}\tilde A(v_1\wedge l,\nabla_y g(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_x\zeta_n. \qquad (3.10)
\end{aligned}$$

This allows to write inequality (3.8) as follows:

$$\begin{aligned}
&\int_{\Omega\times\Omega}-\tilde A(v_1\wedge l\vee k_i^\alpha,\nabla_y g(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_{x+y}\zeta_n\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\wedge l\vee k_i^\alpha\}}\\
&-\int_{\Omega\times\Omega}\{(A(v_1\wedge l\vee k_i^\alpha),0)-A(v_2\wedge l\vee k_i^\alpha),0))\cdot\nabla_y\zeta_n\\
&\qquad -b(v_1\wedge l\vee k_i^\alpha)\zeta_n+f_1\zeta_n\}\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\wedge l\vee k_i^\alpha\}}\\
&\geqslant\lim_{\delta\to 0}\int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}}\{\tilde A(v_1\wedge l\vee k_i^\alpha,\nabla_y g(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_y g(v_1\wedge l\vee k_i^\alpha)\\
&\qquad -\tilde A(v_1\wedge l\vee k_i^\alpha,\nabla_y g(v_1\wedge l\vee k_i^\alpha))\cdot\nabla_x g(v_2\wedge l\vee k_i^\alpha)\\
&\qquad H'_\delta(g(v_1\wedge l\vee k_i^\alpha)-g(v_2\wedge l\vee k_i^\alpha))\zeta_n\}+\int_{\Omega\times\Omega}\zeta_n\,d\mu_l(v_1)
\end{aligned}$$

with $\nabla_{x+y}(\cdot):=\nabla_x(\cdot)+\nabla_y(\cdot)$.

Now, as v_2 is an entropy solution of $P_{b,g}(a_2,f_2)$, choosing $-l$ instead of l, $k=(v_1(y)\wedge l)\vee k_i^\alpha$ and $\xi(x)=\zeta_n(x,y)$ in (2.9) and (3.4), integrating (2.8) in y over Ω and (3.4) in y over $\Omega\setminus\Omega_1$, summing up, we find

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{(\Omega\setminus\Omega_1)\times\Omega}\tilde A(v_2\vee(-l),\nabla_x g(v_2\vee(-l)))\cdot\nabla_x g(v_2\vee(-l))H'_\delta\big(g(v_1\wedge l\vee k_i^\alpha)\\
&\qquad -g(v_2\vee(-l))\big)\zeta_n+\int_{\Omega\times\Omega}\zeta_n\,d\nu_{-l}(v_2)\\
&=\lim_{\delta\to 0}\int_{(\Omega\setminus\Omega_1)\times(\Omega\setminus\Omega_2)}\tilde A(v_2\vee(-l),\nabla_x g(v_2\vee(-l)))\cdot\nabla_x g(v_2\vee(-l))H'_\delta\big(g(v_1\wedge l\vee k_i^\alpha)\\
&\qquad -g(v_2\vee(-l))\big)\zeta_n+\int_{\Omega\times\Omega}\zeta_n\,d\nu_{-l}(v_2)\,dy\\
&\leqslant\int_{\Omega\times\Omega}b(v_2\vee(-l))\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\vee(-l)\}}\zeta_n-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\vee(-l)\}}f_2\zeta_n\\
&\quad-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\vee(-l)\}}\big(A(v_1\wedge l\vee k_i^\alpha,0)\\
&\qquad -A(v_2\vee(-l),\nabla_x g(v_2\vee(-l)))\big)\cdot\nabla_x\zeta_n. \qquad (3.11)
\end{aligned}$$

Arguing as in (3), we get

$$\begin{aligned}
&\int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}}\tilde A(v_2\vee(-l),\nabla_x g(v_2\vee(-l)))\cdot\nabla_x g(v_2\vee(-l))H'_\delta\big(g(v_1\wedge l\vee k_i^\alpha)\\
&\qquad -g(v_2\vee(-l))\big)\zeta_n\\
&=\int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}}\tilde A(T_l v_2\vee k_i^\alpha,\nabla_x g(T_l v_2\vee k_i^\alpha))\cdot\nabla_x g(T_l v_2\vee k_i^\alpha)H'_\delta\big(g(v_1\wedge l\vee k_i^\alpha)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - g(T_l v_2 \vee k_i^\alpha))\zeta_n \\
&+ \int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}} \tilde{A}(T_l v_2 \wedge k_i^\alpha, \nabla_x g(T_l v_2 \wedge k_i^\alpha)) \cdot \nabla_x g(T_l v_2 \wedge k_i^\alpha) H'_\delta\big(g(v_1 \wedge l \vee k_i^\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - g(T_l v_2 \wedge k_i^\alpha))\zeta_n
\end{aligned} \tag{3.12}
$$

and that the second term in the right hand side of (3.12) converges to 0 with $\delta \to 0$. Moreover, the right hand side of (3.11) is equal to

$$
\begin{aligned}
&\int_{\Omega\times\Omega} b(T_l v_2 \vee k_i^\alpha)\chi_{\{v_1\wedge l \vee k_i^\alpha > T_l v_2 \vee k_i^\alpha\}}\zeta_n \\
&\quad - \int_{\Omega\times\Omega} \chi_{\{v_1\wedge l\vee k_i^\alpha > v_2\vee(-l)\vee k_i^\alpha\}}\chi_{\{T_l v_2 \geqslant k_i^\alpha\}} f_2 \zeta_n \\
&\quad + \int_{\Omega\times\Omega} \chi_{\{v_1\wedge l \vee k_i^\alpha > T_l v_2 \vee k_i^\alpha\}} \tilde{A}(T_l v_2 \vee k_i^\alpha, \nabla_x g(T_l v_2 \vee k_i^\alpha)) \cdot \nabla_x \zeta_n \\
&\quad - \int_{\Omega\times\Omega} \chi_{\{v_1\wedge l \vee k_i^\alpha > T_l v_2 \vee k_i^\alpha\}} (A(v_1 \wedge l \vee k_i^\alpha), 0) - A(T_l v_2 \vee k_i^\alpha), 0)) \cdot \nabla_x \zeta_n \\
&\quad - \int_{\Omega\times\Omega} (b(k_i^\alpha) - b(v_2 \vee (-l)))^+ \zeta_n - \int_{\Omega\times\Omega} \chi_{\{k_i^\alpha > v_2 \vee(-l)\}} f_2 \zeta_n \\
&\quad - \int_{\Omega\times\Omega} \chi_{\{k_i^\alpha > v_2\vee(-l)\}} (A(k_i^\alpha, 0) - A(v_2 \vee (-l), \nabla_x g(v_2 \vee (-l))))) \cdot \nabla_x \zeta_n.
\end{aligned} \tag{3.13}
$$

Since $y \mapsto \zeta_n(x,y) H_\delta(g(v_1 \wedge l \vee k_i^\alpha) - g(v_2 \wedge l \vee k_i^\alpha))) \in W_0^{1,p}(\Omega)$ for a.e. $x \in \Omega$, we have

$$
\int_{\Omega\times\Omega} \tilde{A}(T_l v_2 \vee k_i^\alpha, \nabla_x g(T_l v_2 \vee k_i^\alpha)) \cdot \nabla_y (H_\delta(g(v_1 \wedge l \vee k_i^\alpha) - g(T_l v_2 \vee k_i^\alpha))\zeta_n) = 0. \tag{3.14}
$$

Therefore,

$$
\begin{aligned}
&- \lim_{\delta\to 0} \int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}} \tilde{A}(T_l v_2 \vee k_i^\alpha, \nabla_x g(T_l v_2 \vee k_i^\alpha))) \cdot \nabla_y g(v_1 \wedge l \vee k_i^\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad H'_\delta(g(v_1 \wedge l \vee k_i^\alpha) - g(T_l v_2 \vee k_i^\alpha))\zeta_n \\
&\quad = \int_{\Omega\times\Omega} \chi_{\{(g(v_1\wedge l\vee k_i^\alpha) > g(T_l v_2 \vee k_i^\alpha)\}} \tilde{A}(T_l v_2 \vee k_i^\alpha, \nabla_x g(T_l v_2 \vee k_i^\alpha)) \nabla_y \zeta_n.
\end{aligned} \tag{3.15}
$$

Consequently, inequality (3.11) can be equivalently written as follows:

$$
\begin{aligned}
&\int_{\Omega\times\Omega} (b(v_2 \wedge l \vee k_i^\alpha) - f_2)\chi_{\{v_1\wedge l\vee k_i^\alpha > T_l v_2 \vee k_i^\alpha\}}\zeta_n \\
&\quad - \int_{\Omega\times\Omega} \chi_{\{v_1\wedge l \vee k_i^\alpha - T_l v_2 \vee k_i^\alpha\}} \tilde{A}(T_l v_2 \vee k_i^\alpha, \nabla_x g(T_l v_2 \vee k_i^\alpha)) \cdot \nabla_{x+y} \zeta_n \\
&\quad + \int_{\Omega\times\Omega} (A(v_1 \wedge l \vee k_i^\alpha, 0) - A(T_l v_2 \vee k_i^\alpha, 0)) \cdot \nabla_x \zeta_n \chi_{\{(v_1\wedge l\vee k_i^\alpha - T_l v_2 \vee k_i^\alpha\}} \\
&\quad - \int_{\Omega\times\Omega} (b(k_i^\alpha) - b(v_2 \vee (-l)))^+ \zeta_n - \chi_{\{k_i^\alpha > v_2 \vee (-l)\}} f_2 \zeta_n \\
&\quad - \int_{\Omega\times\Omega} \chi_{\{k_i^\alpha > v_2\vee(-l)\}} (A(k_i^\alpha, 0) - A(v_2 \vee (-l), \nabla_x g(v_2 \vee (-l))))) \cdot \nabla_x \zeta_n
\end{aligned}
$$

$$\geqslant \lim_{\delta\to 0}\int_{\{\Omega\setminus\Omega_1\}\times\{\Omega\setminus\Omega_2\}} \tilde{A}(T_l v_2\vee k_i^\alpha,\nabla_x g(T_l v_2\vee k_i^\alpha))\cdot(\nabla_x g(T_l v_2\vee k_i^\alpha)-\nabla_y g(v_1\wedge l\vee k_i^\alpha))$$
$$H'_\delta(g(v_1\wedge l\vee k_i^\alpha)-g(T_l v_2\vee k_i^\alpha))\zeta_n+\int_{\Omega\times\Omega}\zeta_n\, d(\nu_{-l}(v_2)). \tag{3.16}$$

Summing up inequalities (3.11) and (3.16), we get

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{(\Omega\setminus\Omega_1)\times(\Omega\setminus\Omega_2)} (\tilde{A}(T_l v_1\vee k_i^\alpha,\nabla_y g(T_l v_1\vee k_i^\alpha)-\tilde{A}(T_l v_2\vee k_i^\alpha,\nabla_x g(T_l v_2\vee k_i^\alpha)))\\
&\qquad\cdot(\nabla_y g(T_l v_1\vee k_i^\alpha)-\nabla_x g(T_l v_2\vee k_i^\alpha))H'_\delta(g(T_l v_1\vee k_i^\alpha)-g(T_l v_2\vee k_i^\alpha))\zeta_n\\
&\quad+\int_{\Omega\times\Omega}\zeta_n\, d(\mu_l(v_1)+\nu_{-l}(v_2))\\
&\leqslant\int_{\Omega\times\Omega}-(b(T_l v_1\vee k_i^\alpha)-b(T_l v_2\vee k_i^\alpha))^+\zeta_n\\
&\quad+\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l\vee k_i^\alpha>v_2\vee k_i^\alpha\}}\chi_{\{v_1\wedge l>k_i^\alpha\}}(f_1-\chi_{\{v_2\geqslant k_i^\alpha\}}f_2)\zeta_n\\
&\quad-\int_{\Omega\times\Omega}(A(v_1\wedge l\vee k_i^\alpha,\nabla_y g(v_1\wedge l\vee k_i^\alpha))-A(v_2\wedge l\vee k_i^\alpha,\nabla_x g(v_2\wedge l\vee k_i^\alpha)))\cdot\\
&\qquad\qquad(\nabla_{x+y}\zeta_n)\chi_{\{v_1\wedge l\vee k_i^\alpha-v_2\wedge l\vee k_i^\alpha\}}+<\mathscr{M}_{k_i^\alpha,-l}(v_2),\zeta_n>,
\end{aligned}\tag{3.17}$$

where, for $k\in\mathbb{R},\ \xi\in\mathscr{D}(\Omega)$,

$$\begin{aligned}
<\mathscr{M}_{k,-l}(v_2),\xi>=&-\int_\Omega(b(k)-b(v_2\vee(-l)))^+\xi-\int_\Omega\chi_{\{k>v_2\vee(-l)\}}f_2\xi\\
&-\int_\Omega\chi_{\{k>v_2\vee(-l)\}}(A(k,0)-A(v_2\vee(-l),\nabla_x g(v_2\vee(-l))))\cdot\nabla_x\xi.
\end{aligned}\tag{3.18}$$

Denote the integrals on the right hand side of (3.17) by $I_1,\dots,I_4$ successively. Using similar estimations as in [4] and [5], going to the limit with $n\to\infty$, one get

$$\lim_{n\to\infty}I_1=-\int_\Omega(b(T_l v_1\vee k_i^\alpha)-b(T_l v_2\vee k_i^\alpha))^+\xi(x)\phi(x),$$

$$\limsup_{n\to\infty}I_2\leqslant\int_\Omega\kappa_1\chi_{\{v_1\wedge l>k_i^\alpha\}}(f_1-\chi_{\{v_2\geqslant k_i^\alpha\}}f_2)\xi(x)\phi(x)$$

for some

$$\kappa_1\in L^\infty(\Omega)\text{ with }\kappa_1\in\operatorname{sign}^+(v_1\wedge l-v_2\vee k_i^\alpha)\text{ a.e. in }\Omega, \tag{3.19}$$

$$\begin{aligned}
\limsup_{n\to+\infty}I_3=-\int_\Omega\chi_{\{v_1\wedge l\vee k_i^\alpha-v_2\wedge l\vee k_i^\alpha\}}\big(&A(v_1\wedge l,\nabla g(v_1\wedge l\vee k_i^\alpha))\\
&-A(v_2,\nabla g(v_2\wedge l\vee k_i^\alpha))\big)\cdot\nabla(\xi\phi).
\end{aligned}$$

By a simple computation, we prove that

$$<\mathscr{M}_{k_i^\alpha,-l}(v_2),\xi>=<\tilde{\mathscr{M}}_{k_i^\alpha}(v_2),\xi>-<\tilde{\mathscr{M}}_{-l}(v_2),\xi>$$

$$-\int_\Omega \chi_{\{v_2<-l\}}(f_2+b(-l))\xi - \int_\Omega b(k)\chi_{\{v_2\vee(-l)\}}\xi,$$

where, for $k\in\mathbb{R}$ and $\xi\in\mathscr{D}(\Omega)$,

$$\begin{aligned}< \tilde{\mathscr{M}}_k(v_2),\xi >= &\int_\Omega b(v_2)\xi\chi_{\{k>v_2\}} - \chi_{\{k>v_2\}}f_2\xi\\ &-\int_\Omega \chi_{\{k>v_2\}}(A(k,0)-A(v_2,\nabla_x g(v_2)))\cdot\nabla_x\xi. \end{aligned}\tag{3.20}$$

Taking into account (2.11), for every $k\in\mathbb{R}$ and for all $\xi\in\mathscr{D}(\Omega)$ with supp $(\xi)\subset\{|v|<\infty\}$, $<\tilde{\mathscr{M}}_k(v_2),\xi>\geqslant 0$ and by (3.20), it follows that

$$< \mathscr{M}_{k_i^\alpha,-l}(v_2),\xi > \leqslant < \tilde{\mathscr{M}}_{k_i^\alpha}(v_2),\xi > - \int_\Omega \chi_{\{v_2<-l\}}f_2\xi - \int_\Omega \chi_{\{v_2<-l\}}b(k)\xi,$$

for all $\xi\in\mathscr{D}^+(\Omega)$. Since $(\hat{\zeta})_n=(\xi\sigma_n\phi)_n\subset\mathscr{D}(\Omega)$ is an increasing sequence satisfying $0\leqslant\xi\sigma_n\phi\leqslant\xi\phi$, the sequence $(<\mathscr{M}_{k_i^\alpha}(v_2),\xi\sigma_n\phi>)_n$ is a bounded increasing sequence such that $\nabla(\xi\sigma_n\phi)=0$ on $\{|v_2|=+\infty\}$ and thus converges as $n\to\infty$. This in turn implies that $(<\mathscr{M}_{k_i^\alpha,-l}(v_2),\xi\sigma_n\phi>)_n$ converges with $n\to+\infty$.

In order to estimate the term in the left hand side of (3.17), we use our asumptions on the diffusion function g: As $\tilde{A}(r,0)=0$, for small $\delta>0$, we have:

$$\begin{aligned}
&\int_{(\Omega\setminus\Omega_1)\times(\Omega\setminus\Omega_2)}(\tilde{A}(T_lv_1\vee k_i^\alpha,\nabla_y g(T_lv_1\vee k_i^\alpha))-\tilde{A}(T_lv_2\vee k_i^\alpha,\nabla_x g(T_lv_2\vee k_i^\alpha)))\\
&\qquad\cdot\left[\nabla_y g(T_lv_1\vee k_i^\alpha)-\nabla_x g(T_lv_2\vee k_i^\alpha)\right]H'_\delta(g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha))\zeta_n\\
&=\frac{1}{\delta}\int_{\Omega\times\Omega}(\tilde{A}(T_lv_1\vee k_i^\alpha,\nabla_y g(T^2(T_lv_1\vee k_i^\alpha)))-\tilde{A}(T_lv_2\vee k_i^\alpha,\nabla_x g(T^2(T_lv_2\vee k_i^\alpha))))\\
&\qquad\cdot\left[\nabla_y g(T^2(T_lv_1\vee k_i^\alpha))\right.\\
&\qquad\quad\left.-\nabla_x g(T^2(T_lv_2\vee k_i^\alpha))\right]\zeta_n\chi_{\{T_lv_1\vee k_i^\alpha,T_lv_2\vee k_i^\alpha\in(A_2,+\infty),0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\\
&\quad+\frac{1}{\delta}\int_{\Omega\times\Omega}(\tilde{A}(T_lv_1\vee k_i^\alpha,\nabla_y g(T^2(T_lv_1\vee k_i^\alpha))))-\tilde{A}(T_lv_2\vee k_i^\alpha,\nabla_x g(T^1(T_lv_2\vee k_i^\alpha))))\\
&\qquad\cdot\left[\nabla_y g(T^2(T_lv_1\vee k_i^\alpha))\right.\\
&\qquad\quad\left.-\nabla_x g(T^1(T_lv_2\vee k_i^\alpha))\right]\zeta_n\chi_{\{T_lv_1\vee k_i^\alpha\in(A_2,+\infty),T_lv_2\vee k_i^\alpha\in(-\infty,A_1),0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\\
&:=\mathscr{S}_1+\mathscr{S}_2.
\end{aligned}$$

We first estimate $\mathscr{S}_2$ and split this term into

$$\begin{aligned}
\mathscr{S}_2=&\frac{1}{\delta}\int_{\Omega\times\Omega}\tilde{A}(T^2(T_lv_1\vee k_i^\alpha),\nabla_y g(T^2(T_lv_1\vee k_i^\alpha)))\\
&\quad\cdot\nabla_y g(T^2(T_lv_1\vee k_i^\alpha))\chi_{\{0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha))\leqslant\delta\}}\chi_{\{T_lv_2\vee k_i^\alpha\in(-\infty,A_1)\}}\zeta_n\\
&-\frac{1}{\delta}\int_{\Omega\times\Omega}\tilde{A}(T^2(T_lv_1\vee k_i^\alpha),\nabla_y g(T^2(T_lv_1\vee k_i^\alpha)))\\
&\quad\cdot\nabla_x g(T^1(T_lv_2\vee k_i^\alpha))\chi_{\{0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\zeta_n
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\delta}\int_{\Omega\times\Omega}(\tilde{A}(T^1(T_lv_2\vee k_i^\alpha)),\nabla_x g(T^1(T_lv_2\vee k_i^\alpha))\\
&\quad\cdot\nabla_x g(T^1(T_lv_2\vee k_i^\alpha))\chi_{\{0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\chi_{\{T_lv_1\vee k_i^\alpha\in(A_2,+\infty)\}}\zeta_n\\
&-\frac{1}{\delta}\int_{\Omega\times\Omega}(\tilde{A}(T^1(T_lv_2\vee k_i^\alpha),\nabla_x g(T^1(T_lv_2\vee k_i^\alpha)))\\
&\quad\cdot\nabla_y g(T^2(T_lv_1\vee k_i^\alpha))\chi_{\{0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\zeta_n\\
&=\mathscr{S}_2^1+\mathscr{S}_2^2+\mathscr{S}_2^3+\mathscr{S}_2^4.
\end{aligned}
$$

By the weak coerciveness condition (1.5),

$$
\begin{aligned}
\mathscr{S}_2^1&\leqslant\frac{1}{\delta}\int_{\Omega\times\Omega}\zeta_n\tilde{A}(T^2(T_lv_1\vee k_i^\alpha),\nabla_y g(T^2(T_lv_1\vee k_i^\alpha)))\\
&\qquad\qquad\cdot\nabla_y T_\delta(g(T^2(T_lv_1\vee k_i^\alpha))^+\zeta_n\chi_{\{0<(-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\\
&\leqslant\frac{1}{\delta}\int_{\Omega\times\Omega}\zeta_n\tilde{A}(T^2v_1,\nabla_y g(T^2v_1))\\
&\qquad\qquad\cdot\nabla_y T_\delta(g(T^2v_1))^+\zeta_n\chi_{\{0<(-g(v_2)\leqslant\delta\}}\\
&\leqslant\frac{1}{\delta}\delta C(\| f_1\|_{L^1(\Omega)},d)\int_\Omega\chi_{\{0<(-g(v_2)\leqslant\delta\}},
\end{aligned}
$$

where in the last inequality we use the estimations (3.1) and (3.2). Hence $\lim_{\delta\to 0}\mathscr{S}_2^1=0$ and $\mathscr{S}_2^i$, $i=2,3,4$ can be estimated in the same way. Dealing with $\mathscr{S}_1$, we use the additional hypothesis (1.7) on the vector field A, to get

$$
\begin{aligned}
\mathscr{S}_1\geqslant&-\frac{1}{\delta}\int_{\Omega\times\Omega}\chi_{\{T_lv_1\vee k_i^\alpha,T_lv_2\vee k_i^\alpha\in(A_2,+\infty),0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee\eta)\leqslant\delta\}}\zeta_n\\
&\times-(B(g(T^2(T_lv_1\vee k_i^\alpha))-B(g(T^2(T_lv_2\vee k_i^\alpha))))\\
&\times(1+|\nabla_y g(T^2(T_lv_1\vee k_i^\alpha)))|^p+|\nabla_x g(T^2(T_lv_2\vee k_i^\alpha))|^p)\\
&-\frac{1}{\delta}\int_{\Omega\times\Omega}(A(T^2(T_lv_1\vee k_i^\alpha)),0)-A(T^2(T_lv_2\vee k_i^\alpha)),0))\\
&\cdot\nabla_y g(T^2(T_lv_1\vee k_i^\alpha))\chi_{\{T_lv_1\vee k_i^\alpha,T_lv_2\vee k_i^\alpha\in(A_2,+\infty),0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\zeta_n\\
&-\frac{1}{\delta}\int_{\Omega\times\Omega}(A(T^2(T_lv_1\vee k_i^\alpha),0))-A(T^2(T_lv_2\vee k_i^\alpha)),0)))\\
&\cdot\nabla_x g(T^2(T_lv_2\vee k_i^\alpha))\chi_{\{T_lv_1\vee k_i^\alpha,T_lv_2\vee k_i^\alpha\in(A_2,+\infty),0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee k_i^\alpha)\leqslant\delta\}}\zeta_n\\
=&\,\mathscr{S}_1^1+\mathscr{S}_1^2+\mathscr{S}_1^3.
\end{aligned}
$$

Applying the divergence theorem, we get

$$
\begin{aligned}
\mathscr{S}_1^2=-\frac{1}{\delta}\int_{\Omega\times\Omega}\Big(\int_0^{\gamma(T_lv_1,T_lv_2)}&(A_g(g(T^2(T_lv_2\vee k_i^\alpha))+\delta r,0)\\
&-A_g(g(T^2(T_lv_2\vee k_i^\alpha)),0))\,dr\Big)\nabla_y\zeta_n \qquad (3.21)
\end{aligned}
$$

and

$$\mathscr{S}_1^3 = -\frac{1}{\delta}\int_{\Omega\times\Omega}\Big(\int_0^{\gamma(T_lv_1,T_lv_2)}(A_g(g(T^2(v_1\vee k_i^\alpha)),0) - A_g(g(T^2(v_1\vee k_i^\alpha))-\delta r,0))\,dr\Big)\nabla_x\zeta_n, \quad (3.22)$$

where

$$A_g(r,\xi) = A(g^{-1}(r),\xi),\ \ r\in(A_2,+\infty),$$
$$\gamma(T_lv_1,T_lv_2) := \inf\,(g(T^2(T_lv_1\vee k_i^\alpha)) - g(T^2(T_lv_2\vee k_i^\alpha)))^+/\delta,1).$$

Due to the continuity of $A_g(r,\xi)$ in $r\in(A_2,+\infty)$, it follows that

$$\lim_{\delta\to 0}\mathscr{S}_1^2 = \lim_{\delta\to 0}\mathscr{S}_1^3 = 0. \quad (3.23)$$

Finally, as B is locally Lipschitz,

$$\lim_{\delta\to 0}\mathscr{S}_1^1 \geqslant c\lim_{\delta\to 0}\int_{\Omega\times\Omega}\chi_{\{T_lv_1\vee k_i^\alpha,T_lv_2\vee k_i^\alpha\in(A_2,+\infty),0<g(T_lv_1\vee k_i^\alpha)-g(T_lv_2\vee\eta)\leqslant\delta\}}\zeta_n \times(1+|\nabla_y g(T^2(T_lv_1\vee k_i^\alpha)))|^p + |\nabla_x g(T^2(T_lv_2\vee k_i^\alpha))|^p) = 0$$

for some constant c depending independing on δ. Combining all estimates, we get

$$\begin{aligned}
&+\int_\Omega \xi\phi\,d(\mu_l(v_1)+\nu_{-l}(v_2))\\
&\leqslant -\int_\Omega (b(T_lv_1\vee k_i^\alpha) - b(T_lv_2\vee k_i^\alpha))^+\xi\phi\\
&\quad -\int_\Omega \big[A(T_lv_1\vee k_i^\alpha,\nabla_x g(T_lv_1\vee k_i^\alpha))\\
&\qquad\qquad - A(T_lv_2\vee k_i^\alpha,\nabla g(T_lv_2\vee k_i^\alpha))\big]\cdot\nabla_x(\xi\phi)\chi_{\{T_lv_1\vee k_i^\alpha>T_lv_2\vee k_i^\alpha\}}\\
&\quad +\int_\Omega \kappa_1\chi_{\{T_lv_1>k_i^\alpha\}}(f_1-\chi_{\{T_lv_2\geqslant k_i^\alpha\}}f_2)\xi\phi + \lim_{n\to\infty} < \mathscr{M}_{k_i^\alpha,-l}(v_2),\xi\phi\sigma_n > . \quad (3.24)
\end{aligned}$$

Second inequality. We are going to prove the following:

$$\begin{aligned}
&\int_\Omega \xi\phi\,d(\mu_l(v_1)+\nu_{-l}(v_2)) - \int_{\partial\Omega\cap B}\omega^-(x,k_i^\alpha,a_2)\xi\\
&\leqslant \int_\Omega -(b(T_lv_1\wedge k_i^\alpha) - b(T_lv_2\wedge k_i^\alpha))^+\xi\phi\\
&\quad -\int_\Omega \chi_{\{T_lv_1\wedge k_i^\alpha\geqslant T_lv_2\}}\big[A(T_lv_1\wedge k_i^\alpha,\nabla_x g(T_lv_1\wedge k_i^\alpha))\\
&\qquad\qquad - A(T_lv_2\wedge k_i^\alpha,\nabla_x g(T_lv_2\wedge k_i^\alpha))\big]\cdot\nabla_x(\xi\phi)\\
&\quad +\int_\Omega \kappa_2\chi_{\{v_2\vee(-l)<k_i^\alpha\}}(\chi_{\{v_1\wedge l\leqslant k_i^\alpha\}}f_1-f_2)\xi\phi + \lim_{n\to\infty} < \mathscr{L}_{k_i^\alpha,l}(v_1),\xi\phi\sigma_n > \quad (3.25)
\end{aligned}$$

for some $\kappa_2\in\operatorname{sign}^+(v_1\wedge l\wedge k_i^\alpha - v_2\vee(-l))$, where $\mathscr{L}_{k_i^\alpha,l}(v_1)$ is a linear functional which will be defined later (see (3.26)). Then, summing up (3.7) and (3.25), we find in the final step the desired comparison result. To this aim, we choose $\zeta_n := \zeta_n^i : (x,y) \mapsto \phi_i(y)\xi(y)\rho_n(y-x)$ as test function. Then, for n sufficiently large,

$$y \mapsto \zeta_n^i(x,y) \in \mathscr{D}(\Omega), \text{ for any } x \in \Omega,$$
$$x \mapsto \zeta_n^i(x,y) \in \mathscr{D}(\mathbb{R}^N), \text{ for any } y \in \Omega,$$
$$\text{and } \operatorname{supp}_x(\zeta_n^i(y,.)) \subset B_i, \text{ for any } y \in \operatorname{supp}(\phi_i).$$

As $v_1 = v_1(y)$ satisfies (2.8), choosing $k = v_2(x) \vee (-l) \wedge k_i^\alpha$ and $\xi = \zeta_n(x,\cdot)$ in (2.8) and (3.3), integration (2.8) in x over Ω and (3.3) over $\Omega \setminus \Omega_2$, (note that, with the new test function, this choice is admissible), for a.e. $x \in \Omega$, we get

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{(\Omega\setminus\Omega_1)\setminus(\Omega\setminus\Omega_2)} \tilde{A}(v_1\wedge l,\nabla_y g(v_1\wedge l))\cdot\nabla_y g(v_1\wedge l)H'_\delta(g(v_1\wedge l)-g(v_2\vee(-l)\wedge k_i^\alpha))\zeta_n\\
&\qquad+\int_{\Omega\times\Omega}(\zeta_n\,d\mu_l(v_1))\\
&\leqslant -\int_{\Omega\times\Omega} b(v_1\wedge l)\chi_{\{v_1\wedge l>v_2\vee(-l)\wedge k_i^\alpha\}}\zeta_n\\
&\qquad-\int_{\Omega\times\Omega}(A(v_1\wedge l,0)-A(v_2\wedge k_i^\alpha,0))\cdot\nabla_y\zeta_n\chi_{\{v_1\wedge l>v_2\wedge k_i^\alpha\}}\\
&\qquad-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l>v_2\wedge k_i^\alpha\}}\tilde{A}(v_1\wedge l,\nabla_y g(v_1\wedge l)\cdot\nabla_y\zeta_n\\
&= -\int_{\Omega\times\Omega} b(v_1\wedge l\wedge k_i^\alpha)\chi_{\{v_1\wedge l>v_2\wedge k_i^\alpha\}}\zeta_n\\
&\qquad-\int_{\Omega\times\Omega}(A(v_1\wedge l\wedge k_i^\alpha,0)-A(v_2\wedge k_i^\alpha,0))\cdot\nabla_y\zeta_n\chi_{\{v_1\wedge l\wedge k_i^\alpha>v_2\wedge k_i^\alpha\}}\\
&\qquad+\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l\wedge k_i^\alpha>v_2\wedge k_i^\alpha\}}\chi_{\{v_1\wedge l\leqslant k_i^\alpha\}}f_1\zeta_n\\
&\qquad+[-\int_{\Omega\times\Omega}(b(v_1\wedge l)-b(k_i^\alpha))^+\zeta_n+\chi_{\{v_1\wedge l>k_i^\alpha\}}f_1\zeta_n\\
&\qquad-\int_{\Omega\times\Omega}\chi_{\{v_1\wedge l>k_i^\alpha\}}\{(A(v_1\wedge l,\nabla_y g(v_1\wedge l))-A(k_i^\alpha,0))\cdot\nabla_y\zeta_n\}]\\
&:=<\mathscr{L}_{k_i^\alpha,l}(v_1),\zeta_n>,
\end{aligned}\tag{3.26}$$

where for the last equality we have used the fact that:

$$(r-s\wedge k)^+ = (r\wedge k-s\wedge k)^+ + (r-k)^+,$$
$$\chi_{\{r>s\wedge k\}} = \chi_{\{r\wedge k>s\wedge k\}}\chi_{\{r\leqslant k\}} + \chi_{\{r>k\}}, \text{ for all } r,s,k\in\mathbb{R}.$$

As $v_2 = v_2(x)$ is an entropy solution, choosing $k = v_1(y)\wedge k_i^\alpha$, $\xi = \zeta_n$ in (2.9) and (3.4) (this choice is admissible because $g(k_i^\alpha) = 0$), integrating (2.9) in y over Ω_1 and (3.4) over $\Omega\setminus\Omega_1$, we get

$$-\lim_{\delta\to 0}\int_{(\Omega\setminus\Omega_1)\setminus(\Omega\setminus\Omega_2)}\big[A(v_2\vee(-l),\nabla g(v_2\vee(-l)))$$

$$
\begin{aligned}
&\qquad -A(v_2\vee(-l),0))\cdot\nabla g(v_2)H'_\delta(g(v_1\wedge k_i^\alpha)-g(v_2\vee(-l))\big]\\
&\quad -\int_\Omega\int_\Gamma \omega^-(x,v_1(y)\wedge k_i^\alpha,a_2\vee(-l))\zeta_n+\int_{\Omega\times\Omega}\zeta_n\,d\nu_{-l}(v_2))\\
&\leqslant -\int_{\Omega\times\Omega}(b(v_1\wedge k_i^\alpha)-b(v_2\vee(-l)))^+\zeta_n\\
&\quad +\int_{\Omega\times\Omega}\chi_{\{v_1\wedge k_i^\alpha>v_2\}}\big\{(A(v_1\wedge k_i^\alpha,0)-A(v_2\vee(-l),\nabla_x g(v_2\vee(-l)))\cdot\nabla_x\zeta_n\\
&\qquad\qquad -\int_{\Omega\times\Omega}f_2\zeta_n\big\}.
\end{aligned}
$$

Remark that

$$
-\int_\Omega\int_\Gamma\omega^-(x,v_1\wedge k_i^\alpha,a_2\vee(-l))\zeta_n\geqslant-\int_\Omega\int_\Gamma\omega^-(x,k_i^\alpha,a_2\vee(-l))\zeta_n.
$$

Moreover, obviously, $(r\wedge k-s)^+=(r\wedge k-s\wedge k)^+$ for all $r,s,k\in\mathbb{R}$. Therefore, integrating the preceding inequalities in x resp. y over Ω, summing up, using the same arguments as above, passing to the limit with $n\to\infty$ successively, for some $\kappa_2\in L^\infty(\Omega)$ with $\kappa_2\in\operatorname{sign}^+(v_1\wedge k_i^\alpha-v_2\vee(-l))$, we obtain

$$
\begin{aligned}
&-\int_\Gamma\omega^-(x,k_i^\alpha,a_2\vee(-l))\xi\phi_i+\int_\Omega\xi\phi_i\,d(\mu_l(v_1)+\nu_{-l}(v_2))\\
&\quad\leqslant-\int_\Omega(b(T_lv_1\wedge k_i^\alpha)-b(T_lv_2\wedge k_i^\alpha))^+\xi\phi_i\\
&\quad -\int_\Omega\chi_{\{T_lv_1\wedge k_i^\alpha\geqslant T_lv_2\wedge k_i^\alpha\}}\big(A(T_lv_1\wedge k_i^\alpha,\nabla g(T_lv_1\wedge k_i^\alpha))\\
&\qquad\qquad -A(T_lv_2\wedge k_i^\alpha,\nabla g(T_lv_2\wedge k_i^\alpha))\big)\cdot\nabla_x(\xi\phi_i)\\
&\quad +\int_\Omega\kappa_2\chi_{\{T_lv_2<k_i^\alpha\}}(\chi_{\{T_lv_1\leqslant k_i^\alpha\}}f_1-f_2)\xi\phi_i+\lim_{n\to\infty}<\mathscr{L}_{k_i^\alpha,l}(v_1),\xi\sigma_n\phi_i>.
\end{aligned}
\tag{3.27}
$$

Using the same arguments as before, we can prove that $(<\mathscr{L}_{k_i^\alpha,l}(v_1),(\xi\sigma_n\phi_i)>)$ converges with $n\to\infty$. Note also that $(r\vee k-s\vee k)^++(r\wedge k-s\wedge k)^+=(r-s)^+$, for all $r,s,k\in\mathbb{R}$. Moreover, if we define

$$
\kappa:=\kappa_1\chi_{\{v_1>k_i^\alpha\}}+\kappa_2\chi_{\{v_2<k_i^\alpha\}}\chi_{\{v_1\leqslant k_i^\alpha\}},
$$

then

$$
\kappa=\kappa_1\chi_{\{v_2\geqslant k_i^\alpha\}}\chi_{\{v_1>k_i^\alpha\}}+\kappa_2\chi_{\{v_2<k_i^\alpha\}}\in\operatorname{sign}^+(v_1-v_2).
$$

Therefore, summation of (3.24) and (3.27) yields

$$
\begin{aligned}
&-\int_\Gamma\omega^-(x,k_i^\alpha,a_2\vee(-l))\xi\phi_i+2\int_\Omega\xi\phi_i\,d(\mu_l(v_1)+\nu_{-l}(v_2))\\
&\quad\leqslant-\int_\Omega(b(T_lv_1)-b(T_lv_2))^+\xi\phi_i+\int_\Omega\kappa(f_1-f_2)\xi\phi_i\\
&\qquad -\int_\Omega\chi_{\{T_lv_1\geqslant T_lv_2\}}(A(T_lv_1,\nabla g(T_lv_1))-A(T_lv_2,\nabla g(T_lv_2)))\cdot\nabla_x(\xi\phi_i)
\end{aligned}
$$

$$+\lim_{n\to\infty} < \mathscr{L}_{k_i^\alpha,l}(v_1), \xi\phi_i\sigma_n > + \lim_{n\to\infty} < \mathscr{M}_{k_i^\alpha,-l}(v_2), \xi\phi_i\sigma_n > \tag{3.28}$$

for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$, for all $i \in \{1,\dots,m_\alpha\}$.

REMARK 3.3. For $\xi \in \mathscr{D}([0,T)\times\Omega)$, the method of doubling variables allows to prove the following local comparison result:

there exists $\kappa \in L^\infty(\Omega)$ with $\kappa \in \mathrm{sign}^+(T_l v_1 - T_l v_2)$ a.e. in Ω such that, for any $\zeta \in \mathscr{D}(\Omega)$, $\zeta \geqslant 0$,

$$\begin{aligned}2\int_\Omega \xi\phi_i\, d(\mu_l(v_1)+\nu_{-l}(v_2)) &\\ \leqslant -\int_\Omega (b(T_l v_1)-b(T_l v_2))^+\zeta + \int_\Omega \kappa(f_1-f_2)\zeta &\\ -\int_\Omega \chi_{\{T_l v_1 > T_l v_2\}}(A(T_l v_1, \nabla g(T_l v_1)) - A(T_l v_2, \nabla g(T_l v_2))\cdot\nabla\zeta.&\end{aligned} \tag{3.29}$$

The proof in this case is easier than the global comparison result. Indeed, as $\xi = 0$ on Γ, we can choose $k = v_2(x)$ (resp $k = v_1(x)$) in (2.8) (resp in 2.9) and we have only to add the obtained inequalities, then to go to the limit on n in order to get (3.29).

As $\xi = \xi(1-\sigma_n)+\xi\sigma_n$ and $\xi\sigma_n \in \mathscr{D}(\Omega)$ for n sufficiently large, applying the local comparison principle (3.29) with $\zeta = \xi\sigma_n$, the global estimate (3.28) with $\xi(1-\sigma_n)$, we obtain

$$\begin{aligned}-\int_\Omega (b(T_l v_1)-b(T_l v_2))^+\xi\phi_i - \chi_{\{T_l v_1 \geqslant T_l v_2\}}\big(A(T_l v_1, \nabla g(T_l v_1))&\\ - A(T_l v_2, \nabla g(T_l v_2))\big)\cdot\nabla_x(\xi\phi_i)&\\ + \int_\Omega \kappa(f_1-f_2)\xi\phi_i - 2\int_\Omega \xi\phi_i\, d(\mu_l(v_1)+\nu_{-l}(v_2))&\\ \geqslant -\int_\Omega (b(T_l v_1)-b(T_l v_2))^+(\xi(1-\sigma))\phi_i&\\ -\int_\Omega \chi_{\{T_l v_1 \geqslant T_l v_2\}}(A(T_l v_1, \nabla g(T_l v_1)) - A(T_l v_2, \nabla g(T_l v_2)))\cdot\nabla_x(\xi(1-\sigma)\phi_i)&\\ +\int_\Omega \kappa(f_1-f_2)\xi(1-\sigma)\phi_i&\\ \geqslant -\int_\Gamma \omega^-(x,k_i^\alpha, a_2\vee(-l))\xi\phi_i(1-\sigma) + \lim_{n\to\infty} < \mathscr{L}_{k_i^\alpha,l}(v_1), (\xi\phi_i(1-\sigma)\sigma_n) >&\\ + \lim_{n\to\infty} < \mathscr{M}_{k_1^\alpha,-l}(v_2), (\xi\phi_i(1-\sigma)\sigma_n) >&\\ = -\int_\Gamma \omega^-(x,k_i^\alpha, a_2\vee(-l))\xi\phi_i + \lim_{n\to\infty} < \mathscr{L}_{k_i^\alpha,l}(v_1), (\xi\phi_i(\sigma_n - \sigma_n\sigma) >&\\ + \lim_{n\to\infty} < \mathscr{M}_{k_1^\alpha,-l}(v_2), (\xi\phi_i(\sigma_n - \sigma_n\sigma) > .&\end{aligned}$$

Note that $\phi_i\sigma_n\sigma = \phi_i\sigma$ for n sufficiently large. Therefore,

$$\lim_{m\to\infty}\lim_{n\to\infty} < \mathscr{L}_{k_i^\alpha,l}(v_1), (\xi\phi_i(\sigma_n - \sigma\sigma_n)) >$$

$$= \lim_{m\to\infty}\lim_{n\to\infty} < \mathscr{M}_{k_i^\alpha,-l}(v_2), (\xi\phi_i(\sigma_n - \sigma\sigma_n)) >= 0,$$

and thus, passing to the limit with $m \to \infty$ in the preceding inequality yields

$$\begin{aligned}
-\int_\Omega (b(T_l v_1) - b(T_l v_2))^+ \xi\phi_i - \chi_{\{T_l v_1 \geqslant T_l v_2\}} \big(A(T_l v_1, \nabla g(T_l v_1)) \\
- A(T_l v_2, \nabla g(T_l v_2))\big) \cdot \nabla_x(\xi\phi_i) \\
+ \int_\Omega \kappa(f_1 - f_2)\xi\phi_i \\
\geqslant -\int_\Gamma \omega^-(x, k_i^\alpha, a_2 \vee (-l))\xi\phi_i + 2\int_\Omega \xi\phi_i\, d(\mu_l(v_1) + \nu_{-l}(v_2))
\end{aligned}$$

for all $i = 1,\ldots,m_\alpha$. Summing up over $i = 0,\ldots,m_\alpha$, we find

$$\begin{aligned}
-\int_\Omega (b(T_l v_1) - b(T_l v_2))^+ \xi - \chi_{\{T_l v_1 \geqslant T_l v_2\}} \big(A(T_l v_1, \nabla g(T_l v_1)) \\
- A(T_l v_2, \nabla g(T_l v_2))\big) \cdot \nabla_x(\xi) \\
+ \int_\Omega \kappa(f_1 - f_2)\xi - 2\int_\Omega \xi\, d(\mu_l(v_1) + \nu_{-l}(v_2)) \\
\geqslant -\sum_{i=1}^{m_\alpha} \int_\Gamma \omega^-(x, k_i^\alpha, a_2 \vee (-l))\xi\phi_i \\
\geqslant -\sum_{i=1}^{m_\alpha} \int_\Gamma \omega^-(x, T_l a_1 + \frac{\varepsilon}{2}, T_l a_2)\xi\phi_i \\
= -\sum_{i=1}^{m_\alpha} \int_{\Gamma\cap\partial B_i} \omega^-(x, T_l a_1 + \frac{\varepsilon}{2}, T_l a_2)\xi.
\end{aligned}$$

By continuity of ω, letting $\varepsilon \to 0$ and after summation over i, we get (2.17) for all $\xi \in \mathscr{D}^+(\mathbb{R}^N)$ with $\operatorname{supp}(\xi) \subset \{|v_1| < \infty\} = \{|v_2| < \infty\}$. As $T_l v_1 = T_l v_2 = l$ on $\{v_1 = +\infty\} = \{v_2 = +\infty\}$ and $T_l v_1 = T_l v_2 = -l$ on $\{v_1 = -\infty\} = \{v_2 = -\infty\}$, it follows that

$$\begin{aligned}
\int_\Omega (b(T_l v_1) - b(T_l v_2))^+ \xi \leqslant \int_\Omega \kappa(f_1 - f_2)\xi \\
-\int_\Omega \chi_{\{T_l v_1 \geqslant T_l v_2\}} (A(T_l v_1, \nabla g(T_l v_1)) - A(T_l v_2, \nabla g(T_l v_2))) \cdot \nabla_x \xi \\
+ \int_\Gamma \omega^-(x, a_1, a_2 \vee (-l))\xi
\end{aligned}$$

for all $\xi \in \mathbb{R}^N$. $\square$

4. Existence of a renormalized entropy solution

The proof of existence consists in two steps: in the first step, we consider the problem

$$P_{b_\alpha,g}(f,a) \begin{cases} b_\alpha(v) - \operatorname{div} A(v, \nabla g(v)) = f & \text{in } \Omega, \\ g(v) = g(a) & \text{on } \Gamma, \end{cases}$$

with $f \in L^\infty(\Omega)$, $a \in L^\infty(\Gamma)$ $g(a) = 0$ and with $b_\alpha(r) = b(r) + \alpha r$, $\alpha > 0$. Existence and uniqueness results for this problem are already proved in the non stationary case (see [3]) by means of weak entropy solutions. The same results remain true for the stationary problem. The definition of the weak entropy solution in this case is given in Proposition 4.1 below.

In the second step, proceeding by approximation, we pass to the limit with $\alpha \to 0$ and solve the problem $P_{b,g}(f,a)$ in the L^1-setting.

4.1. First step

PROPOSITION 4.1. *Let $f \in L^\infty(\Omega)$ and $a \in L^\infty(\Gamma)$ such that $g(a) = 0$ a.e on Γ. Then, there exists a unique $v \in L^\infty(\Omega)$ entropy solution of $P_{b_\alpha,g}(f,a)$ i.e. $g(v) \in W_0^{1,p}(\Omega)$ and v satisfies the following entropy inequalities:*

For all $k \in \mathbb{R}$, for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ such that $\xi \geqslant 0$ and $sign^+(-g(k))\xi = 0$ a.e. on Γ,

$$-\int_\Gamma \omega^+(x,k,a)\xi + \int_\Omega b_\alpha(v)\chi_{\{v>k\}}\xi \leqslant \int_\Omega \chi_{\{v>k\}}\big[f\xi - (A(v,\nabla g(v)) - A(k,0))\cdot\nabla\xi\big] \tag{4.1}$$

and for all $k \in \mathbb{R}$, for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ such that $\xi \geqslant 0$ and $sign^+(g(k))\xi = 0$ a.e. on Γ,

$$-\int_\Gamma \omega^-(x,k,a)\xi - \int_\Omega b_\alpha(v)\chi_{\{k>v\}}\xi \leqslant -\int_\Omega \chi_{\{k>v\}}\big[f\xi - (A(v,\nabla g(v)) - A(k,0))\cdot\nabla\xi\big]. \tag{4.2}$$

Moreover, the following comparison principle holds true.

THEOREM 4.2. *For $i = 1,2$, let $f_i \in L^\infty(\Omega)$ and $a_i \in L^\infty(\Gamma)$ such that $g(a_i) = 0$ $i = 1,2$ a.e. on Γ. Let $v_i \in L^\infty(\Omega)$ be a weak entropy solution of $P_{b_\alpha,g}(a_i,f_i)$. Then there exist $\kappa \in L^\infty(\Omega)$ with $\kappa \in sign^+(v_1 - v_2)$ a.e. in Ω such that, for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$,*

$$\int_\Omega (b(v_1) - b(v_2))^+\xi + \int_\Omega \chi_{\{v_1>v_2\}}(A(v_1,\nabla g(v_1)) - A(v_2,\nabla g(v_2)))\cdot\nabla\xi \leqslant \int_\Omega \kappa(f_1 - f_2)\xi + \int_\Gamma \omega^-(x,a_1,a_2)\xi. \tag{4.3}$$

For the proof of the above results, we refer to Theorem 2.3 and Remark 2.2 (ii) in [3]. Remark that the range condition $R(b_\alpha + g) = \mathbb{R}$ is satisfied here because $b_\alpha : \mathbb{R} \to \mathbb{R}$ is bijective.

4.2. Second step

The comparison principle is the main tool in this second step. Let $f \in L^1(\Omega)$ and $a \in M(\Gamma)$ with $g(a)=0$ a.e. on Γ. For $m,n \in \mathbb{N}$, let $f_{m,n} = f \wedge m \vee (-n)$, and define $b_{m,n} : r \mapsto b(r) + \frac{1}{m} r^+ - \frac{1}{n} r^-$. Denote by $v_{m,n}$ the unique weak entropy solution of $P_{b_{m,n},g}(f_{m,n},a)$ (which exists by the result of the first step). Then,

$$-\int_\Gamma \omega^+(x,k,a)\xi \leqslant \int_\Omega -\chi_{\{v_{m,n}>k\}}\{(A(v_{m,n},\nabla g(v_{m,n})) - A(k,0))\cdot\nabla\xi \\ - f_{m,n}\xi + b_{m,n}(v_{m,n})\}\xi \quad (4.4)$$

for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$, for all $k \in \mathbb{R}$ such that $\mathrm{sign}^+(g(a)-g(k))\xi = 0$ on Γ and

$$-\int_\Gamma \omega^-(x,k,a)\xi \leqslant \int_\Omega \chi_{\{k>v_{m,n}\}}\{(A(v_{m,n},\nabla g(v_{m,n})) - A(k,0))\cdot\nabla\xi \\ - f_{m,n}\xi + b_{m,n}(v_{m,n})\xi\} \quad (4.5)$$

for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$, for all $k \in \mathbb{R}$ such that $\mathrm{sign}^+(g(k)-g(a))\xi = 0$ on Γ. Recall that $v_{m,n}$ is a weak solution of $b(v) -$ div $A(v,\nabla g(v)) = f$ i.e.,

$$\int_\Omega b(v_{m,n})\xi + \int_\Omega A(v_{m,n},\nabla g(v_{m,n}))\cdot\nabla\xi = \int_\Omega f\xi \quad (4.6)$$

for all $\xi \in \mathscr{D}(\Omega)$. By Theorem 4.2, there exists $\kappa_{m_1,m_2} \in L^\infty(\Omega)$ and $\tilde{\kappa}_{n_1,n_2} \in L^\infty(\Omega)$ with $\kappa_{m_1,m_2} \in \mathrm{sign}^+(v_{m_1,n} - v_{m_2,n})$, $\tilde{\kappa}_{n_1,n_2} \in \mathrm{sign}^+(v_{m,n_1} - v_{m,n_2})$ such that, for all $\xi \in \mathscr{D}^+(\mathbb{R}^N)$, $\xi \geqslant 0$,

$$\int_\Omega (\frac{1}{m_2}(v^+_{m_1,n}) - \frac{1}{m_2}(v^+_{m_2,n}))^+\xi + \frac{1}{n}(-v^-_{m_1,n} + v^-_{m_2,n})^+\xi \\ + \int_\Omega (b(v_{m_1,n}) - b(v_{m_2,n}))^+\xi \\ \leqslant -\int_\Omega \chi_{\{v_{m_1,n}>v_{m_2,n}\}}(A(v_{m_1,n},\nabla g(v_{m_1,n})) - A(v_{m_2,n},\nabla g(v_{m_2,n})))\cdot\nabla\xi \\ + \int_\Omega \kappa_{m_1,m_2}(\frac{1}{m_2} - \frac{1}{m_1})v^+_{m_1,n}\xi + \int_\Omega \kappa_{m_1,m_2}(f_{m_1,n} - f_{m_2,n})\xi \quad (4.7)$$

and

$$\int_\Omega (\frac{1}{n_2}v^-_{m,n_1}\frac{1}{n_2}v^-_{m,n_2})^+\xi + \frac{1}{m}(v^+_{m,n_1} - v^+_{m,n_2})^+\xi \\ + \int_\Omega (b(v_{m,n_1}) - b(v_{m,n_2}))^+\xi \\ \leqslant \int_\Omega \chi_{\{v_{m,n_1}>v_{m,n_2}\}}(A(v_{m,n_1},\nabla g(v_{m,n_1})) - A(v_{m,n_2},\nabla g(v_{m,n_2})))\cdot\nabla\xi \\ - \int_\Omega \tilde{\kappa}_{n_1,n_2}(\frac{1}{n_2} - \frac{1}{n_1})v^-_{m,n_1} + \int_\Omega \tilde{\kappa}_{n_1,n_2}(f_{m,n_1} - f_{m,n_2})\xi. \quad (4.8)$$

This yields that $v_{m_1,n} \leqslant v_{m_2,n}$ for $m_1 \leqslant m_2$ and $v_{m,n_1} \leqslant v_{m,n_2}$ for $n_1 \geqslant n_2$. i.e. $(v_{m,n})_m$ is increasing in m and $(v_{m,n})_n$ is decreasing in n.

In the case where $R(b+g) = \mathbb{R}$, we can proceed as in [2] to prove the convergence (up to a subsequence) of $(v_{m,n})$ a.e. to a function $v : \Omega \to \mathbb{R}$ with the following properties:

i) v is finite a.e. in Ω,

ii) $b(v) \in L^1(\Omega)$,

iii) $T_k g(v) \in W_0^{1,p}(\Omega)$, for all $k > 0$,

iv) v satisfies the entropy inequalities (2.8) and (2.9).

We are interested here in the case where

$$\lim_{r\to+\infty} (b+g)(r) < +\infty \text{ and } \lim_{r\to-\infty} (b+g)(r) > -\infty.$$

Suppose first that $\lim_{r\to+\infty} (b+g)(r) < +\infty$. We choose

$$(g(v_{m,n}) - T_{g(L)} g(v_{m,n}))^+ \in W_0^{1,p}(\Omega)$$

with $L > d$ and $k > 0$ as test function in (4.6) to find

$$\int_{\{g(v_{m,n})>g(L)\}} A(v_{m,n}, \nabla g(v_{m,n})) \cdot \nabla g(v_{m,n}) \leqslant C$$

with $C > 0$ independant of m and n and k. By (1.5), it follows that

$$\lambda \int_{\{g(v_{m,n})>g(L)\}} |\nabla g(v_{m,n})|^p \leqslant C$$

for some positive constant C. This implies that

$$(g(v_{m,n}) - T_{g(L)} g(v_{m,n}))^+ \text{ is bounded in } W^{1,p}(\Omega) \tag{4.9}$$

and passing to a subsequence if necessary, we can suppose that

$$(g(v_{m,n}) - T_{g(L)} g(v_{m,n}))^+ \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } V \in W^{1,p}(\Omega), \tag{4.10}$$

$$(g(v_{m,n}) - T_{g(L)} g(v_{m,n}))^+ \text{ converges strongly in } L^p(\Omega) \text{ and a.e. to } V, \tag{4.11}$$

with $V \geqslant 0$ a.e. on Ω. As g is increasing in $\mathbb{R} \setminus [-A_1, A_2]$, it follows that

$$(v_{m,n} \vee L) \text{ converges a.e. to } v^L \tag{4.12}$$

where $v^L : \Omega \to \overline{\mathbb{R}}$ is nonnegative, measurable with $g(v^L) = V + g(L)$ a.e. in $\{V < +\infty\}$. Now, as $v_{m,n}$ is increasing in m and decreasing in n, by classical arguments, we can extract a diagonal subsequence $(v_{m,n(m)})$ such that

$$v^+_{m,n(m)} \wedge L \text{ is convergent a.e. to some measurable function } v_L : \Omega \to \mathbb{R}^+. \tag{4.13}$$

For simplicity, we will omit the second index and simply denote $v_{m,n(m)} = v_m$ and $f_{m,n(m)} = f_m$. Combining (4.12) and (4.13), we deduce that v_m^+ is convergent a.e. to some v^+ with $v^+ \vee L = v^L$ and $v^+ \wedge L = v_L$.

Using similar arguments as above, we can prove the a.e. convergence of v_m^- to some measurable function $v^- : \Omega \to \mathbb{R} \cup \{+\infty\}$. Defining $v = v^+ - v^-$, we deduce that

$$(v_m)_m \text{ converges (again up to a subsequence) a.e. in } \Omega \text{ to } v. \tag{4.14}$$

Moreover, from (4.7) und (4.8), it follows that

$$b(v_m)\chi_{\{|v|<\infty\}} \to b(v)\chi_{\{|v|<\infty\}} \text{ in } L^1(\Omega). \tag{4.15}$$

It remains only to prove the inequalities (2.8) and (2.9).

To this end, let us first verify that v_m satisfies (2.8) and (2.9) for all $m \in \mathbb{N}$. For all $k \in I\!R$, for all $l \geqslant k$, for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$ with $(g(a \wedge l) - g(k))^+ \xi = 0$ on Γ, we have

$$\begin{aligned}
&\int_\Omega \chi_{\{v_m \wedge l > k\}} \{-b_m(v_m \wedge l)\xi + f_m \xi \\
&\qquad - (A(v_m \wedge l, \nabla g(v_m \wedge l)) - A(k,0)) \cdot \nabla \xi\} + \int_\Gamma \omega^+(x,k,a \wedge l)\xi \\
&= \int_\Omega \chi_{\{v_m > k\}} \{-(b_m(v_m) - f_m)\xi \\
&\qquad - (A(v_m, \nabla g(v_m)) - A(k,0)) \cdot \nabla \xi\} + \int_\Gamma \omega^+(x,k,a)\xi \\
&\qquad + \int_\Omega \chi_{\{v_m > l\}} \{(b_m(v_m) - b_m(l))\xi \\
&\qquad + (A(v_m, \nabla g(v_m)) - A(l,0)) \cdot \nabla \xi\} - \int_\Gamma \omega^+(x,l,a)\xi \\
&\qquad + \int_\Gamma \omega^+(x,k,a \wedge l) - \int_\Gamma \omega^+(x,k,a)\xi + \int_\Gamma \omega^+(x,a,l)\xi \\
&\geqslant [\int_\Omega \chi_{\{v_m > l\}} \{(b_m(v_m) - b_m(l))\xi + (A(v_m, \nabla g(v_m)) - A(l,0)) \cdot \nabla \xi\} \\
&\qquad - \int_\Gamma \omega^+(x,l,a)\xi] \\
&=: < \mu_l(v_m), \xi >
\end{aligned} \tag{4.16}$$

We split the right hand side of inequality (4.16) into

$$\begin{aligned}
< \mu_l(v_m), \xi > &= \Big[\int_\Omega \chi_{\{v_m > l\}} \{(b_m(v_m) - f_m)\xi + (A(v_m, \nabla g(v_m)) - A(l,0)) \cdot \nabla \xi\} \\
&\qquad - \int_\Gamma \omega^+(x,l,a)\xi \Big] + \int_\Omega \chi_{\{v_m > l\}} f_m \xi - \int_\Omega \chi_{\{v_m > l\}} \xi b_m(l)\xi \\
&:= \int_\Omega \xi \, d\tilde{\mu}_l + \int_\Omega \chi_{\{v_m > l\}} f_m \xi - \int_\Omega \chi_{\{v_m > l\}} b_m(l)\xi.
\end{aligned}$$

Thus, $\mu_l(v_m)$ is the sum of the negative measure $\tilde{\mu}_l$ and the operator $\xi \mapsto \int_\Omega \chi_{\{v_m>l\}} f_m \xi - \int_\Omega \chi_{\{v_m>l\}} b_m(l)\xi$. In particular, $\mu_l(v_m) \equiv 0$ for $l \geqslant \| v_m \|_{L^\infty(\Omega)} + d$. Moreover, for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $0 \leqslant \xi \leqslant 1$,

$$\int_\Omega \xi d(\mu_l(v_m))^+ \leqslant \int_\Omega \chi_{\{v_m>l\}} f_m^+ \xi + \int_\Omega \chi_{\{v_m>l\}} \xi |b_m(l)| \xi$$

and

$$\int_\Omega \xi d(\tilde{\mu}_l(v_m))^- \leqslant \int_\Omega \chi_{\{v_m>l\}} f_m^+ \xi + \int_\Gamma \omega^+(x,l,a)\xi.$$

This in turn applies that

$$\int_\Omega \xi d(\mu_l(v_m))^- \leqslant \int_\Omega \chi_{\{v_m>l\}} (|f_m| + b_m(l))\xi + \int_\Gamma \omega^+(x,l,a)\xi.$$

Thus, $(\mu_l(v_m))_m$ is uniformly bounded with respect to m. Therefore, we can extract a subsequence still denoted by $(\mu_l(v_m))_m$ which is convergent with respect to the weak $-*$ topology on $C(\overline{\Omega})$ to some Radon measure $\mu_l(v)$. We are going to prove that for $\xi \in \mathscr{D}(\mathbb{R}^N)$ with $\nabla\xi = 0$ on $\{v = +\infty\}$,

$$\lim_{l\to\infty} < \mu_l(v), \xi >= 0. \tag{4.17}$$

Indeed, for $l > d$, $\omega^+(x,l,a) = 0$ a.e. on Γ and as $\lim_{r\to\infty} b(r) < +\infty$ and $b(v) \in L^1(\Omega)$,

$$\lim_{l\to+\infty} \lim_{m\to\infty} \int_\Omega (b_m(v_m) - b_m(l))^+ \xi \leqslant \lim_{l\to+\infty} \lim_{m\to\infty} \int_\Omega (b(v_m) - b(l))^+ \xi = 0.$$

Moreover, for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\lim_{m\to+\infty} \int_\Omega \chi_{\{v_m>l\}} (A(v_m, \nabla_x g(v_m)) - A(l,0)) \cdot \nabla_x \xi \\
&\quad = \lim_{m\to+\infty} \int_\Omega \chi_{\{v_m>l\}} \tilde{A}(v_m, \nabla_x g(v_m)) \cdot \nabla_x \xi \\
&\qquad + \lim_{m\to+\infty} \int_\Omega \chi_{\{v_m>l\}} (A(v_m,0) - A(l,0)) \cdot \nabla_x \xi \\
&\quad = \mathscr{T}_1 + \mathscr{T}_2.
\end{aligned}$$

In order to estimate $\mathscr{T}_1$, we use hypothesis (1.4). For $m \in \mathbb{N}$, let

$$w_m := \int_d^{v_m \vee d} (C(r))^{\frac{p}{p-1}} \, dg(r).$$

Using w_m as test function in (4.6), applying the divergence theorem, we get

$$\begin{aligned}
&\int_\Omega \tilde{A}(v_m, \nabla_x g(v_m)) \cdot \nabla_x (w_m) \\
&\quad = \int_\Omega \chi_{\{v_m>d\}} A(v_m, \nabla_x g(v_m)) \cdot \nabla_x (g(v_m)) (C(v_m))^{\frac{p}{p-1}}
\end{aligned}$$

$$= \int_\Omega (-b(v_m) + f_m) w_m$$
$$\leqslant \Big(\int_0^{+\infty} (C(r))^{\frac{p}{p-1}}\, dg(r)\Big)(\| f \|_{L^1(\Omega)} + \sup_{r\in\mathbb{R}} |b(r)|). \tag{4.18}$$

Hence, by (1.5), it follows that

$$\int_\Omega \chi_{\{v_m>d\}} \lambda(v_m) |\nabla_x g(v_m)|^{\mathrm{p}} (C(v_m))^{\frac{p}{p-1}}$$
$$\leqslant \Big(\int_0^{+\infty} (C(r))^{\frac{p}{p-1}}\, dg(r)\Big)(\| f \|_{L^1(\Omega)} + \sup_{r\in\mathbb{R}} |b(r)|)$$

and by (1.2), we deduce that

$$\int_\Omega |\tilde{A}(v_m, \nabla g(v_m))|^{p'} \chi_{\{v_m>d\}} \leqslant C \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1, \tag{4.19}$$

and with $C > 0$ depending only on the following:

$$\int_0^{+\infty} (C(r))^{\frac{p}{p-1}}\, dg(r),\ \| f \|_{L^1(\Omega)} \ \text{ and } \lambda.$$

Therefore, for every $l > d$,

$$(\tilde{A}(v_m, \nabla g(v_m))\chi_{\{v_m>l\}})_m \text{ converges weakly in } L^{p'}(\Omega) \text{ to some } X_l \tag{4.20}$$

and by (4.10)-(4.14) and the classical pseudo-monotonicity argument, it follows that $X_l = \tilde{A}(v, \nabla g(v))\chi_{\{v>l\}}$ a.e. on $\chi_{\{|v|<+\infty\}}$. By assumption (1.4) and (4.20), we now deduce that

$$\lim_{l\to+\infty} \lim_{m\to+\infty} \int_\Omega \chi_{\{v_m>l\}} \tilde{A}(v_m, \nabla_x g(v_m)) \cdot \nabla_x \xi = \lim_{l\to+\infty} \int_{\{v>l\}} \tilde{A}(v, \nabla g(v)) \cdot \nabla \xi = 0$$

(because $\nabla\xi = 0$ a.e. in $\{v = +\infty\}$). Now, by assumption (1.3),

$$\lim_{l\to+\infty} \lim_{m\to+\infty} \int_\Omega \chi_{\{v_m>l\}} (A(v_m,0) - A(l,0)) \cdot \nabla_x \xi$$

$$= \lim_{l\to+\infty} \lim_{m\to+\infty} \int_\Omega \chi_{\{v<\infty\}} \chi_{\{v_m>l\}} (A(v_m,0) - A(l,0)) \cdot \nabla_x \xi$$

$$= \lim_{l\to+\infty} \int_\Omega \chi_{\{v<\infty\}} \chi_{\{v>l\}} (A(v,0) - A(l,0)) \cdot \nabla_x \xi = 0.$$

Thus,

$$\lim_{l\to+\infty} \int_\Omega \chi_{\{v>l\}} (A(v, \nabla_x g(v)) - A(l,0)) \cdot \nabla_x \xi = 0$$

for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ with $\nabla\xi = 0$ a.e. on $\{v = +\infty\}$. Therefore, we can pass to the limit with $m \to \infty$ in inequality (4.16) to obtain (2.8). Working on the second entropy inequality, we construct a family of bounded measures $(\nu_l(v_m))_l$ on $\overline{\Omega}$ such that

$$< \nu_l(v_m), \xi > := -\int_\Omega \chi_{\{l>v_m\}} \{(b(v_m) - b(l))\xi + (A(v_m, \nabla g(v_m)) - A(l,0)) \cdot \nabla \xi\}$$

$$- \int_\Gamma \omega^-(x,l,a)\xi,$$

and

$$\int_\Omega \chi_{\{k>v_m\vee l\}}\{b_m(v_m\vee l)\xi - f_m\xi + (A(v_m\vee l,\nabla g(v_m\vee l)) - A(l,0))\cdot\nabla\xi\}$$
$$\geqslant -<\nu_l(v_m),\xi> - \int_\Gamma \omega^-(x,k,a\vee l)\xi$$

for all $\xi\in\mathscr{D}^+(\mathbb{R})$ and $k\in\mathbb{R}$ such that $(g(k)-g(a\vee l))^+\xi=0$ on Γ. Moreover, we can extract a subsequence still denoted by $(\nu_l(v_m))_m$ which is convergent with respect to the weak$-*$ topology on $C(\overline{\Omega})$ to some measure $\nu_l(v)$ such that for all $\xi\in\mathscr{D}(\overline{\Omega})$ with supp $(\nabla\xi)\subset\{v>-\infty\}$,

$$\lim_{l\to+\infty}\int_\Omega \xi\, d\nu_l(v) = 0.$$

This yields to (2.9). $\square$

REMARK 4.3. In a forthcoming paper, we study the evolution problem

$$b(v)_t - \text{ div } a(v,\nabla g(v)) = f,\ \ v=a \text{ on the boundary},$$

with the same assumptions on the vector field A and the functions b,g.

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