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LOSS OF REGULARITY OF WEAK SOLUTIONS OF p–LAPLACE EQUATIONS FOR $p \neq 2$

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Abstract. If $1 < p < \infty$ and $p \neq 2$ then the exponent $\gamma_c = p/|p-2|$ is critical for the pointwise loss of regularity of the p-Laplace equation $-\Delta_p u = F(x)$, $u \in W_0^{1,p}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$, and $F \in L^{p'}(\Omega)$. By this we mean the following: if $1 < p < 2$ and N is large enough, and the right-hand side F has a singularity of order $\gamma > \gamma_c$ at some point $a \in \Omega$, that is, $F(x) \simeq |x-a|^{-\gamma}$ in a neighbourhood of a, then at the same point the weak solution u has singularity of order which is larger than γ. The value of γ_c is optimal. For $p > 2$ we have the loss of regularity in the sense that if $F(x) = C|x|^m$ with $m > 0$, then $u(x) = u(0) + D|x|^\mu$ with $\mu < m$, provided $m > \gamma_c$. We show that the p-Laplace operator is not hypoelliptic for $p \in (1,\infty) \setminus \{1 + 1/n : n \in 2\mathbb{N} - 1\}$.

1. Introduction

Singularities of Sobolev functions have been studied extensively since the 1950s, see a short historical survey in [18]. Various aspects of singularities appearing within the context of nonlinear elliptic equations and related problems have been studied in numerous papers and several research monographs published in the course of the last two decades. Let us mention only a few of them: Acciaio and Pucci [1], Borghol and Véron [2], Drábek, Kufner and Nicolosi [3], Fonseca, Malý and Mingione [5], Ghoussoub and Robert [6], Grillot [7], Heinonen, Kilpeläinen and Martio [8], Mingione [10], Pucci, García-Huidobro, Manásevich and Serrin [11], Pucci and Servadei [12], Simon [13], de Thélin [14], Véron [15, 16], and Žubrinić [24]. See also the references therein. In this paper we are interested in the study of the pointwise loss of regularity of weak solutions for the simplest p-Laplace equation, see (1) below, and in estimating the Hausdorff dimension of the set of points on which the loss of regularity occurs. The main results are stated in Theorems 2.1 and 2.6 dealing with the case of $1 < p < 2$, in Theorem 3.3 dealing with the case of $p > 2$, and in Example 4.1.

Let $F : \Omega \to \mathbb{R}$ be a Lebesgue measurable function, where Ω is an open set in $\mathbb{R}^N$. We say that $\gamma = \gamma(a) > 0$ is the order of singularity of F at $a \in \Omega$ if $F(x) \simeq |x-a|^{-\gamma}$ a.e. in a neighbourhood of a, that is, there exist two positive constants A and B such that $A|x-a|^{-\gamma} \leqslant F(x) \leqslant B|x-a|^{-\gamma}$ a.e. in a neighbourhood of a.

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We say that F has singularity of order at least $\gamma > 0$ at $a \in \Omega$ if there exists a positive constant C such that $F(x) \geqslant C|x-a|^{-\gamma}$ a.e. in a neighbourhood of a.

We consider the p-Laplace equation

$$-\Delta_p u = F(x), \quad u \in W_0^{1,p}(\Omega), \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, where $1 < p < \infty$. The p-Laplace operator is defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We assume that $F \in L^{p'}(\Omega)$, so that there exists the unique weak solution u. We view (1) as the family of p-Laplace equations indexed by $F \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ is the conjugate exponent of p.

In [17, Theorem 4] we proved the following result dealing with generating of singularities of weak solutions of (1). Note that the condition $p < N$ is natural in Theorem 1.1, since for $p \geqslant N$ the Sobolev functions do not possess singularities in the sense introduced above.

THEOREM 1.1. ([17, Theorem 4]) *Assume that $p < N$, $p < \gamma < 1 + N/p'$, and $a \in \Omega$ is fixed. If $F \in L^{p'}(\Omega)$ has the order of singularity γ at $a \in \Omega$, then the corresponding weak solution u of (1) has the order of singularity equal to $\frac{\gamma-p}{p-1}$ at a.*

We are especially interested in the case when $\frac{\gamma-p}{p-1} > \gamma$, that is, when the solution u is "more singular" at $a \in \Omega$ than the input function F in (1). In this sense we speak about the loss of regularity of weak solution at $a \in \Omega$ with respect to the regularity of input function at the same point.

DEFINITION 1.2. We say that (1) has the *loss of regularity at a given point $a \in \Omega$* if there exists $F \in L^{p'}(\Omega)$ which is singular at a, such that the corresponding weak solution has larger order of singularity at this point than the right-hand side F.

Assume that F has singularity of order $\gamma > 0$ at a point $a \in \Omega$. It is easy to see that the loss of regularity of (1) at a point $a \in \Omega$ cannot occur for $p \geqslant 2$. Indeed, defining the difference of the orders of singularities of output and input functions (that is, of u and F) by

$$\delta = \delta(F) := \frac{\gamma - p}{p-1} - \gamma = \frac{\gamma(2-p) - p}{p-1}, \tag{2}$$

we see that $\delta(F) < 0$ if $p \geqslant 2$. Hence, Theorem 1.1 implies that the weak solution u of (1) has singularity at a with the corresponding order which is smaller than γ.

Assuming that $\Omega = B_R(0)$ is a ball of radius R centered at the origin, let $F(x) = C|x|^{-\gamma}$, $C > 0$, such that γ satisfies the assumptions of Theorem 1.1. Then the corresponding weak solution of (1) can be written explicitely as

$$u(x) = \left(\frac{C}{m+N}\right)^{p'-1} \frac{|x|^{-\mu} - R^{-\mu}}{\mu}, \tag{3}$$

where $\mu = (\gamma - p)/(p-1) > 0$, see [17, Lemma 1]. As we see, it has the form $u(x) \simeq |x|^{-\mu}$ near $x = 0$. Compare with Lemma 3.1 below.

REMARK 1.3. A regularity result stated in Pucci and Servadei [12, Theorem 2.4] shows that the condition $p < \gamma$ in Theorem 1.1 cannot be relaxed. Compare with Lemma 3.1 for $m = -p$. Also conversely, Theorem 1.1 shows that the condition $a(x) \in L^{N/p(1-\varepsilon)}(\Omega)$ in [12, Theorem 2.4] cannot be relaxed.

2. Loss of regularity in the case of $1 < p < 2$

Let us assume that $1 < p < 2$. We have $\delta = 0$ for

$$\gamma_c = \frac{p}{2-p} \in (p, \infty), \tag{4}$$

which we call the *critical exponent* for the loss of regularity. Note that $\gamma_c > p$ and $\gamma_c \to \infty$ as $p \to 2-0$. The value of δ, where δ is defined by (2), will be called the *loss of regularity* at $a \in \Omega$ associated with F if $\delta > 0$. The value of $|\delta|$ will be called the *gain of regularity* at a associated with F if $\delta < 0$. If $\gamma = \gamma_c$ then there is no change of regularity. If for example $p = 2$, then we have $\delta = -2$, so the gain of regularity is equal to 2. The case of $p = 2$ is the only one in which the gain of regularity $|\delta|$ does not depend on γ.

Theorem 1.1 implies the following result concerning the property of loss of regularity of (1) at a given point $a \in \Omega$.

THEOREM 2.1. *Assume that $1 < p < 2$, $N > 2\gamma_c$, where γ_c is defined by (4). Let $a \in \Omega$ be fixed, and denote by $\mathscr{F}(a)$ the family of all functions $F \in L^{p'}(\Omega)$ such that there exists γ, $\gamma \in (\gamma_c, 1 + N/p')$, for which $F(x) \simeq |x-a|^{-\gamma}$ in a neighbourhood of a.*

(*a*) *For each $F \in \mathscr{F}(a)$ the corresponding weak solution u of (1) has the loss of regularity at the point a. More precisely, the order of singularity of u at a is $(\gamma - p)/(p-1)$, which is larger than γ.*

(*b*) *The supremum of losses of regularity at $a \in \Omega$, corresponding to all $F \in \mathscr{F}(a)$, is equal to*

$$\sup_{F \in \mathscr{F}(a)} \delta(F) = \frac{N}{\gamma_c} - 2. \tag{5}$$

Proof. (a) Note that $N > 2\gamma_c$ is equivalent to $\gamma_c < 1 + N/p'$, so that the interval $(\gamma_c, 1 + N/p')$ for γ is nonempty, and therefore the family $\mathscr{F}(a)$ is nonempty. Using Theorem 1.1, see (2), we have that for any $F \in \mathscr{F}(a)$,

$$\delta(F) = \frac{\gamma(2-p) - p}{p-1} > \frac{\gamma_c(2-p) - p}{p-1} = 0.$$

Therefore we have the loss of regularity of (1) at a.

(b) Using the definition of $\delta = \delta(F)$ in (2) we have that

$$\sup_{F \in \mathscr{F}(a)} \delta(F) = \lim_{\gamma \to 1 + N/p'} \delta(F)$$
$$= \frac{(1 + N/p')(2-p) - p}{p-1} = \frac{N(2-p)}{p} - 2 = \frac{N}{\gamma_c} - 2.$$

As we see, for any fixed $p \in (1,2)$ and N sufficiently large, the loss of regularity in a point can be made arbitrarily large, choosing a suitable function $F \in L^{p'}(\Omega)$. Now we would like to study the loss of regularity of (1) on subsets of Ω.

DEFINITION 2.2. Let A be a given nonempty subset of Ω. We say that (1) has the *loss of regularity on* A if there exists $F \in L^{p'}(\Omega)$ with singularity at least of order $\gamma = \gamma(a) > 0$ at each $a \in A$, such that the corresponding weak solution u has the loss of regularity for all points $a \in A$. In other words, the order of singularity of u at any $a \in A$ is larger than $\gamma = \gamma(a)$.

We would like to see how large can be a subset $A \subset \Omega$ in the sense of Hausdorff dimension, on which (1) has the loss of regularity. To answer this question, we shall need the following result from [22, Theorem 1], which represents a partial generalization of Theorem 1.1. By $d(x,A)$ we denote the Euclidean distance from x to A, that is, $d(x,A) = \inf\{|x-a| : a \in A\}$, while $\overline{\dim}_B A$ is the upper box dimension of A, see Falconer [4].

THEOREM 2.3. ([22, Theorem 1]) *Assume that $p < N$, and A is a compact subset of Ω such that*

$$p < \gamma < \frac{1}{p'}(N - \overline{\dim}_B A). \tag{6}$$

If $F \in L^{p'}(\Omega)$ is such that $F(x) \geqslant C d(x,A)^{-\gamma}$ for a.e. $x \in \Omega$, where C is a positive constant, then the corresponding weak solution u of (1) has the order of singularity at least $\frac{\gamma-p}{p-1}$ at each $a \in A$. Moreover, there exist two positive constants D and E such that

$$u(x) \geqslant D d(x,A)^{-(\gamma-p)/(p-1)} - E \tag{7}$$

a.e. in Ω.

REMARK 2.4. Note that the function $x \mapsto C d(x,A)^{-\gamma}$ is in $L^{p'}(\Omega)$ since $\gamma < \frac{1}{p'}(N - \overline{\dim}_B A)$ and Ω is bounded, see [19], and also [20] for a more detailed discussion. The condition $p < N$ in Theorem 2.3 implies by the Sobolev imbedding theorem that u may have singularities. From (6) we see that even stronger condition has been imposed: $pp' < N$.

REMARK 2.5. The conclusion of Theorem 2.3 holds also for any supersolution u of (1). By a supersolution of (1) we mean a function $u \in W^{1,p}(\Omega)$ such that $-\Delta_p u \geqslant F(x)$ in Ω and $u \geqslant 0$ on $\partial\Omega$ in the weak sense. We do not know if under the conditions of Theorem 2.3 we have that $F(x) \simeq d(x,A)^{-\gamma}$ implies $u(x) \simeq d(x,A)^{-(\gamma-p)/(p-1)}$ in a neighbourhood of A.

In the following theorem we show that there exist subsets A of Ω on which (1) has the loss of regularity, and such that their Hausdorff dimension is arbitrarily close to $N - p'\gamma_c$.

THEOREM 2.6. *Assume that* $1 < p < 2$ *and* $N > p'\gamma_c$. *Let* $\mathscr{L}$ *be the family of all subsets of* Ω *on which the* p*-Laplace equation* (1) *has the loss of regularity. Then*

$$\sup\{\dim_H A : A \in \mathscr{L}\} \geqslant N - p'\gamma_c. \tag{8}$$

Proof. Let us construct a compact set A in Ω satisfying $\overline{\dim}_B A < N - p'\gamma_c$, so that its Hausdorff dimension $\dim_H A$ is arbitrarily close to $N - p'\gamma_c$. To this end, assume that $N - p'\gamma_c$ is not an integer (otherwise the proof can be obtained by a slight modification of the construction below). It suffices to define A as the Cartesian product of generalized Cantor set $C^{(\alpha)} \subset [0,1]$, $\alpha \in (0,1/2)$, and the set of the form $[0,1]^k$, where $k = \lfloor N - p'\gamma_c \rfloor$ is the integer part of $N - p'\gamma_c$. The generalized Cantor set $C^{(\alpha)}$ is obtained similarly as the standard Cantor set, by consecutive removal of middle open intervals. First we remove the middle interval of length $1 - 2\alpha$ from $[0,1]$, then we do analogously with the two remaining intervals scaling by factor α, etc. Both Hausdorff and box dimensions of $C^{(\alpha)}$ are equal to $(\log 2)/(\log 1/\alpha) \in (0,1)$, see Falconer [4]. The set A has the form of a 'Cantor grill', and after using scaling and rigid motion it can be considered as a subset of Ω. Then

$$\dim_H A = \dim_B A = k + \frac{\log 2}{\log 1/\alpha},$$

where we have used the additivity property of box dimension and Hausdorff dimension, see [4]. Also, both dimensions are unaffected by the scaling and rigid motion. Now $\dim_H A$ tends to $N - p'\gamma_c$ from the left when $\alpha \uparrow \alpha_0$, where α_0 is defined by

$$\frac{\log 2}{\log 1/\alpha_0} = (N - p'\gamma_c) - \lfloor N - p'\gamma_c \rfloor \in (0,1).$$

Let $\mathscr{L}_0$ be the family of all sets A constructed above, with $k = \lfloor N - p'\gamma_c \rfloor$ and $\alpha < \alpha_0$. As we have just seen, for any $\varepsilon > 0$ there exists $A \in \mathscr{L}_0$ such that

$$N - p'\gamma_c - \varepsilon < \dim_H A = \dim_B A < N - p'\gamma_c. \tag{9}$$

The right-hand side inequality implies $\gamma_c < \frac{1}{p'}(N - \dim_B A)$. Hence, the open interval

$$I = (\gamma_c, \frac{1}{p'}(N - \dim_B A))$$

is nonempty, and we can choose $\gamma \in I$. Therefore, since also $\gamma_c > p$, all conditions of Theorem 2.3 are fulfilled. Defining $F(x) = d(x,A)^{-\gamma}$ and using Theorem 2.3, from (7) we conclude that (1) has the loss of regularity on A, since $\frac{\gamma - p}{p-1} > \gamma$ due to $\gamma > \gamma_c$. Taking the supremum in (9) over the family $\mathscr{L}_0$ we get

$$N - p'\gamma_c - \varepsilon < \sup\{\dim_H A : A \in \mathscr{L}_0\}.$$

By letting $\varepsilon \to 0$ we obtain

$$\sup\{\dim_H A : A \in \mathscr{L}_0\} \geqslant N - p'\gamma_c. \tag{10}$$

Then (8) follows from $\mathscr{L}_0 \subset \mathscr{L}$.

REMARK 2.7. The supremum in (8) can be named *the loss of regularity dimension* for the p-Laplace equation (1). It would be interesting to find its precise value. We do not know if the supremum can be achieved by some $A \in \mathscr{L}$.

REMARK 2.8. Since $p'\gamma_c = \frac{p^2}{(p-1)(2-p)}$, we see that for p close to 1 or 2 in Theorem 2.6 the condition $N > p'\gamma_c$ means that N should be large. The expression $p'\gamma_c$, viewed as a function of $p \in (1,2)$, attains its minimum for $p = 4/3$, so that in Theorem 2.6 the value of N should be at least 9 in this case. For $N = 9$ and $p = 4/3$ the corresponding supremum in (8) is at least 1.

REMARK 2.9. In [19] we introduced the notion of singular dimension of arbitrary nonempty set X of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ by

$$\text{s-dim}\,X = \sup\{\dim_H(\text{Sing}\,u) : u \in X\}, \tag{11}$$

where $\text{Sing}\,u$ is the singular set of u, that is, the set of all $a \in \Omega$ for which there exist positive constants C and γ such that $u(x) \geqslant C|x-a|^{-\gamma}$ a.e. in a neighbourhood of a. For example $\text{s-dim}\,W^{k,p}(\Omega) = (N-kp)_+$, see [18] and a survey article [23].

Let $X(\Omega, p)$ be the set of all weak solutions of p-Laplace equations (1) generated by $F \in L^{p'}(\Omega)$, that is,

$$X(\Omega,p) = \{u \in W_0^{1,p}(\Omega) : -\Delta_p u = F(x),\ F \in L^{p'}(\Omega)\},$$

In [24, Theorem 4(a)] we have shown that for $1 < p < 2$ the following estimate holds:

$$(N-pp')_+ \leqslant \text{s-dim}\,X(\Omega,p) \leqslant (N-2p)_+. \tag{12}$$

Here $t_+ = \max\{t, 0\}$ is positive part of a real number t. Therefore, in Theorem 2.6 we have also an upper bound:

$$\sup\{\dim_H A : A \in \mathscr{L}\} \leqslant \text{s-dim}\,X(\Omega,p) \leqslant N-2p.$$

We do not know the precise value of $\text{s-dim}\,X(\Omega,p)$ when $1 < p < 2$. If $p \geqslant 2$ then

$$\text{s-dim}\,X(\Omega,p) = (N-pp')_+, \tag{13}$$

see [24, Theorem 4(c)]. For $p = 2$ we have $\text{s-dim}\,X(\Omega,2) = (N-4)_+$, and the supremum in (11) is achieved, see [9, Theorem 2].

REMARK 2.10. Let us mention two regularity results for weak solutions of (1) when $F \in L^{p'}(\Omega)$ and Ω is bounded. For $p > 2$ we have the regularity formulated in terms of Besov spaces: $u \in B^{p,\infty}_{p',loc}(\Omega)$, see Simon [13, Chapter V, Theorem 1] or [14, Remark 4]. When $1 < p < 2$, we have $u \in W^{2,p}_{loc}(\Omega)$, see de Thélin [14]. These results were important in proving (12) and (13), exploiting our results dealing with singular dimension of Besov spaces, see [21]. An interesting generalization of de Thélin's result to more general elliptic differential operators of Leray-Lions type and on domains which are not necessarily bounded, has been proved by Pucci and Servadei in [12, Theorem 2.4].

3. Loss of Hölder regularity in the case of $p > 2$

Our discussion of the loss of Hölder regularity for $p > 2$ is based on the following lemma from [17]. It has been stated there under slightly different conditions, suited to generating singularities, see (3). The proof of Lemma 3.1 below is very simlar to that of [17, Lemma 1], therefore we omit it.

LEMMA 3.1. ([17, Lemma 1]) *Let Ω be the ball of radius $R > 0$ in $\mathbb{R}^N$ centered at the origin. Assume that $1 < p < \infty$, $m > \max\{-p, -N\}$, and let $F(x) = C|x|^m$, $C > 0$. Then the p-Laplace equation (1) possesses the unique weak solution in $W_0^{1,p}(\Omega)$, and it is given by*

$$u(x) = \left(\frac{C}{m+N}\right)^{p'-1} \frac{R^\mu - |x|^\mu}{\mu}, \tag{14}$$

where $\mu = (m+p)/(p-1)$.

As we see, the weak solution u in Lemma 3.1 is of the form $u(x) = u(0) + D|x|^\mu$, where $D < 0$. If $m < 0$ then we have the gain of regularity at $x = 0$: in this case the function F is singular at $x = 0$, while the solution is uniformly bounded due to $\mu > 0$.

DEFINITION 3.2. Assume therefore that $m \geqslant 0$. We say that the p-Laplace equation (1) has *the loss of Hölder regularity at $x = a$* if there exists a continuous function F of the form $F(x) \simeq |x-a|^m$ as $x \to a$, such that the corresponding weak solution u of (1) satisfies the condition $|u(x) - u(a)| \simeq |x-a|^\mu$ as $x \to a$, with $\mu < m$.

It is natural to measure the loss of Hölder regularity at a given point $a \in \Omega$ by the following quantity, which we call the loss of regularity of (1) associated with F having the form $F(x) \simeq |x-a|^m$:

$$\delta(F) := m - \mu. \tag{15}$$

According to Lemma 3.1 for $F(x) = C|x|^m$ we have $\mu = (m+p)/(p-1)$, therefore

$$\delta(F) = m - \frac{m+p}{p-1} = \frac{m(p-2)-p}{p-1}. \tag{16}$$

Note that for $1 < p \leqslant 2$ we have $\delta(F) < 0$, that is, we have the gain of Hölder regularity since $m < \mu$. The case of $p = 2$ is the only one in which $\delta(F)$ does not depend on m: here $\delta(F) = -2$, that is, we have the gain of regularity equal to 2. We are interested in the case of $\delta(F) > 0$, which is equivalent to (17) below.

THEOREM 3.3. *Let $\Omega = B_R(0)$ and assume that $p > 2$. Denote by $\mathscr{F}(0)$ the set of all functions $F : \Omega \to \mathbb{R}$ of the form $F(x) = C|x|^m$ with m satisfying*

$$m > m_c := \frac{p}{p-2}, \tag{17}$$

(a) *For any $F \in \mathscr{F}(0)$ the corresponding weak solution of (1) has the loss of Hölder regularity at $x = 0$.*

(b) The supremum of losses of Hölder regularity of (1) *in the class of right-hand sides of* (1) *from* $\mathscr{F}(0)$ *is equal to infinity:*

$$\sup_{F\in\mathscr{F}(0)} \delta(F) = \infty.$$

Proof. (a) Note that F is uniformly bounded in this case. Using Lemma 3.1, from (16) and (17) we see that $\delta(F) > 0$, that is, $m > \mu$. The claim in (b) follows at once since m can be arbitrarily large, see (16).

REMARK 3.4. Note that the value of critical exponent m_c in (17) for $p > 2$ is defined analogously as the critical exponent γ_c in (4) for $1 < p < 2$. Therefore it has sense to define the critical exponent $\gamma_c = \frac{p}{|p-2|}$ for $p \neq 2$ as in the Abstract of this paper.

REMARK 3.5. If the constant m in $F(x) = C|x|^m$ is such that $u(x) = u(0) + D|x|^\mu$ with $0 < \mu < 1$, that is, u is Hölderian near $x = 0$, then from $\mu = \frac{m+p}{p-1} < 1$ necessarily $m < -1$. For $m > -1$ the corresponding weak solution u in Lemma 3.1 is at least of class C^1, since

$$|\nabla u(0)| = u'(r)|_{r=0} = \lim_{r\to 0} D\, r^{(m+1)/(p-1)} = 0.$$

REMARK 3.6. Assume that $p > 2$. Analogously as in Section 2, we say that the p-Laplace equation (1) has the *loss of Hölderian regularity on a given subset* $A \subset \Omega$ if it has the loss of Hölderian regularity in any of its points. It would be interesting to know the supremum of Hausdorff dimensions of sets on which (1) has the loss of regularity.

4. Absence of hypoellipticity for p-Laplace operators

It is well known that the classical Laplace operator is hypoelliptic, that is, if $-\Delta u = F(x)$ in the weak sense, and $F \in C^\infty(\Omega)$, then for the weak solution we also have $u \in C^\infty(\Omega)$. We show that this is not the case for general p-Laplace operators. This also represents a phenomenon of the loss of regularity of (1), but different from the ones discussed in previous sections, though related.

EXAMPLE 4.1. Let $p \in (1,\infty) \setminus \{1 + 1/n : n \text{ is odd}\}$. Then the corresponding p-Laplace operator is not hypoelliptic.

To prove this we exploit Lemma 3.1. Let us consider simply the constant input function $F(x) = C > 0$, that is $m = 0$. The corresponding weak solution of (1) is $u(x) = u(0) + D|x|^\mu$ where

$$\mu = \frac{m+p}{p-1} = \frac{p}{p-1} = p' \in (1,\infty). \tag{18}$$

If $p \in (1,\infty)$ is not of the form $1 + n^{-1}$, $n \in \mathbb{N}$, then due to (18) the value of μ is not of the form $(1 + n^{-1})' = n + 1$. This means that μ is either noninteger or $\mu = 1$, and therefore the corresponding weak solution $u(x)$ is not of class C^∞.

If p is of the form $1+n^{-1}$ with even n, then $\mu = n+1$ is odd, so that again the solution $u(x)$ of (1) corresponding to constant input function $F(x) = C > 0$ is not of the class C^∞. Therefore, the operator Δ_p is not hypoelliptic for any $p \in (1,\infty)$ which is not of the form $1+n^{-1}$ with odd n.

REMARK 4.2. We do not know if the p-Laplace operator is hypoelliptic for p's of the form $p = 1+\frac{1}{n}$, where $n \geqslant 3$ is an odd integer (the case of $n = 1$ corresponds to the classical Laplace operator, which *is* hypoelliptic). We note by the way that for any function of the form $F(x) = C|x|^m$, where m is a nonnegative even integer, the corresponding solution $u(x) = u(0) + D|x|^\mu$ of (1) is of class C^∞ for such p's, since then $\mu = nm+n+1$ is even, and hence $u(x)$ is just a polynomial in variables x_k, $k = 1,\dots,N$.

REMARK 4.3. We believe it would be of interest to find structural conditions on more general elliptic operators of Leray-Lions type, which imply the phenomenon of loss of regularity of weak solutions, analogous to the one described in this paper for the p-Laplace operator.

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