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EXISTENCE AND NON–EXISTENCE OF ENTIRE POSITIVE SOLUTIONS FOR QUASILINEAR SYSTEMS WITH SINGULAR AND SUPER–LINEAR TERMS

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Abstract. We establish the results concerning existence and non-existence of entire positive solutions for the nonlinear elliptic systems

$$\begin{cases} -\Delta_p u = a(x)u^m + \lambda c(x)v^n, & x \in \mathbb{R}^N, \\ -\Delta_q v = b(x)v^l + \theta c(x)u^n, & x \in \mathbb{R}^N, \\ u, v > 0,\ x \in \mathbb{R}^N \text{ and } u \to 0, v \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $1 < p, q < N$ and $\lambda, \theta \geqslant 0$ are nonnegative parameters, $a, b, c : \mathbb{R}^N \to [0, \infty)$ are locally Hölder continuous functions not identically zero, and $-\infty < m < p-1, -\infty < l < q-1, \max\{p-1, q-1\} < n$. The main purpose of this paper is to extend the principal theorem of Xu and Yang in [23] which concerned single equation.

1. Introduction

In this paper we consider some new results concerning the existence and non-existence of solutions for quasilinear system of the type

$$\begin{cases} -\Delta_p u = a(x)u^m + \lambda c(x)v^n, & x \in \mathbb{R}^N, \\ -\Delta_q v = b(x)v^l + \theta c(x)u^n, & x \in \mathbb{R}^N, \\ u, v > 0,\ x \in \mathbb{R}^N \text{ and } u \to 0, v \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, denotes the p-Laplacian operator, and Δ_q has the same meaning, $1 < p, q < N$ and $\lambda, \theta \geqslant 0$ are nonnegative parameters, $a, b, c : \mathbb{R}^N \to [0, \infty)$ are locally Hölder continuous functions not identically zero, and we assume $-\infty < m < p-1, -\infty < l < q-1, \max\{p-1, q-1\} < n$.

Problem (1.1) appears in many nonlinear phenomena, for instance, in the theory of quasiregular and quasiconformal mappings, in the generalized reaction-diffusion theory or in the study of non-Newtonian fluids, see [9], [19], [21]. In the latter case, the quantity (p, q) is a characteristic of the medium. Media with $(p, q) > (2, 2)$ are called

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dilatant fluids and those with $(p,q) < (2,2)$ are called pseudoplastics. If $(p,q) = (2,2)$, they are Newtonian fluids.

Since 1980s, many important results have been obtained for quasilinear elliptic systems. We will introduce some results in the following. Existence and non-existence of solutions of the quasilinear elliptic system

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u,v) = 0, \ x \in \mathbb{R}^N, \\ \operatorname{div}(|\nabla v|^{q-2}\nabla v) + g(u,v) = 0, \ x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

has gained much attention recently. See, for example, [8], [15], [25] and [26].

When $p = q = 2$, system (1.2) becomes

$$\begin{cases} \Delta u + f(u,v) = 0, \ x \in \mathbb{R}^N, \\ \Delta v + g(u,v) = 0, \ x \in \mathbb{R}^N, \end{cases}$$

for which the existence and the non-existence of positive solutions and positive boundary blow-up solutions have been investigated extensively. We list here, for example, [4], [6], [8], [18], [20], [26] and refer to the references therein.

When to single equation, that is for equation

$$\Delta u + f(x,u) = 0, \ x \in \mathbb{R}^N,$$

there had been many results about the existence or uniqueness of the positive solutions, see [5], [10] and [11]. Recently A. V. Lair and A. Mohammed in [15] considered the existence and nonexistence of positive entire large solutions of the semilinear elliptic equation

$$\Delta u = p(x)u^{\alpha} + q(x)u^{\beta}, \ 0 < \alpha \leqslant \beta.$$

Before their work, Xu and Yang in [23] established the existence for single equation

$$\begin{cases} -\Delta_p u = a(x)(u^m + \lambda u^n), \quad x \in \mathbb{R}^N, \\ u > 0, \quad x \in \mathbb{R}^N, \\ u \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{1.3}$$

where $0 < m < p-1 < n$, they proved there exists a $\lambda^* > 0$ such that (1.3) has a positive solution for $0 < \lambda < \lambda^*$. For more results we refer the reader to the works [16], [17], [21], [29] and the references therein.

Motivated by the results of the papers [1], [7], [12], [15], [23], and [28]. In this paper, we consider the quasilinear elliptic system (1.1). The main object of the present paper is to extend the principal result of [23] and complement results in [1], [7], [12], [15], and [28] to show that there exists $(0,0) < (\lambda_*, \theta_*)$, $(\lambda^*, \theta^*) < (\infty, \infty)$ such that when $(0,0) \leqslant (\lambda, \theta) \leqslant (\lambda_*, \theta_*)$ system (1.1) has at least one solution, but no position solution when $(\lambda^*, \theta^*) < (\lambda, \theta)$. We use $(\lambda, \theta) > (\lambda^*, \theta^*)$ to denote $\lambda > \lambda^*, \theta > \theta^*$ and the same meaning for other cases in this paper.

The paper is organized as follows. In section 2, we recall some facts that will be needed in the paper. In section 3, we give the proofs of the main results in this paper.

2. Preliminaries

In order to establish our results, we introduce some notations. We denote

$$M(x) = \max\{a(x), c(x)\},\ x \in \mathbb{R}^N,\ \ \widetilde{m}(x) = \min\{a(x), c(x)\},\ x \in \mathbb{R}^N,$$

$$N(x) = \max\{b(x), c(x)\},\ x \in \mathbb{R}^N,\ \ \widetilde{n}(x) = \min\{b(x), c(x)\},\ x \in \mathbb{R}^N.$$

A sub-solution of (1.1) is meant as a pair of positive functions $(\underline{u}, \underline{v}) \in C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ with $\underline{u} \to 0, \underline{v} \to 0$ as $|x| \to \infty$ and

$$\int_{\mathbb{R}^N} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi dx \leqslant \int_{\mathbf{R}^N} (a(x)\underline{u}^m + \lambda c(x)\underline{v}^n)\varphi dx,$$

$$\int_{\mathbb{R}^N} |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \varphi dx \leqslant \int_{\mathbf{R}^N} (b(x)\underline{v}^l + \theta c(x)\underline{u}^n)\varphi dx,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geqslant 0$ and $(\overline{u}, \overline{v})$ to be the super-solution if it satisfied the inverse inequality above.

The following lemma is well known.

LEMMA 2.1. (see [3]) *Suppose there exist a sub-solution* $(\underline{u}, \underline{v})$ *and a super-solution* $(\overline{u}, \overline{v})$ *of system* (1.1) *such that* $(\underline{u}, \underline{v}) \leqslant (\overline{u}, \overline{v})$. *Then there exist at least one solution* (u, v) *of* (1.1) *such that* $(\underline{u}, \underline{v}) \leqslant (u, v) \leqslant (\overline{u}, \overline{v})$.

In this paper, we use the following definition.

DEFINITION 2.2. Say that a function $\rho(x) \in C(\mathbb{R}^N), \rho(x) \geqslant 0$ has the property (H_p) if the problem

$$\begin{cases} -\Delta_p u = \rho(x), & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \\ u \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{2.1}$$

has an entire bounded positive solution $\omega_\rho \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

In fact, from [28] we know that if $\rho(x)$ satisfies

$$H_\infty = \int_0^\infty (s^{1-N} \int_0^s t^{N-1} \psi(t) dt)^{\frac{1}{p-1}} ds < \infty, \tag{2.2}$$

where $\psi(t) = \max_{|x|=t} \rho(x), t > 0$, then $\rho(x)$ has the property (H_p).

REMARK 1. If $M(x)$ has the property (H_p) and $\widetilde{m}(x) \neq 0$, then $\widetilde{m}(x)$ also has the property (H_p), additionally, we can easy to verify $0 < \omega_{\widetilde{m}} \leqslant \omega_M$.

REMARK 2. If $N \geqslant 3, p < N$, then condition (2.2) can be replaced by

$$0 < \int_1^\infty t^{\frac{1}{p-1}} \psi(t)^{\frac{1}{p-1}} dt < \infty, \text{ if } 1 < p \leqslant 2, \tag{A}$$

$$0 < \int_1^\infty t^{\frac{(p-2)N+1}{p-1}} \psi(t) dt < \infty, \text{ if } p \geqslant 2, \tag{B}$$

where $\psi(t) = \max_{|x|=t} \rho(x) \geqslant 0$.

For the prove of non-existence of positive solutions for system (1.1), we also consider the eigenvalue problem on a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = \lambda \rho(x)|u|^{p-2}u, & x \in \Omega, \\ u(x) = 0 \quad x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $\rho(x) \in C^\alpha(\overline{\Omega}, (0,\infty))$ for some $0 < \alpha < 1$. The first eigenvalue of the problem (2.3) will be denoted by $\lambda_\Omega(\rho)$. It is well known that the following result holds true.

LEMMA 2.3. (see [2]) *Suppose that $\Omega_1 \subset \Omega_2$, and $\Omega_1 \neq \Omega_2$. Then $\lambda_{\Omega_1}(\rho) > \lambda_{\Omega_2}(\rho)$ if both exist.*

So there exists

$$\lambda_0(\rho) = \lim_{k\to\infty} \lambda_{B_k(0)}(\rho) \in [0,\infty),$$

where $B_k(0)$ is the ball centered at the origin and radius $k = 1,2,\cdots$.

THEOREM 2.4. ([2, Theorem 2.1, p.821]) *Let $v \in C^1$ satisfy $v > 0$ in Ω and $-\Delta_p v \geqslant \lambda g v^{p-1}$ for some $\lambda > 0$. Then for $u \geqslant 0$ in X_p (the completion of $C_0^\infty(\Omega)$), we have*

$$\int_\Omega |\nabla u|^p dx \geqslant \lambda \int_\Omega g|u|^p dx, \tag{2.4}$$

and $\lambda \leqslant \lambda_1$. The equality in (2.4) holds if and only if $\lambda = \lambda_1, u = kv$ and $v = cu_1$ (on each component of Ω if Ω is not connected) for some constants k, c. In particular, the principal eigenvalue λ_1 is simple if Ω is connected.

LEMMA 2.5. *Given $0 < \lambda < \infty$, assume that there exists a $u \in C^1(\mathbb{R}^N)$ such that*

$$\begin{cases} -\Delta_p u \geqslant \lambda \rho(x) u^{p-1}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N \\ u \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{2.5}$$

holds, then $\lambda \leqslant \lambda_0(\rho)$.

Proof. Because u satisfies (2.5), then we have

$$\begin{cases} -\Delta_p u \geqslant \lambda \rho(x) u^{p-1}, & x \in B_k(0), \\ u > 0, & x \in B_k(0), \end{cases}$$

for $k = 1,2,\cdots$ and $B_k(0)$ is as above. Assume the first eigenvalue of (2.3) with Ω be replaced by $B_k(0)$ is $\lambda_{B_k(0)}$. By theorem 2.4, we have

$$\lambda \leqslant \lambda_{B_k(0)}(\rho),\ k = 1,2,\cdots.$$

That is $\lambda \leqslant \lim_{k\to\infty} \lambda_{B_k(0)}(\rho) = \lambda_0(\rho)$, this end the proof.

REMARK 3. By Lemma 2.4 we can easily verify $\lambda_0(\rho)$ is positive if (2.2) is satisfied.

Now we obtain our main result.

THEOREM 2.6. *Assume* $M(x), N(x)$ *have the property* $(H_p), (H_q)$ *and*

$$\widetilde{m}(x) \neq 0,\ \widetilde{n}(x) \neq 0,\ -\infty < m < p-1,\ -\infty < l < q-1,\ \max\{p-1, q-1\} < n.$$

Then there exist $(0,0) < (\lambda_*, \theta_*)$, $(\lambda^*, \theta^*) < (\infty, \infty)$ *such that the system* (1.1) *has:*
(1) *at least one solution, if* $(0,0) \leqslant (\lambda, \theta) \leqslant (\lambda_*, \theta_*)$ *and*
(2) *no position solution, if* $(\lambda^*, \theta^*) < (\lambda, \theta)$.

3. Proof of Main Result

In this section we will give the proof of Theorem 2.6. First let us study another system:

$$\begin{cases} -\Delta_p u = a(x)(u^m + \lambda v^n), \quad x \in \mathbb{R}^N, \\ -\Delta_q v = b(x)(v^l + \theta u^s), \quad x \in \mathbb{R}^N, \\ u, v > 0,\ x \in \mathbb{R}^N \text{ and } u \to 0, v \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{3.1}$$

where $a, b : \mathbb{R}^N \to [0, \infty)$ are locally Hölder continuous functions, and we assume $-\infty < m < p-1 < n, -\infty < l < q-1 < s$. Then we have.

THEOREM 3.1. *Assume that* $a(x), b(x)$ *have the property* $(H_p), (H_q)$*, and* $-\infty < m < p-1 < n, -\infty < l < q-1 < s$. *Then there exist* $(\lambda_*, \theta_*) > (0,0)$ *such that system* (3.1) *has at least one positive solution for each* $(\lambda_*, \theta_*) \geqslant (\lambda, \theta) \geqslant (0,0)$.

Proof. First we may assume ω_a satisfies (2.1) with $\rho(x)$ be replaced by $a(x)$ and ω_b satisfies (2.1) with $\rho(x), p$ be replaced by $b(x), q$. Consider

$$\lambda(t) = \frac{t^{p-1} - t^m \|\omega_a\|_\infty^m}{t^n \|\omega_b\|_\infty^n} = \frac{1}{\|\omega_b\|_\infty^n}(t^{p-1-n} - t^{m-n}\|\omega_a\|_\infty^m),$$

$$\theta(t) = \frac{t^{q-1} - t^l \|\omega_b\|_\infty^l}{t^s \|\omega_a\|_\infty^s} = \frac{1}{\|\omega_a\|_\infty^s}(t^{q-1-s} - t^{l-s}\|\omega_b\|_\infty^l).$$

It is easy to see $\lambda(t)$ reaches its maximum value at $t_1 = (\frac{(m-n)\|\omega_a\|_\infty^m}{p-1-n})^{\frac{1}{p-1-m}}$, and $\theta(t)$ reaches its maximum value at $t_2 = (\frac{(l-s)\|\omega_b\|_\infty^l}{q-1-s})^{\frac{1}{p-1-l}}$, set $t_0 = \max\{t_1, t_2\}$, and denote

$$\lambda_* = \frac{1}{\|\omega_b\|_\infty^n}(t_0^{p-1-n} - t_0^{m-n}\|\omega_a\|_\infty^m),\ \theta_* = \frac{1}{\|\omega_a\|_\infty^s}(t_0^{q-1-s} - t_0^{l-s}\|\omega_b\|_\infty^l).$$

Then for $(0,0) \leqslant (\lambda, \theta) \leqslant (\lambda_*, \theta_*)$, we take $\overline{u} = t_0\omega_a$, $\overline{v} = t_0\omega_b$. Then there is

$$\begin{aligned} -\Delta_p \overline{u} = t_0^{p-1} a(x) &\geqslant a(x)[t_0^m \|\omega_a\|_\infty^m + \lambda t_0^n \|\omega_b\|_\infty^n] \\ &\geqslant a(x)[t_0^m \omega_a^m + \lambda t_0^n \omega_b^n] \\ &\geqslant a(x)(\overline{u}^m + \overline{v}^n). \end{aligned}$$

It follows that we have

$$-\Delta_q \overline{v} \geqslant b(x)(\overline{v}^l + \overline{u}^s).$$

Thus $(\overline{u},\overline{v}) \in C^1(\mathbb{R}^N)\times C^1(\mathbb{R}^N)$ is a supper-solution of system (3.1).

On the other hand, let us consider

$$\begin{cases} -\Delta_p u = a(x)u^m, \quad x\in B_k(0), \\ -\Delta_q v = b(x)v^l \quad x\in B_k(0) \\ u,v>0,\ x\in B_k(0) \text{ and } u=0, v=0,\ x\in\partial B_k(0), \end{cases} \tag{3.2}$$

where $B_k(0)$ as above. In fact, the existence of positive solutions for system (3.2) is equivalent to the existence of positive solutions for the following two elliptic problems:

$$\begin{cases} -\Delta_p u = a(x)u^m, \quad x\in B_k(0), \\ u>0,\ x\in B_k(0) \text{ and } u=0,\ x\in\partial B_k(0), \end{cases}$$

and

$$\begin{cases} -\Delta_q v = b(x)v^l \quad x\in B_k(0), \\ v>0,\ x\in B_k(0) \text{ and } v=0,\ x\in\partial B_k(0). \end{cases}$$

From [21] we know there exist $u_k, v_k \in C^1(B_k(0))\cap C(\overline{B_k(0)})$ satisfy the above two problems, that's satisfy system (3.2). Taking $u_k = v_k = 0$ for $|x|>k$ and using a weak comparison principle (see [13]), for any $x\in\mathbb{R}^N$ we have:

$$u_1(x)\leqslant u_2(x)\leqslant\cdots\leqslant u_k(x)\leqslant u_{k+1}(x)\leqslant\cdots\leqslant\overline{u}(x),$$

$$v_1(x)\leqslant v_2(x)\leqslant\cdots\leqslant v_k(x)\leqslant v_{k+1}(x)\leqslant\cdots\leqslant\overline{v}(x).$$

Setting $\underline{u}(x)=\lim_{k\to\infty}u_k(x)$, $\underline{v}(x)=\lim_{k\to\infty}v_k(x)$ and using some standard computations we show that $\underline{u}(x), \underline{v}(x)\in C^1(\mathbb{R}^N)$ and satisfy

$$\begin{cases} -\Delta_p u = a(x)u^m \leqslant a(x)(\underline{u}^m+\lambda\underline{v}^n), \quad x\in\mathbb{R}^N, \\ -\Delta_q v = b(x)v^l \leqslant b(x)(\underline{v}^l+\theta\underline{u}^s) \quad x\in\mathbb{R}^N, \\ u,v>0,\ x\in\mathbb{R}^N \text{ and } u\to 0, v\to 0 \text{ as } |x|\to\infty, \end{cases} \tag{3.3}$$

and $\underline{u}\leqslant\overline{u}, \underline{v}\leqslant\overline{v}$. By lemma 2.1 we complete the proof. □

REMARK 4. Theorem 3.1 can be considered as an improvement and generalization of the result in [23].

PROOF OF THEOREM 2.6. First we denote. Similar to the proof of Theorem 3.1, we can find $(\lambda_*,\theta_*)>(0,0)$ corresponding to $M(x), N(x)$ such that for any

$$(\lambda_*,\theta_*)\geqslant(\lambda,\theta)\geqslant(0,0),$$

there is $(u_\lambda, v_\theta)\in C^1(\mathbb{R}^N)\times C^1(\mathbb{R}^N)$ satisfy

$$\begin{cases} -\Delta_p u = M(x)(u^m+\lambda v^n)\geqslant a(x)u^m+\lambda c(x)v^n\geqslant a(x)u^m, \quad x\in\mathbb{R}^N, \\ -\Delta_q v = N(x)(v^l+\theta u^n)\geqslant b(x)v^l+\theta c(x)u^n\geqslant b(x)v^l, \quad x\in\mathbb{R}^N, \\ u,v>0,\ x\in\mathbb{R}^N \text{ and } u\to 0, v\to 0 \text{ as } |x|\to\infty. \end{cases} \tag{3.4}$$

In particular, $(\overline{u},\overline{v})=(u_\lambda,v_\theta)$ is a supper-solution for system (1.1).

Now we consider $(\underline{u},\underline{v})$ being a solution for system (3.3), then $(\underline{u},\underline{v})$ is a subsolution of (1.1), and $(\underline{u},\underline{v}) \leqslant (\overline{u},\overline{v})$. Then by Lemma 1.1, there exists a positive solution $(u,v) \in C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ with $(\underline{u},\underline{v}) \leqslant (u,v) \leqslant (\overline{u},\overline{v})$.

This shows (1) of Theorem 2.6.

On the other hand, given $\lambda,\theta > 0$, define

$$h_\lambda(t) = t^{m-p+1} + \lambda t^{n-p+1} \text{ and } h_\theta(t) = t^{l-q+1} + \theta t^{n-q+1}.$$

Consider

$$t_\lambda = \Big(\frac{p-1-m}{n-p+1}\Big)^{\frac{1}{n-m}} \frac{1}{\lambda^{\frac{1}{n-m}}},\ \lambda > 0,$$

$$t_\theta = \Big(\frac{q-1-l}{n-q+1}\Big)^{\frac{1}{n-l}} \frac{1}{\theta^{\frac{1}{n-l}}},\ \theta > 0.$$

We easily to see that

$$h_\lambda(t) \geqslant h_\lambda(t_\lambda) = \Big(\frac{n-m}{n-p+1}\Big)\Big(\frac{p-1-m}{n-p+1}\Big)^{\frac{m-p+1}{n-m}} \lambda^{\frac{n-p+1}{n-m}},\ \forall t > 0,$$

and

$$h_\theta(t) \geqslant h_\theta(t_\theta) = \Big(\frac{n-l}{n-q+1}\Big)\Big(\frac{q-1-l}{n-q+1}\Big)^{\frac{l-q+1}{n-l}} \theta^{\frac{n-q+1}{n-l}},\ \forall t > 0,$$

since $h_\lambda(t) \to \infty$ as $\lambda \to \infty$, and $h_\theta(t) \to \infty$ as $\theta \to \infty$, we choose

$$\lambda > \lambda_1(\widetilde{m})^{\frac{n-m}{p-1-m}} \Big(\frac{p-1-m}{n-p+1}\Big)^{\frac{p-1-n}{p-1-m}} \Big(\frac{n-m}{n-p+1}\Big)^{\frac{m-n}{p-1-m}} = \lambda^*, \tag{3.5}$$

$$\theta > \theta_1(\widetilde{n})^{\frac{n-l}{q-1-l}} \Big(\frac{q-1-l}{n-q+1}\Big)^{\frac{q-1-n}{q-1-l}} \Big(\frac{n-l}{n-q+1}\Big)^{\frac{l-n}{q-1-l}} = \theta^*, \tag{3.6}$$

by contradiction, system (1.1) has a solution (u,v), so for any $\varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geqslant 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_{\mathbf{R}^N} h_\lambda(t_\lambda)\widetilde{m}(x)u^{p-1}\varphi dx \\
&\quad \geqslant \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_{\mathbf{R}^N} h_\lambda(u)\widetilde{m}(x)u^{p-1}\varphi dx \\
&\quad = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_{\mathbb{R}^N} \widetilde{m}(x)(u^m + \lambda u^n)\varphi dx \\
&\quad \geqslant \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_{\mathbb{R}^N} a(x)u^m\varphi + \lambda c(x)u^n\varphi dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \nabla\varphi dx - \int_{\mathbb{R}^N} h_\theta(t_\theta)\widetilde{n}(x)u^{q-1}\varphi dx \\
&\qquad \geqslant \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \nabla\varphi dx - \int_{\mathbb{R}^N} b(x)v^l\varphi + \theta c(x)v^n\varphi dx,
\end{aligned}$$

but we have

$$\int_{\mathbb{R}^N} c(x)u^n\varphi dx \geqslant \int_{\mathbf{R}^N} c(x)v^n\varphi dx \tag{3.7}$$

or

$$\int_{\mathbb{R}^N} c(x)u^n\varphi dx \geqslant \int_{\mathbb{R}^N} c(x)v^n\varphi dx \tag{3.8}$$

holds. If (3.8) holds, then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi dx - \int_{\mathbf{R}^N} h_\lambda(t_\lambda)\widetilde{m}(x)u^{p-1}\varphi dx \\ \geqslant \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi dx - \int_{\mathbb{R}^N} a(x)u^m\varphi + \lambda c(x)v^n\varphi dx = 0.\end{aligned}$$

By lemma 2.5 we have $h_\lambda(t_\lambda) \leqslant \lambda_1(\widetilde{m}(x))$, but its impossible from (3.5).

If (3.7) holds, then we also get the contradiction with (3.6). That's end the proof of Theorem 2.6 □

REMARK 5. If we still have $m,l \geqslant 0$ of system (1.1) or (3.1), and there is a positive solution (u,v) for (1.1) or (3.1), similar as the argument in [23], define $u_a = \frac{p-1}{p-1-m}u^{\frac{p-1-m}{p-1}}(x)$ and $v_b = \frac{q-1}{q-1-l}v^{\frac{q-1-l}{p-1}}(x)$. It's easy to see that u_a is a supper-solution of (2.1) with $\rho = a$, so $a(x)$ has the property (H_p), also we have $b(x)$ has the property (H_q).

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