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ON THE RESOLUTION OF A PARABOLIC EQUATION IN A NONREGULAR DOMAIN OF $\mathbb{R}^3$

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Abstract. In this work we give new results of existence, uniqueness and maximal regularity of a solution to a parabolic equation set in a nonregular domain Q with Cauchy-Dirichlet boundary conditions, where $Q = \{(t,x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\} \times]0,b[\subseteq \mathbb{R}^3$ with some assumptions on the functions $(\varphi_i)_{i=1,2}$. The right-hand side term of the equation is taken in $L^2(Q)$. The method used is based on the approximation of the domain Q by a sequence of subdomains $(Q_n)_n$ which can be transformed into regular domains. This work is an extension of the one space variable case studied in [12].

1. Introduction

Let Ω be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \{(t,x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\}$$

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined on $[0,T]$, Lipschitz continuous on $]0,T[$, and such that

$$\varphi_1(t) < \varphi_2(t)$$

for $t \in]0,T[$. φ_1 is allowed to coincide with φ_2 for $t = 0$ and for $t = T$. We also assume that $(\varphi_2 - \varphi_1)$ is a bounded function on $]0,T[$. For a fixed positive number b, let Q be the three-dimensional domain defined by

$$Q = \{(t,x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\} \times]0,b[,$$

with boundary $\partial Q = (\Gamma \times]0,b[) \cup (\Omega \times \{0\}) \cup (\Omega \times \{b\})$, Γ is the boundary of Ω (see Fig.1).

In this work, we study the existence and the regularity of the solution of the parabolic equation with Cauchy-Dirichlet boundary conditions

$$(P)\begin{cases} D_t u - D_{x_1}^2 u - D_{x_2}^2 u = f \text{ in } Q \\ u = 0 \text{ on } \partial Q - \Gamma_T, \end{cases}$$

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where Γ_T is the part of the boundary of Q where $t = T$. The right-hand side term f of the equation lies in $L^2(Q)$.

In Baderko [3] we can find domains of the same kind but which can not include our domain. In Sadallah [12] the same problem has been studied for a $2m$-parabolic operator in the case of one space variable. Further references on the analysis of parabolic problems in non-cylindrical domains are: Savaré [13], Aref'ev and Bagirov [2], Hoffmann and Lewis [6], Labbas, Medeghri and Sadallah [7], [8], and Alkhutov [1]. There are many other works concerning boundary-value problems in nonsmooth domains (see, for example, Grisvard [5] and the references therein).

We are especially interested in question of what conditions the functions $(\varphi_i)_{i=1,2}$ must verify in order that Problem (P) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_\gamma^{1,2}(Q) = \left\{u \in H^{1,2}(Q) : u_{/\partial Q - \Gamma_T} = 0\right\}$$

with

$$H^{1,2}(Q) = \left\{u \in L^2(Q) : D_t u, D_{x_1}^j u, D_{x_2}^j u, D_{x_1} D_{x_2} u \in L^2(Q), j = 1,2\right\}?$$

An idea to solve Problem (P) consists in transforming the parabolic equation in the nonregular domain Q into a variable-coefficient equation in a regular domain. However, in order to perform this, one must assume that $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$.

So, in Section 2, we prove that Problem (P) admits a (unique) solution when Q could be transformed into a regular domain by means of a regular change of variable, i.e., we suppose that $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$.

In Section 3 we approximate Q by a sequence (Q_{α_n}) of such domains and we establish an *a priori* estimate of the type

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \leqslant K \|f\|_{L^2(Q_{\alpha_n})},$$

where u_n is the solution of Problem (P) in Q_{α_n} and K is a constant independent of n. Finally, in Section 4 we take limits in (Q_{α_n}) in order to reach the domain Q.

The main assumptions on the functions $(\varphi_i)_{i=1,2}$ are

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \to 0 \quad \text{as } t \to 0,\ i = 1,2 \tag{1.1}$$

and

$$\varphi_i'(t)(\varphi_2(t) - \varphi_1(t)) \to 0 \quad \text{as } t \to T,\ i = 1,2 \tag{1.2}$$

2. Resolution of Problem (P) in a reference domain Q_α

In this section, we replace Q by

$$Q_\alpha = \left\{(t, x_1) \in \mathbb{R}^2 : \alpha < t < T - \alpha, \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0, b[,$$

with $\alpha > 0$. Thus, we have

$$\begin{cases} \varphi_1(\alpha) < \varphi_2(\alpha) \\ \varphi_1(T-\alpha) < \varphi_2(T-\alpha), \end{cases}$$

(see Fig.1).

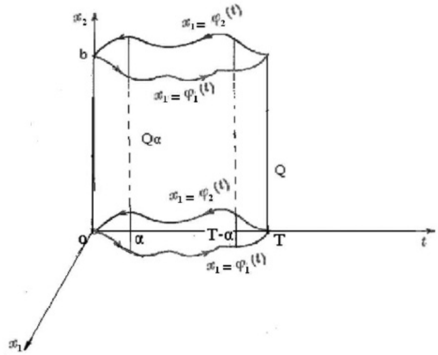

Figure 1.

We can find a change of variable ψ mapping Q_α into the parallelepiped

$$P_\alpha =]\alpha, T-\alpha[\times]0,1[\times]0,b[,$$

which leaves the variable t unchanged. ψ is defined as $\psi : Q_\alpha \longrightarrow P_\alpha$ such that:

$$(t,x_1,x_2) \longmapsto \psi(t,x_1,x_2) = \Big(\tau, y_1, y_2\Big) = (t, \frac{x_1 - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}, x_2\Big).$$

The mapping ψ transforms the parabolic equation in the domain Q_α into a variable-coefficient parabolic equation in the parallelepiped P_α. Indeed, the equation

$$D_t u - D_{x_1}^2 u - D_{x_2}^2 u = f$$

in Q_α is equivalent to the following

$$D_\tau v + a(\tau, y_1) D_{y_1} v - c(\tau) D_{y_1}^2 v - D_{y_2}^2 v = g$$

in P_α, where a and c are defined by

$$a(\tau, y_1) = \frac{\Big(\varphi_1'(\tau) - \varphi_2'(\tau)\Big) y_1 - \varphi_1'(\tau)}{\varphi_2(\tau) - \varphi_1(\tau)} \text{ and } c(\tau) = \frac{1}{(\varphi_2(\tau) - \varphi_1(\tau))^2},$$

and

$$\begin{aligned} f(t,x_1,x_2) &= g(\tau,y_1,y_2), \\ u(t,x_1,x_2) &= v(\tau,y_1,y_2). \end{aligned}$$

Since the functions a, c and $\varphi_2 - \varphi_1$ are bounded, it is easy to check the following.

LEMMA 2.1. *$u \in H^{1,2}(Q_\alpha)$ if and only if $v \in H^{1,2}(P_\alpha)$.*

In other words, ψ preserves the Sobolev space $H^{1,2}$.

The boundary conditions on v which correspond to the boundary conditions on u are the following

$$v_{/\partial P_\alpha - \Gamma_{T-\alpha}} = 0,$$

where $\Gamma_{T-\alpha}$ is the part of the boundary of P_α where $t = T - \alpha$.

In the sequel, the variables (τ, y_1, y_2) will be denoted again by (t, x_1, x_2).

THEOREM 2.1. *The operator*

$$L' = D_t + aD_{x_1} - cD_{x_1}^2 - D_{x_2}^2$$

is an isomorphism from $H_\gamma^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$, with

$$H_\gamma^{1,2}(P_\alpha) = \left\{ u \in H^{1,2}(P_\alpha) : u_{/\partial P_\alpha - \Gamma_{T-\alpha}} = 0 \right\}.$$

Consider the simplified problem

$$(P') \begin{cases} D_t v - c(t) D_{x_1}^2 v - D_{x_2}^2 v = g & \text{in } P_\alpha, \\ v_{/\partial P_\alpha - \Gamma_{T-\alpha}} = 0. \end{cases}$$

Note that $g \in L^2(P_\alpha)$ if and only if $f \in L^2(Q_\alpha)$.

LEMMA 2.2. *For every $g \in L^2(P_\alpha)$, there exists a unique $v \in H_\gamma^{1,2}(P_\alpha)$ solution of (P').*

Proof. Since the coefficient $c(t)$ is continuous in $\overline{P_\alpha}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [9].

LEMMA 2.3. *Let B be an operator, $B : H_\gamma^{1,2}(P_\alpha) \to L^2(P_\alpha)$, defined by*

$$Bv = a(t, x_1) D_{x_1} v.$$

The operator B is compact.

Proof. P_α has the "horn property" of Besov [4], so

$$\begin{array}{rccc} D_{x_1} : & H_\gamma^{1,2}(P_\alpha) & \to & H^{\frac{1}{2},1}(P_\alpha) \\ & v & \mapsto & D_{x_1} v \end{array}$$

is continuous. Since P_α is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(P_\alpha)$ into $L^2(P_\alpha)$, where (see [10]):

$$H^{\frac{1}{2},1}(P_\alpha) = L^2\left(\alpha, T-\alpha; H^1(]0,1[\times]0,b[)\right) \cap H^{\frac{1}{2}}\left(\alpha, T-\alpha; L^2(]0,1[\times]0,b[)\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces see for instance [4]. Consider the composition

$$\begin{array}{rcccc} D_{x_1}: H_\gamma^{1,2}(P_\alpha) & \rightarrow & H^{\frac{1}{2},1}(P_\alpha) & \rightarrow & L^2(P_\alpha) \\ v & \mapsto & D_{x_1}v & \mapsto & D_{x_1}v, \end{array}$$

then D_{x_1} is a compact operator from $H_\gamma^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Since $a(.,.)$ is a bounded function, the operator aD_{x_1} is also compact from $H_\gamma^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$.

Lemma 2.2 shows that the operator $L_1 = D_t - cD_{x_1}^2 - D_{x_2}^2$ is an isomorphism from $H_\gamma^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$; on the other hand the operator aD_{x_1} is compact, consequently, $L_1 + aD_{x_1}$ is a Fredholm operator from $H_\gamma^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Thus the invertibility of $L_1 + aD_{x_1}$ follows from its injectivity.

LEMMA 2.4. *The space*

$$D\left(]\alpha, T-\alpha[; H^4(]0,1[\times]0,b[) \cap H_0^1(]0,1[\times]0,b[)\right), (\textit{see } [10, \text{p.}13]),$$

is dense in the subspace of $H^{1,2}(]\alpha, T-\alpha[\times]0,1[\times]0,b[)$ *defined by*

$$u = 0 \textit{ on }]\alpha, T-\alpha[\times\{0\}\times]0,b[\textit{ and }]\alpha, T-\alpha[\times\{1\}\times]0,b[.$$

It is a particular case of Theorem 2.1 [10].

We shall need the following result in order to justify the calculus of Section 3.

LEMMA 2.5. *The space*

$$\left\{u \in H^4(P_\alpha);\ u_{/\partial P_\alpha - \Gamma_{T-\alpha}} = 0\right\}$$

is dense in the space

$$\left\{u \in H^{1,2}(P_\alpha);\ u_{/\partial P_\alpha - \Gamma_{T-\alpha}} = 0\right\}.$$

Proof. Let Γ_α be the part of the boundary of P_α where $t = \alpha$. The previous lemma shows that the space

$$\left\{u \in H^4(P_\alpha); u_{/\partial P_\alpha - \Gamma_{T-\alpha} - \Gamma_\alpha} = 0\right\}$$

is dense in the space

$$\left\{u \in H^{1,2}(P_\alpha);\ u_{/\partial P_\alpha - \Gamma_{T-\alpha} - \Gamma_\alpha} = 0\right\}.$$

So, if

$$u \in \left\{u \in H^{1,2}(P_\alpha);\ u_{/\partial P_\alpha - \Gamma_{T-\alpha} - \Gamma_\alpha} = 0\right\},$$

then there exists a sequence

$$(u_n) \in \left\{u \in H^4(P_\alpha); u_{/\partial P_\alpha - \Gamma_{T-\alpha} - \Gamma_\alpha} = 0\right\}$$

such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^{1,2}(P_\alpha), \quad \text{as } n \to \infty.$$

Let (e_n) be a sequence of $C^\infty([\alpha, T-\alpha])$ such that

$$e_n(t) = \begin{cases} 1, & \text{if } t \geqslant \alpha + \dfrac{1}{n}, \\ 0, & \text{if } t \leqslant \alpha + \dfrac{1}{2n}. \end{cases}$$

The sequence $(e_n u_n)$ belongs to

$$\left\{ u \in H^4(P_\alpha); u_{/\partial P_\alpha - \Gamma_{T-\alpha}} = 0 \right\}.$$

In addition

$$e_n u_n \rightharpoonup u \text{ weakly in } H^{1,2}(P_\alpha), \text{ as } n \to \infty.$$

REMARK 2.1. In Lemma 2.5, we can replace P_α by Q_α with the help of the change of variable ψ defined above.

3. *A priori* estimate

Now, we shall prove an *a priori* estimate which will allow us to take limits in α_n. We denote by $u_n \in H^{1,2}(Q_{\alpha_n})$ the solution of Problem (P) corresponding to the right-hand side $f_n = f_{/Q_{\alpha_n}} \in L^2(Q_{\alpha_n})$ in

$$Q_{\alpha_n} = \Omega_{\alpha_n} \times]0, b[,$$

where

$$\Omega_{\alpha_n} = \left\{ (t, x_1) \in \mathbb{R}^2 : \alpha_n < t < T - \alpha_n, \varphi_1(t) < x_1 < \varphi_2(t) \right\},$$

with $(\alpha_n)_n$ a sequence decreasing to zero.

PROPOSITION 3.1. *There exists a constant K_1 independent of n such that*

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \leqslant K_1 \|f_n\|_{L^2(Q_{\alpha_n})} \leqslant K_1 \|f\|_{L^2(Q)}.$$

In order to prove Proposition 3.1, we need some preliminary results.

LEMMA 3.1. *Let $]\alpha, \beta[\subset \mathbb{R}$. There exists a constant K_2 (independent of α and β) such that*

$$\left\| u^{(j)} \right\|^2_{L^2(]\alpha,\beta[)} \leqslant (\beta - \alpha)^{2(2-j)} K_2 \left\| u^{(2)} \right\|^2_{L^2(]\alpha,\beta[)}, \; j = 0, 1,$$

for every $u \in H^2(\alpha, \beta) \cap H^1_0(\alpha, \beta)$, where $u^{(1)}$ (respectively $u^{(2)}$) is the first (respectively the second) derivative of u on $]\alpha, \beta[$ and $u^{(0)} = u$.

The proof of this inequality may be found, for instance, in Nečas [11].

LEMMA 3.2. *For every $\varepsilon > 0$, chosen such that $(\varphi_2(t) - \varphi_1(t)) \leqslant \varepsilon$, there exists a constant C_1 independent of n such that*

$$\left\|D_{x_1}^j u_n\right\|_{L^2(Q_{\alpha_n})}^2 \leqslant C_1 \varepsilon^{2(2-j)} \left\|D_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2, j = 0,1.$$

Proof. Replacing in Lemma 3.1 u by u_n and (α, β) by $(\varphi_1(t), \varphi_2(t))$, for a fixed t, we obtain

$$\begin{aligned}\int_{\varphi_1(t)}^{\varphi_2(t)} \left(D_{x_1}^j u_n\right)^2 dx_1 &\leqslant K_2 (\varphi_2(t) - \varphi_1(t))^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(D_{x_1}^2 u_n\right)^2 dx_1 \\ &\leqslant K_2 \varepsilon^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(D_{x_1}^2 u_n\right)^2 dx_1.\end{aligned}$$

Integrating in the previous inequality with respect to t, then with respect to x_2, we get the desired result with $C_1 = K_2$.

Proof of Proposition 3.1. Let us denote the inner product in $L^2(Q_{\alpha_n})$ by $\langle .,. \rangle$, then we have

$$\begin{aligned}\|f_n\|_{L^2(Q_{\alpha_n})}^2 &= \langle D_t u_n - D_{x_1}^2 u_n - D_{x_2}^2 u_n, D_t u_n - D_{x_1}^2 u - D_{x_2}^2 u_n \rangle \\ &= \|D_t u_n\|_{L^2(Q_{\alpha_n})}^2 + \left\|D_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2 + \left\|D_{x_2}^2 u_n\right\|_{L^2(Q_{\alpha_n})}^2 \\ &\quad - 2\langle D_t u_n, D_{x_1}^2 u_n \rangle - 2\langle D_t u_n, D_{x_2}^2 u_n \rangle + 2\langle D_{x_1}^2 u_n, D_{x_2}^2 u_n \rangle.\end{aligned}$$

1) *Estimation of* $-2\langle D_t u_n, D_{x_1}^2 u_n \rangle$. We have

$$D_t u_n D_{x_1}^2 u_n = D_{x_1}(D_t u_n D_{x_1} u_n) - \frac{1}{2} D_t (D_{x_1} u_n)^2.$$

Then

$$\begin{aligned}-2\langle D_t u_n, D_{x_1}^2 u_n \rangle &= -2 \int_{Q_{\alpha_n}} D_t u_n D_{x_1}^2 u_n dt dx_1 dx_2 \\ &= -2 \int_{Q_{\alpha_n}} D_{x_1}(D_t u_n D_{x_1} u_n) dt dx_1 dx_2 \\ &\quad + \int_{Q_{\alpha_n}} D_t (D_{x_1} u_n)^2 dt dx_1 dx_2 \\ &= \int_{\partial Q_{\alpha_n}} \left[(D_{x_1} u_n)^2 \nu_t - 2 D_t u_n D_{x_1} u_n \nu_{x_1} \right] d\sigma,\end{aligned}$$

where $\nu_t, \nu_{x_1}, \nu_{x_2}$ are the components of the unit outward normal vector at ∂Q_{α_n}. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_{α_n} where $t = \alpha_n, x_2 = 0$ and $x_2 = b$, we have $u_n = 0$ and consequently $D_{x_1} u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t=T-\alpha_n$, we have $\nu_{x_1}=0$ and $\nu_t=1$. Accordingly the corresponding boundary integral

$$A=\int_0^b\int_{\varphi_1(T-\alpha_n)}^{\varphi_2(T-\alpha_n)}(D_{x_1}u_n)^2\,dx_1dx_2$$

is nonnegative. On the part of the boundary where $x_1=\varphi_i(t)$, $i=1,2$, we have $u_n=0$. By differentiating with respect to t we obtain

$$D_tu_n=-\varphi_i^{'}(t)\,D_{x_1}u_n.$$

Consequently, the corresponding boundary integrals I_1 and I_2 are the following:

$$I_1=-\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_1^{'}(t)\left[D_{x_1}u_n(t,\varphi_1(t),x_2)\right]^2\,dtdx_2$$
$$I_2=\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_2^{'}(t)\left[D_{x_1}u_n(t,\varphi_2(t),x_2)\right]^2dtdx_2.$$

We have

$$-2\langle D_tu_n,D_{x_1}^2u_n\rangle\geqslant-|I_1|-|I_2|\,. \tag{3.1}$$

LEMMA 3.3. *There exists a constant K_4 independent of n such that*

$$|I_i|\leqslant K_4\varepsilon\left\|D_{x_1}^2u_n\right\|_{L^2(Q_{\alpha_n})}^2,\ \ i=1,2.$$

Proof. We convert the boundary integral I_1 into a surface integral by setting

$$\begin{aligned}\left[D_{x_1}u_n(t,\varphi_1(t),x_2)\right]^2&=-\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\left[D_{x_1}u_n(t,x_1,x_2)\right]^2\Big|_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)}\\&=-\int_{\varphi_1(t)}^{\varphi_2(t)}D_{x_1}\left\{\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\left[D_{x_1}u_n\right]^2\right\}dx_1\\&=-2\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}D_{x_1}u_n.D_{x_1}^2u_ndx_1\\&\quad+\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{1}{\varphi_2(t)-\varphi_1(t)}\left[D_{x_1}u_n\right]^2dx_1.\end{aligned}$$

Then, we have

$$\begin{aligned}I_1&=-\int_0^b\int_{\alpha_n}^{T-\alpha_n}\varphi_1^{'}(t)\left[D_{x_1}u_n(t,\varphi_1(t),x_2)\right]^2\,dtdx_2\\&=-\int_{Q_{\alpha_n}}\frac{\varphi_1^{'}(t)}{\varphi_2(t)-\varphi_1(t)}\left[D_{x_1}u_n(t,x_1,x_2)\right]^2dtdx_1dx_2\\&\quad+2\int_{Q_{\alpha_n}}\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\varphi_1^{'}(t)\left(D_{x_1}u_n\right)\left(D_{x_1}^2u_n\right)\,dtdx_1dx_2.\end{aligned}$$

Thanks to Lemma 3.2, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} [D_{x_1} u_n]^2 \, dx_1 \leqslant K_2 [\varphi_2(t) - \varphi_1(t)]^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[D_{x_1}^2 u_n\right]^2 dx_1.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)} [D_{x_1} u_n]^2 \frac{\left|\varphi_1'\right|}{\varphi_2 - \varphi_1} dx_1 \leqslant K_2 \left|\varphi_1'\right| [\varphi_2 - \varphi_1] \int_{\varphi_1(t)}^{\varphi_2(t)} \left[D_{x_1}^2 u_n\right]^2 dx_1,$$

consequently

$$\begin{aligned} |I_1| \leqslant K_2 & \int_{Q_{\alpha_n}} \left|\varphi_1'\right| [\varphi_2 - \varphi_1] \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2 \\ & + 2 \int_{Q_{\alpha_n}} \left|\varphi_1'\right| |D_{x_1} u_n| \left|D_{x_1}^2 u_n\right| \, dt dx_1 dx_2, \end{aligned}$$

since $\left|\dfrac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)}\right| \leqslant 1.$ Using the inequality

$$2\left|\varphi_1' D_{x_1} u_n\right| \left|D_{x_1}^2 u_n\right| \leqslant \varepsilon \left(D_{x_1}^2 u_n\right)^2 + \frac{1}{\varepsilon} \left(\varphi_1'\right)^2 (D_{x_1} u_n)^2$$

for all $\varepsilon > 0,$ we obtain

$$\begin{aligned} |I_1| \leqslant K_2 & \int_{Q_{\alpha_n}} \left|\varphi_1'\right| [\varphi_2 - \varphi_1] \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2 \\ & + \int_{Q_{\alpha_n}} \varepsilon \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2 + \frac{1}{\varepsilon} \int_{Q_{\alpha_n}} \left(\varphi_1'\right)^2 (D_{x_1} u_n)^2 dt dx_1 dx_2. \end{aligned}$$

Lemma 3.2 yields

$$\frac{1}{\varepsilon} \int_{Q_{\alpha_n}} \left(\varphi_1'\right)^2 (D_{x_1} u_n)^2 dt dx_1 dx_2 \leqslant K_2 \frac{1}{\varepsilon} \int_{Q_{\alpha_n}} \left(\varphi_1'\right)^2 [\varphi_2 - \varphi_1]^2 \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2.$$

Thus,

$$\begin{aligned} |I_1| & \leqslant K_2 \int_{Q_{\alpha_n}} \left[\left|\varphi_1'\right| |\varphi_2 - \varphi_1| + \frac{1}{\varepsilon} \left(\varphi_1'\right)^2 |\varphi_2 - \varphi_1|^2\right] \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2 \\ & \quad + \int_{Q_{\alpha_n}} \varepsilon \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2 \\ & \leqslant (2K_2 + 1) \varepsilon \int_{Q_{\alpha_n}} \left(D_{x_1}^2 u_n\right)^2 dt dx_1 dx_2, \end{aligned}$$

since $\left|\varphi_1' (\varphi_2 - \varphi_1)\right| \leqslant \varepsilon.$ Finally, taking $K_4 = (2K_2 + 1),$ we obtain

$$|I_1| \leqslant K_4 \varepsilon \left\|D_{x_1}^2 u_n\right\|_{L^2(Q_{\alpha_n})}.$$

The inequality

$$|I_2| \leqslant K_4 \varepsilon \left\| D_{x_1}^2 u_n \right\|_{L^2(Q_{\alpha_n})},$$

can be proved by a similar argument.

2) *Estimation of* $-2\langle D_t u_n, D_{x_2}^2 u_n \rangle$. We have

$$D_t u_n D_{x_2}^2 u_n = D_{x_2}\left(D_t u_n D_{x_2} u_n\right) - \frac{1}{2} D_t \left(D_{x_2} u_n\right)^2.$$

Then

$$\begin{aligned} -2\langle D_t u_n, D_{x_2}^2 u_n \rangle &= -2 \int_{Q_{\alpha_n}} D_t u_n D_{x_2}^2 u_n dt dx_1 dx_2 \\ &= -2 \int_{Q_{\alpha_n}} D_{x_2}\left(D_t u_n D_{x_2} u_n\right) \, dt dx_1 dx_2 \\ &\quad + \int_{Q_{\alpha_n}} D_t \left(D_{x_2} u_n\right)^2 dt dx_1 dx_2 \\ &= \int_{\partial Q_{\alpha_n}} \left[\left(D_{x_2} u_n\right)^2 \nu_t - 2 D_t u_n D_{x_2} u_n \nu_{x_2} \right] d\sigma. \end{aligned}$$

Using the Cauchy-Dirichlet boundary conditions, we see that the above boundary integral is nonnegative. Consequently

$$-2\left\langle D_t u_n, D_{x_2}^2 u_n \right\rangle \geqslant 0. \tag{3.2}$$

3) *Estimation of* $2\langle D_{x_1}^2 u_n, D_{x_2}^2 u_n \rangle$. We have

$$D_{x_1}^2 u_n . D_{x_2}^2 u_n = D_{x_1}\left(D_{x_1} u_n . D_{x_2}^2 u_n\right) - D_{x_2}\left(D_{x_1} u_n . D_{x_1} D_{x_2} u_n\right) + \left(D_{x_1} D_{x_2} u_n\right)^2.$$

Then

$$\begin{aligned} 2\langle D_{x_1}^2 u_n, D_{x_2}^2 u_n \rangle &= 2 \int_{Q_{\alpha_n}} D_{x_1}^2 u_n . D_{x_2}^2 u_n dt dx_1 dx_2 \\ &= 2 \int_{Q_{\alpha_n}} D_{x_1}\left(D_{x_1} u_n . D_{x_2}^2 u_n\right) dt dx_1 dx_2 \\ &\quad - 2 \int_{Q_{\alpha_n}} D_{x_2}\left(D_{x_1} u_n . D_{x_1} D_{x_2} u_n\right) dt dx_1 dx_2 \\ &\quad + 2 \int_{Q_{\alpha_n}} \left(D_{x_1} D_{x_2} u_n\right)^2 dt dx_1 dx_2 \\ &= 2 \int_{Q_{\alpha_n}} \left(D_{x_1} D_{x_2} u_n\right)^2 dt dx_1 dx_2 \\ &\quad + 2 \int_{\partial Q_{\alpha_n}} \left[D_{x_1} u_n D_{x_2}^2 u_n \nu_{x_1} - D_{x_1} u_n . D_{x_1} D_{x_2} u_n \nu_{x_2} \right] d\sigma. \end{aligned}$$

Thanks to the boundary conditions, the above boundary integral vanishes. Consequently

$$2\langle D_{x_1}^2 u_n, D_{x_2}^2 u_n \rangle = 2 \left\| D_{x_1} D_{x_2} u_n \right\|_{L^2(Q_{\alpha_n})}^2. \tag{3.3}$$

Summing up the estimates (3.1), (3.2) and (3.3) of the inner products and making use of Lemma 3.3, we then obtain

$$\begin{aligned}\|f_n\|^2_{L^2(Q_{\alpha_n})} &\geqslant \|D_t u_n\|^2_{L^2(Q_{\alpha_n})} + \left\|D^2_{x_1} u_n\right\|^2_{L^2(Q_{\alpha_n})} + \left\|D^2_{x_2} u_n\right\|^2_{L^2(Q_{\alpha_n})} \\ &\quad - |I_1| - |I_2| + 2\|D_{x_1} D_{x_2} u_n\|^2_{L^2(Q_{\alpha_n})} \\ &\geqslant \|D_t u_n\|^2_{L^2(Q_{\alpha_n})} + (1-2K_4\varepsilon)\left\|D^2_{x_1} u_n\right\|^2_{L^2(Q_{\alpha_n})} \\ &\quad + \left\|D^2_{x_2} u_n\right\|^2_{L^2(Q_{\alpha_n})} + 2\|D_{x_1} D_{x_2} u_n\|^2_{L^2(Q_{\alpha_n})}.\end{aligned}$$

Then, it is sufficient to choose ε such that $(1-2K_4\varepsilon) > 0$ to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L^2(Q_{\alpha_n})} \geqslant K_0 \|u_n\|_{H^{1,2}(Q_{\alpha_n})},$$

and since

$$\|f_n\|_{L^2(Q_{\alpha_n})} \leqslant \|f\|_{L^2(Q)},$$

there exists a constant $K_1 > 0$, independent of n satisfying

$$\|u_n\|_{H^{1,2}(Q_{\alpha_n})} \leqslant K_1 \|f_n\|_{L^2(Q_{\alpha_n})} \leqslant K_1 \|f\|_{L^2(Q)}.$$

This completes the proof of Proposition 3.1.

4. Passage to the limit

We are now able to prove the main result of this work

THEOREM 4.1. *Assume that φ_1 and φ_2 fulfil conditions (*1.1*) and (*1.2*). Then the heat operator*

$$L = D_t - D^2_{x_1} - D^2_{x_2}$$

is an isomorphism from $H^{1,2}_\gamma(Q)$ into $L^2(Q)$.

Proof. Choose a sequence Q_{α_n} $n = 1,2,\dots$ of reference domains (see Section 2) such that $Q_{\alpha_n} \subseteq Q$ with (α_n) a sequence decreasing to 0, as $n \to \infty$. Then we have $Q_{\alpha_n} \to Q$, as $n \to \infty$.

Consider the solution $u_{\alpha_n} \in H^{1,2}(Q_{\alpha_n})$ of the Cauchy-Dirichlet problem

$$f(x) = \begin{cases} D_t u_{\alpha_n} - D^2_{x_1} u_{\alpha_n} - D^2_{x_2} u_{\alpha_n} = f & \text{in } Q_{\alpha_n}, \\ u_{\alpha_n / \partial Q - \Gamma_{T-\alpha_n}} = 0, \end{cases}$$

with $\Gamma_{T-\alpha_n}$ is the part of the boundary of Q_{α_n} where $t = T - \alpha_n$. Such a solution u_{α_n} exists by Theorem 2.1. Let $\widetilde{u_{\alpha_n}}$ the 0-extension of u_{α_n} to Q. In virtue of Proposition 3.1, we know that there exists a constant C such that

$$\|\widetilde{u_{\alpha_n}}\|_{L^2(Q)} + \left\|\widetilde{D_t u_{\alpha_n}}\right\|_{L^2(Q)} + \sum_{\substack{i,j=0 \\ 1 \leqslant i+j \leqslant 2}}^{2} \left\|\widetilde{D^i_{x_1} D^j_{x_2} u_{\alpha_n}}\right\|_{L^2(Q)} \leqslant C \|f\|_{L^2(Q)}.$$

This means that $\widetilde{u_{\alpha_n}}$, $\widetilde{D_t u_{\alpha_n}}$, $\widetilde{D_{x_1}^i D_{x_2}^j u_{\alpha_n}}$ for $1 \leqslant i+j \leqslant 2$ are bounded functions in $L^2(Q)$. So for a suitable increasing sequence of integers n_k, $k = 1,2,...,$ there exist functions

$$u, v \quad \text{and} \quad v_{i,j}, \quad 1 \leqslant i+j \leqslant 2$$

in $L^2(Q)$ such that

$$\begin{gathered} \widetilde{u_{\alpha_{n_k}}} \rightharpoonup u \text{ weakly in } L^2(Q),\ k \to \infty, \\ \widetilde{D_t u_{\alpha_{n_k}}} \rightharpoonup v \text{ weakly in } L^2(Q), k \to \infty, \\ \widetilde{D_{x_1}^i D_{x_2}^j u_{\alpha_{n_k}}} \rightharpoonup v_{i,j} \text{ weakly in } L^2(Q), \quad 1 \leqslant i+j \leqslant 2, k \to \infty. \end{gathered}$$

Clearly,

$$v = D_t u,\ v_{i,j} = D_{x_1}^i D_{x_2}^j u,\ 1 \leqslant i+j \leqslant 2$$

in the sense of distributions in Q. So, $u \in H^{1,2}(Q)$ and

$$D_t u - D_{x_1}^2 u - D_{x_2}^2 u = f \quad \text{in } Q.$$

On the other hand, the solution u satisfies the boundary conditions $u_{/\partial Q - \Gamma_T} = 0$ since

$$\forall n \in \mathbb{N}, u_{/Q_{\alpha_n}} = u_{\alpha_n}.$$

This proves the existence of a solution to Problem (P).

Notice that we have the estimate

$$\|u\|_{H^{1.2}(Q)}^2 \leqslant K \|f\|_{L^2(Q)}^2,$$

which implies the uniqueness of the solution.

REMARK 4.1. The result given in Theorem 4.1 holds true only under the assumption (1.1) (respectively, (1.2)), if $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(T) < \varphi_2(T)$ (respectively, if $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) = \varphi_2(T)$).

REMARK 4.2. Note that this work may be extended at least in the following directions:

1. The nonregular domain Q may be replaced by a non-cylindrical domain (conical domain, for example).
2. The function f on the right-hand side of the equation of Problem (P), may be taken in $L^p(\Omega)$, where $p \in]1,\infty[$. The method used here does not seem to be appropriate for the space $L^p(\Omega)$ when $p \neq 2$.
3. The operator L may be replaced by a high order operator.

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