Differential
Equations
& Applications
Volume 2, Number 2 (2010), 283–298

POSITIVE PERIODIC SOLUTIONS FOR THE NONLINEAR WAVE EQUATION

SVETLIN GEORGIEV GEORGIEV

(*Communicated by C.-H. Hsu*)

Abstract. In this paper we prove that nonlinear wave equation

$$u_{tt}-\Delta u = f(t,x,u,u_t,u_x)$$

has unique positive solution $u(t,x)$ which is ω-periodic with respect to the time variable t. The period $\omega>0$ is arbitrarily chosen and fixed.

1. Introduction

In this paper we consider the periodicity problem for the nonlinear wave equation

$$u_{tt}-\Delta u = f(t,x,u,u_t,u_x), \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^n, \tag{1.1}$$

$$u \quad \text{is periodic in} \quad t. \tag{1.2}$$

For a positive real number ω, a function $u=u(t,x):\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is called ω-periodic in the first variable t if $u(t+\omega,x)=u(t,x)$ for every $(t,x)\in\mathbb{R}\times\mathbb{R}^n$.

We will search a positive ω-periodic solution to the equation (1.1) in the form of a series $\left\{\left\{v(t)g(x)\right\}_{t\in[0,\omega]}\right\}_{x\in\mathbb{R}^n}$, where $g(x)\in\mathscr{C}^2(\mathbb{R}^n)$ and $v(t)\in\mathscr{C}^2(\mathbb{R})$. This notation is an original notation by S. Georgiev.

Let

$$u=\left\{\left\{v(t)g(x)\right\}_{t\in[0,\omega]}\right\}_{x\in\mathbb{R}^n}. \tag{1.1$'$}$$

We have that $u(t,x)\in\mathscr{C}^2([0,\omega]\times\mathbb{R}^n)$.

When we say that u, which is defined with $(1.1')$, is a solution to the equation (1.1) we understand: for every fixed $x\in\mathbb{R}^n$ the function $v(t)$ satisfies the equation

$$v''(t)=v(t)\frac{1}{g(x)}\sum_{i=1}^{n}\frac{\partial^2 g(x)}{\partial x_i^2}+\frac{1}{g(x)}f(t,x,v(t)g(x),v'(t)g(x),v(t)\nabla g(x)), \tag{1.1$''$}$$

where $g(x)\neq 0$ for every $x\in\mathbb{R}^n$.

Mathematics subject classification (2010): 35L05, 35L20, 35L40.
Keywords and phrases: Nonlinear wave equation, periodic solutions.

Paper DEA-02-19

Below we will prove if the function $v(t)g(x)$ satisfies the equation $(1.1'')$ for every fixed $x \in \mathbb{R}^n$ and for every $t \in [0,\omega]$ then $v(t)g(x)$ satisfies the equation (1.1) for every fixed $x \in \mathbb{R}^n$ and for every $t \in [0,\omega]$. For convenience we will note the series $\left\{\left\{v(t)g(x)\right\}_{t\in[0,\omega]}\right\}_{x\in\mathbb{R}^n}$ with $v(t)g(x)$, i.e. below we will use the notation $u = v(t)g(x)$.

When we say that the function $v(t)g(x)$ is a positive solution to the equations (1.1), which is continuous ω-periodic with respect to the time variable t and continuous with respect to the variable x we understand: $v(t)$ is positive continuous ω-periodic , $g(x)$ is positive continuous and for every fixed $x \in \mathbb{R}^n$ the function $v(t)g(x)$ satisfies the equation $(1.1'')$ for every $t \in [0,\omega]$. When we say that the function $v(t)g(x)$ is an unique positive solution to the equation (1.1) which is continuous ω-periodic with respect to the time variable t and continuous with respect to the variable x we understand: the function $v(t)g(x)$ is a positive solution to the equation (1.1), which is continuous ω-periodic with respect to time variable t and continuous with respect to the variable x, and for every fixed $x \in \mathbb{R}^n$ the function $v(t)$ is unique.

The search of such solutions is motivated from the way in which is obtained every model with partial differential equations. Usually the models are based on the every numerical value $v(t)g(x)$ when x is fixed or t is fixed and physical laws. This is the reason for which we consider here separable solutions. For instance we will give the seismic modeling and imaging (see [12] and references given therein). Also the analyze of the amplitude variation as a function of reflection angle for angle domain common image gathers produced via wave equation (see [1], [10] and the references given therein).

Here the period $\omega > 0$ is arbitrary chosen and fixed.

Here we propose new approach for investigation of the problem (1.1), (1.2) which is based on the theory of completely continuous vector field presented by M. Krasnosel'skii and P. Zabreiko [6]. This method is used for investigation of the periodicity problem for the Korteweg de Vries equation [3] and for the nonlinear parabolic equation [4]. In the accessible literature there are too many methods for investigations of the periodicity problem (1.1), (1.2)(see [2], [8], [9], [11] and the references therein) which are different than the method which we propose in this paper. This method gives new results for the problem (1.1), (1.2).

The equation (1.1) is equivalent to the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_0, \\ \dfrac{\partial u_j}{\partial t} = \dfrac{\partial u_0}{\partial x_j}, \quad j = 1,2,\cdots,n, \\ \dfrac{\partial u_0}{\partial t} = \displaystyle\sum_{j=1}^{n} \dfrac{\partial u_j}{\partial x_j} + f(t,x,u,u_t,u_x), \end{cases} \tag{1.3}$$

where $u_j = \frac{\partial u}{\partial x_j}$. Therefore we will search positive ω-periodic solutions to the system (1.3) with respect to the time variable t in the form $u(t,x) = v(t)g(x)$, $u_j(t,x) = v_j(t)g_j(x)$, $j = 0,1,\cdots,n$.

Our main result is

THEOREM 1.1. *Let $n \geqslant 2$ and $\omega > 0$ be fixed; the constants $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, $m_3 > 0$ are taken so that*

$$\begin{cases} m_3 \leqslant M_3, \quad \dfrac{e^{-M_3\omega}\left(1-e^{-m_3\omega}\right)^2}{\left(1-e^{-M_3\omega}\right)e^{2(M_3-m_3)\omega}} > (M_3+nM_1+M_2)\omega, \\ \dfrac{e^{-M_3\omega}\left(1-e^{-m_3\omega}\right)^2}{\left(1-e^{-M_3\omega}\right)e^{2(M_3-m_3)\omega}} > (M_3+1)\omega, \\ m_3\omega\dfrac{e^{-M_3\omega}}{1-e^{-M_3\omega}} > 1. \end{cases} \tag{1.4}$$

Let $f(t,x,u,u_t,u_x) \in \mathscr{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n)$ be ω - periodic function with respect to the time variable t, $f(t,x,u_1,u_{1t},u_{1x}) \geqslant f(t,x,u_2,u_{2t},u_{2x})$ for every $t \in [0,\omega]$, $x \in \mathbb{R}^n$, $u_1 \geqslant u_2$, $0 \leqslant f(t,x,u,u_t,u_x) \leqslant M_2$, $f(t,x,\lambda u,\lambda u_t,\lambda u_x) \geqslant \sqrt{\lambda} f(t,x,u,u_t,u_x)$ for every $(t,x,u,u_t,u_x) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ and $\lambda \in [0,1]$, $f(t,x,0,u_t,u_x) = 0$ for every $(t,x,u_t,u_x) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $g(x)$, $g_j(x)$, $j = 0,1,\cdots,n$ be fixed functions for which $g(x) \in \mathscr{C}^2(\mathbb{R}^n)$, $g_j(x) \in \mathscr{C}^1(\mathbb{R}^n)$, $j = 0,1,\cdots,n$, $g(x) \neq 0$, $g_j(x) \neq 0$, $j = 0,1,\cdots,n$,

$$\begin{aligned} M_1 \geqslant \frac{1}{g_0(x)}\frac{\partial g_0(x)}{\partial x_j} > 0, \qquad M_1 \geqslant \frac{1}{g_j(x)}\frac{\partial g_j(x)}{\partial x_j} > 0, \\ M_1 \geqslant \frac{1}{g(x)}\frac{\partial g(x)}{\partial x_j} > 0, \quad j = 1,\cdots,n, \end{aligned} \tag{1.5}$$

for every $x \in \mathbb{R}^n$. Then the system (1.3) has exactly one nontrivial solution

$$\begin{aligned} \bar{u}(t,x) &= (u(t,x),u_1(t,x),\cdots,u_n(t,x),u_0(t,x)) \\ &= (v(t)g(x),v_1(t)g_1(x),\cdots,v_n(t)g_n(x),v_0(t)g_0(x)), \end{aligned} \tag{1.6}$$

which is positive continuous ω-periodic with respect to the time variable t and positive continuous with respect to the variable x.

A function $f : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is ω-periodic with respect to the time variable t if $f(t+\omega,x,u,u_t,u_x) = f(t,x,u,u_t,u_x)$ for every $(t,x,u,u_t,u_x) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$.

EXAMPLE. **1)** There exist constants M_1, M_2, M_3, m_3, which satisfy the conditions (1.4) for every fixed $n \geqslant 2$ and $\omega > 0$. Really,

$$M_1\omega = M_2\omega = \frac{1}{10n}, \qquad M_3\omega = \frac{1}{10}, \qquad m_3\omega = \frac{1}{10^2},$$

where $\omega > 0$ is fixed, $n \geqslant 2$ is arbitrary chosen and fixed, satisfy the conditions (1.4).

2) Let M_1, ω, n be the same constants as in above example. Then the functions

$$g(x) = g_j(x) = e^{M_1\sum_{j=1}^n x_j}, \quad j = 0,1,\ldots,n,$$

satisfy all conditions of the Theorem 1.1.

The function

$$f(t,x,u,u_t,u_x)=f_0(t)\frac{u^{\frac{2}{7}}}{1+u^{\frac{2}{7}}},$$

where $f_0(t)$ is given positive continuous ω-periodic function, $f_0\leqslant M_2$, satisfies all conditions of the Theorem 1.1.

The author prepare a paper with applications of the Theorem 1.1 which are connected with seismic modeling (see [5]), more precisely, accurate descriptions of the lateral variation of reservoir heterogeneities.

The paper is organized as follows. In section 2 we will prove some preliminary results. In section 3 we will prove existence of positive continuous ω-periodic solutions. In section 4 we will prove uniqueness of the positive continuous ω-periodic solution of the equation (1.3).

2. Preliminary results

In this section we will prove some preliminary results which are connected with the system (1.3).

LEMMA 2.1. *Let $a(t)$, $a_j(t)$, $j=0,1,\cdots,n$, be fixed positive ω-periodic functions, $g(x)$, $g_j(x)$, $j=0,1,\cdots,n$, be fixed continuous-differentiable functions for which $g(x)\neq 0$, $g_j(x)\neq 0$, $j=0,1,\cdots,n$, for every $x\in\mathbb{R}^n$. Let also $f(t,x,u,u_t,u_x)\in\mathscr{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n)$ be ω-periodic function with respect to the time variable t. If the system (1.3) has a solution in the form (1.6) which is ω-periodic with respect to the time variable t, then for every fixed $x\in\mathbb{R}^n$ it satisfies the following system*

$$\begin{cases} u(t,x)=\dfrac{e^{-[a]\omega}}{1-e^{-[a]\omega}}\displaystyle\int_0^\omega e^{\int_t^{t+s}a(\tau)d\tau}\Big[a(t+s)u(t+s,x)+u_0(t+s,x)\Big]ds\\ u_j(t,x)=\dfrac{e^{-[a_j]\omega}}{1-e^{-[a_j]\omega}}\displaystyle\int_0^\omega e^{\int_t^{t+s}a_j(\tau)d\tau}\Big[a_j(t+s)u_j(t+s,x)+\frac{1}{g_0(x)}\frac{\partial g_0(x)}{\partial x_j}u_0(t+s,x)\Big]ds\\ u_0(t,x)=\dfrac{e^{-[a_0]\omega}}{1-e^{-[a_0]\omega}}\displaystyle\int_0^\omega e^{\int_t^{t+s}a_0(\tau)d\tau}\Big[a_0(t+s)u_0(t+s,x)+\sum_{j=1}^n\frac{1}{g_j(x)}\frac{\partial g_j(x)}{\partial x_j}u_j(t+s,x)\\ \qquad +f(t+s,x,u(t+s,x),\dfrac{v'(t+s)}{v(t+s)}u(t+s,x),\dfrac{\partial g}{\partial x_1}\dfrac{1}{g(x)}u(t+s,x),\\ \qquad \cdots,\dfrac{\partial g}{\partial x_n}\dfrac{1}{g(x)}u(t+s,x))\Big]ds. \end{cases} \tag{2.1}$$

Conversely, if for every fixed $x\in\mathbb{R}^n$ the function (1.6) is a continuous ω-periodic solution to the system (2.1), then for every fixed $x\in\mathbb{R}^n$ it is a continuous ω-periodic solution to the system (1.3).

REMARK. Here we note $[a]=\frac{1}{\omega}\int_0^\omega a(s)ds$ for continuous ω-periodic function $a(t)$.

Proof. **1.** Let $x \in \mathbb{R}^n$ be fixed. Let also (1.6) be a continuous ω-periodic solution to the system (1.3). Then

$$\begin{cases} v'(t) = -a(t)v(t) + a(t)v(t) + \dfrac{1}{g(x)} v_0(t) g_0(x) \\ v_j'(t) = -a_j(t)v_j(t) + a_j(t)v_j(t) + v_0(t)\dfrac{1}{g_j(x)}\dfrac{\partial g_0(x)}{\partial x_j}, \quad j = 1, 2, \cdots, n, \\ v_0'(t) = -a_0(t)v_0(t) + a_0(t)v_0(t) + \dfrac{1}{g_0(x)}\displaystyle\sum_{j=1}^{n} v_j(t)\dfrac{\partial g_j}{\partial x_j} + \dfrac{1}{g_0(x)} f(t, x, u, u_t, u_x). \end{cases}$$

The last system we can consider as a system of ordinary linear differential equations of first order with respect to $v(t)$, $v_j(t)$, $j = 0, 1, \cdots, n$. Let us consider the first equation

$$v'(t) = -a(t)v(t) + a(t)v(t) + \frac{1}{g(x)} v_0(t) g_0(x).$$

Then

$$v(t) = e^{-\int_0^t a(\tau)d\tau}\left(C + \int_0^t e^{\int_0^\tau a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau\right)$$

for every constant C. Let

$$C + \int_0^{-\infty} e^{\int_0^\tau a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau = 0,$$

from here

$$\begin{aligned} C &= \int_{-\infty}^{0} e^{\int_0^\tau a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau, \\ v(t) &= e^{-\int_0^t a(s)ds}\int_{-\infty}^{t} e^{\int_0^\tau a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau \\ &= \int_{-\infty}^{t} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau \\ &= \int_{t-\omega}^{t} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau \\ &\quad + \int_{t-2\omega}^{t-\omega} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau \\ &\quad + \int_{t-3\omega}^{t-2\omega} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau + \cdots. \end{aligned}$$

Let

$$J_1 = \int_{t-\omega}^{t} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau) + \frac{1}{g(x)} v_0(\tau) g_0(x)]d\tau.$$

Then

$$\int_{t-2\omega}^{t-\omega} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau)+\frac{1}{g(x)}v_0(\tau)g_0(x)]d\tau = (\tau = y-\omega)$$
$$= \int_{t-\omega}^{t} e^{-\int_{y-\omega}^{y} a(s)ds} e^{-\int_y^t a(s)ds}[a(y)v(y)+\frac{1}{g(x)}v_0(y)g_0(x)]dy$$
$$= e^{-[a]\omega}J_1.$$

As in above we can see that

$$\int_{t-3\omega}^{t-2\omega} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau)+\frac{1}{g(x)}v_0(\tau)g_0(x)]d\tau = (\tau = y-\omega)$$
$$= \int_{t-2\omega}^{t-\omega} e^{-\int_{y-\omega}^{y} a(s)ds} e^{-\int_y^t a(s)ds}[a(y)v(y)+\frac{1}{g(x)}v_0(y)g_0(x)]dy$$
$$= e^{-[a]\omega}\int_{t-2\omega}^{t-\omega} e^{-\int_y^t a(s)ds}[a(y)v(y)+\frac{1}{g(x)}v_0(y)g_0(x)]d\tau = e^{-2[a]\omega}J_1.$$

Consequently

$$v(t) = J_1 + e^{-[a]\omega}J_1 + e^{-2[a]\omega}J_1 + \cdots$$
$$= \frac{1}{1-e^{-[a]\omega}}J_1$$
$$= \frac{1}{1-e^{-[a]\omega}}\int_{t-\omega}^{t} e^{-\int_\tau^t a(s)ds}[a(\tau)v(\tau)+\frac{1}{g(x)}v_0(\tau)g_0(x)]d\tau$$

now we make the change $\tau = y-\omega+t$

$$= \frac{1}{1-e^{-[a]\omega}}\int_0^{\omega} e^{-\int_{y-\omega+t}^{y+t} a(s)ds} e^{-\int_{y+t}^{t} a(s)ds}[a(y-\omega+t)v(y-\omega+t)$$
$$+\frac{1}{g(x)}v_0(y-\omega+t)g_0(x)]dy$$
$$= \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}}\int_0^{\omega} e^{\int_t^{y+t} a(s)ds}[a(y+t)v(y+t)+\frac{1}{g(x)}v_0(y+t)g_0(x)]dy,$$

from here

$$u(t,x) = \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}}\int_0^{\omega} e^{\int_t^{y+t} a(s)ds}[a(y+t)u(y+t,x)+u_0(y+t,x)]dy.$$

Similarly, we can check that the functions $u_j(t,x)$, $j = 0,1,\cdots,n$ satisfy the system (2.1).

2. Let $x \in \mathbb{R}^n$ be fixed and the function (1.6) be a solution to the system (2.1) which is continuous-differentiable with respect to the variable x and continuous ω-periodic with respect to the time variable t. Then

$$u(t,x) = \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}}\int_0^{\omega} e^{\int_t^{y+t} a(s)ds}[a(y+t)u(y+t,x)+u_0(y+t,x)]dy$$

now we make the change $y+t=\tau$

$$
\begin{aligned}
&= \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \int_t^{t+\omega} e^{\int_t^\tau a(s)ds}[a(\tau)u(\tau,x)+u_0(\tau,x)]d\tau \\
&= e^{-\int_0^t a(s)ds} \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \int_t^{t+\omega} e^{\int_0^\tau a(s)ds}[a(\tau)u(\tau,x)+u_0(\tau,x)]d\tau, \\
u_t(t,x) &= -a(t)e^{-\int_0^t a(s)ds} \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \int_t^{t+\omega} e^{\int_0^\tau a(s)ds}[a(\tau)u(\tau,x)+u_0(\tau,x)]d\tau \\
&\quad + e^{-\int_0^t a(s)ds} \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \Big(e^{\int_0^{t+\omega} a(s)ds}[a(t+\omega)u(t+\omega,x)+u_0(t+\omega,x)] \\
&\quad - e^{\int_0^t a(s)ds}[a(t)u(t,x)+u_0(t,x)]\Big) \\
&= -a(t)u(t,x) + \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} e^{[a]\omega}\left(1-e^{-[a]\omega}\right)[a(t)u(t,x)+u_0(t,x)] \\
&= -a(t)u(t,x)+a(t)u(t,x)+u_0(t,x) = u_0(t,x).
\end{aligned}
$$

Consequently for every fixed $x \in \mathbb{R}^n$ we have

$$\frac{\partial u}{\partial t}(t,x) = u_0(t,x).$$

As in above we can see that our assertion is valid for the functions $u_j(t,x)$, $j=0,1,\cdots,n$. $\square$

Let

$$
\begin{aligned}
D^- &= \min_{0\leqslant t,s\leqslant\omega} \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} e^{\int_t^{t+s} a(\tau)d\tau}, \\
D^+ &= \max_{0\leqslant t,s\leqslant\omega} \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} e^{\int_t^{t+s} a(\tau)d\tau}, \\
D_j^- &= \min_{0\leqslant t,s\leqslant\omega} \frac{e^{-[a_j]\omega}}{1-e^{-[a_j]\omega}} e^{\int_t^{t+s} a_j(\tau)d\tau}, \\
D_j^+ &= \max_{0\leqslant t,s\leqslant\omega} \frac{e^{-[a_j]\omega}}{1-e^{-[a_j]\omega}} e^{\int_t^{t+s} a_j(\tau)d\tau}, \quad j=0,1,\cdots,n, \\
P^- &= \min\{D^-,D_1^-,D_2^-,\cdots,D_n^-,D_0^-\}, \\
P^+ &= \max\{D^+,D_1^+,D_2^+,\cdots,D_n^+,D_0^+\}.
\end{aligned}
$$

We put

$$
\begin{aligned}
\chi(\bar{u}) &= \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \int_0^\omega e^{\int_t^{t+s} a(\tau)d\tau}\Big[a(t+s)u(t+s,x)+u_0(t+s,x)\Big]ds, \\
\chi_j(\bar{u}) &= \frac{e^{-[a_j]\omega}}{1-e^{-[a_j]\omega}} \int_0^\omega e^{\int_t^{t+s} a_j(\tau)d\tau}\Big[a_j(t+s)u_j(t+s,x)+\frac{1}{g_0(x)}\frac{\partial g_0(x)}{\partial x_j}u_0(t+s,x)\Big]ds, \\
&\quad j=1,2,\cdots,n,
\end{aligned}
$$

$$\chi_0(\bar{u}) = \frac{e^{-[a_0]\omega}}{1-e^{-[a_0]\omega}} \int_0^{\omega} e^{\int_t^{t+s} a_0(\tau)d\tau} \Big[a_0(t+s)u_0(t+s,x) + \sum_{j=1}^{n} \frac{1}{g_j(x)} \frac{\partial g_j(x)}{\partial x_j} u_j(t+s,x)$$
$$+ f(t+s,x,u(t+s,x), \frac{v'(t+s)}{v(t+s)} u(t+s,x), \frac{\partial g}{\partial x_1} \frac{1}{g(x)} u(t+s,x), \cdots,$$
$$\frac{\partial g}{\partial x_n} \frac{1}{g(x)} u(t+s,x)) \Big] ds,$$

where $\bar{u}$ is (1.6), $g(x) \neq 0$, $g_j(x) \neq 0$ for every $x \in \mathbb{R}^n$, $g(x)$, $g_j(x)$, $j = 0,1,\cdots,n$, are continuous-differentiable functions.

We put

$$\mathscr{G}(\bar{u}) = (\chi(\bar{u}), \chi_1(\bar{u}), \cdots, \chi_n(\bar{u}), \chi_0(\bar{u})).$$

Let $\mathscr{C}(\omega)$ ($\mathscr{C}_+(\omega)$) be the space of real (real positive) continuous ω-periodic functions which are defined on the whole real axis.

When we write $||\bar{u}||$ for $\bar{u} \in \Big[\mathscr{C}(\omega)\Big]^{n+2}$ we understand

$$\max_{t\in[0,\omega]} |\bar{u}| = ||\bar{u}|| := \max \left\{ \max_{t\in[0,\omega]} |u(t)|, \max_{t\in[0,\omega]} |u_1(t)|, \cdots, \max_{t\in[0,\omega]} |u_n(t)|, \max_{t\in[0,\omega]} |u_0(t)| \right\}.$$

When we write $\min_t |\bar{u}|$ for $\bar{u} \in \Big[\mathscr{C}(\omega)\Big]^{n+2}$ we understand

$$\min_t |\bar{u}| := \min \left\{ \min_{t\in[0,\omega]} |u(t)|, \min_{t\in[0,\omega]} |u_1(t)|, \cdots, \min_{t\in[0,\omega]} |u_n(t)|, \min_{t\in[0,\omega]} |u_0(t)| \right\}.$$

When we write $u_1 \geqslant u_2$ for $u_1 = (u_{11}, \cdots, u_{1n})$, $u_2 = (u_{21}, \cdots, u_{2n})$ we understand $u_{1i} \geqslant u_{2i}$ for every $i = 1,2,\cdots,n$.

Let also

$$\mathscr{C}_+^{\circ}(\omega) = \Big\{ x \in \Big[\mathscr{C}_+(\omega)\Big]^{n+2} : \min_t x(t) \geqslant \frac{P^-}{P^+} ||x|| \Big\}.$$

PROPOSITION 2.1. *The space $\mathscr{C}_+^{\circ}(\omega)$ is a cone. Let $f(t,x,u,u_t,u_x) \in \mathscr{C}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$, $f(t,x,u,u_t,u_x) \geqslant 0$ for every $(t,x,u,u_t,u_x) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, $g(x)$, $g_j(x)$, $j = 0,1,\cdots,n$ be fixed functions for which $g(x) \neq 0$, $g_j(x) \neq 0$, $j = 0,1,\cdots,n$ for every $x \in \mathbb{R}^n$, $g(x) \in \mathscr{C}^1(\mathbb{R}^n)$, $g_j(x) \in \mathscr{C}^1(\mathbb{R}^n)$, $j = 0,1,\cdots,n$,*

$$\frac{1}{g_0(x)} \frac{\partial g_0(x)}{\partial x_j} > 0, \qquad \frac{1}{g_j(x)} \frac{\partial g_j(x)}{\partial x_j} > 0, \qquad \frac{1}{g(x)} \frac{\partial g(x)}{\partial x_j} > 0, \quad j = 1,2,\cdots,n,$$

for every $x \in \mathbb{R}^n$. Let also $a(t)$, $a_j(t)$, $j = 0,1,\ldots,n$, be fixed positive continuous ω-periodic functions. Then

$$\mathscr{G} : \mathscr{C}_+^{\circ}(\omega) \longrightarrow \mathscr{C}_+^{\circ}(\omega).$$

Proof. The space $\mathscr{C}_+^{\circ}(\omega)$ is closed and convexed. Indeed, let $y \in \mathscr{C}_+^{\circ}(\omega)$ and $k > 0$ be fixed constant. Then

$$k \min_t y(t) \geqslant \frac{P^-}{P^+} k||y|| \Longleftrightarrow \min_t (ky) \geqslant \frac{P^-}{P^+} ||ky||,$$

i.e. $ky \in \mathscr{C}_+^\circ(\omega)$. If $k<0$ then

$$\min_t(ky) \leqslant \frac{P^-}{P^+}||ky||.$$

Consequently $\mathscr{C}_+^\circ(\omega)$ is a cone.

Let $x \in \mathbb{R}^n$ be fixed, $\bar{u}(t,x) \in \mathscr{C}_+^\circ(\omega)$. Then from the definition of the operator χ we have

$$\chi(\bar{u}) \geqslant D^- \int_0^\omega [a(t+s)u(t+s,x)+u_0(t+s,x)]ds.$$

Since $a(t)$, $u(t,x)$, $u_0(t,x)$ are continuous ω-periodic functions with respect to the time variable t we have

$$\chi(\bar{u}) \geqslant D^- \int_0^\omega [a(s)u(s,x)+u_0(s,x)]ds.$$

From here

$$\min_{0\leqslant t\leqslant\omega} \chi(\bar{u}) \geqslant D^- \int_0^\omega [a(s)u(s,x)+u_0(s,x)]ds. \tag{2.2}$$

On the other hand

$$\chi(\bar{u}) \leqslant D^+ \int_0^\omega [a(t+s)u(t+s,x)+u_0(t+s,x)]ds = D^+ \int_0^\omega [a(s)u(s,x)+u_0(s,x)]ds,$$

from where

$$\max_{0\leqslant t\leqslant\omega} \chi(\bar{u}) \leqslant D^+ \int_0^\omega [a(s)u(s,x)+u_0(s,x)]ds. \tag{2.3}$$

From (2.2), (2.3) we get

$$\min_{t\in[0,\omega]} \chi(\bar{u}) \geqslant \frac{D^-}{D^+} \max_{t\in[0,\omega]} \chi(\bar{u}) \geqslant \frac{P^-}{P^+} \max_{t\in[0,\omega]} \chi(\bar{u}).$$

As in above we can see that

$$\min_{t\in[0,\omega]} \chi_j(\bar{u}) \geqslant \tfrac{D_j^-}{D_j^+} \max_{t\in[0,\omega]} \chi_j(\bar{u}) \geqslant \tfrac{P^-}{P^+} \max_{t\in[0,\omega]} \chi_j(\bar{u}), \quad j=0,1,\cdots,n. \qquad \square$$

PROPOSITION 2.2. *Let $f(t,x,u,u_t,u_x) \in \mathscr{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n)$, $|f(t,x,u,u_t,u_x)| \leqslant M_2$ for every $(t,x,u,u_t,u_x) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$, $g(x)$, $g_j(x)$, $j=0,1,\cdots,n$ be fixed functions for which $g(x)\neq 0$, $g_j(x)\neq 0$, $j=0,1,\cdots,n$ for every $x\in\mathbb{R}^n$, $g(x)\in\mathscr{C}^1(\mathbb{R}^n)$, $g_j(x)\in\mathscr{C}^1(\mathbb{R}^n)$, $j=0,1,\cdots,n$,*

$$\left|\frac{1}{g_0(x)}\frac{\partial g_0(x)}{\partial x_j}\right| \leqslant M_1, \quad \left|\frac{1}{g_j(x)}\frac{\partial g_j(x)}{\partial x_j}\right| \leqslant M_1, \quad \left|\frac{1}{g(x)}\frac{\partial g(x)}{\partial x_j}\right| \leqslant M_1, \quad j=1,2,\cdots,n,$$

for every $x\in\mathbb{R}^n$. Let also $a(t)$, $a_j(t)$ be fixed continuous ω-periodic functions and $|a(t)|\leqslant M_3$, $|a_j(t)|\leqslant M_3$, $j=0,1,\cdots,n$ for every $t\in[0,\omega]$. Then the operator $\mathscr{G}$ is *a completely continuous operator in* $\left[\mathscr{C}(\omega)\right]^{n+2}$ *for every fixed* $x \in \mathbb{R}^n$ *and for every* $t \in [0, \omega]$.

Proof. Let $x \in \mathbb{R}^n$ be fixed and $\bar{u}(t,x) = (u(t,x), u_1(t,x), u_2(t,x), \cdots, u_n(t,x), u_0(t,x))$ $\in \left[\mathscr{C}(\omega)\right]^{n+2}$, $\max_{t\in[0,\omega]} |\bar{u}(t,x)| = r$, $r > 0$ be fixed constant. From the definition of the operator $\mathscr{G}$, for every fixed $x \in \mathbb{R}^n$, we have

$$|\chi(\bar{u})| \leqslant P^+ \int_0^\omega (M_3+1) r ds = P^+(M_3+1) r\omega, \tag{2.4}$$

$$|\chi_j(\bar{u})| \leqslant P^+ \int_0^\omega (M_3+M_1) r ds = P^+(M_3+M_1) r\omega, \quad j = 1, 2, \cdots, n, \tag{2.5}$$

$$|\chi_0(\bar{u})| \leqslant P^+ \int_0^\omega (M_3 r + M_1 n r + M_2) ds = P^+(M_3 r + M_1 n r + M_2)\omega. \tag{2.6}$$

Consequently the functions $\chi(\bar{u})(t)$, $\chi_j(\bar{u})(t)$, $j = 0, 1, \cdots, n$, are uniformly bounded in the space $\left[\mathscr{C}(\omega)\right]^{n+2}$ for every fixed $x \in \mathbb{R}^n$.

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for every t_1 and t_2 for which $|t_1 - t_2| < \delta$ we have

$$\begin{aligned}
&\Big| e^{\int_{t_1}^{t_1+s} a(\tau)d\tau}\Big[a(t_1+s)u(t_1+s,x) + u_0(t_1+s,x)\Big] \\
&\quad - e^{\int_{t_2}^{t_2+s} a(\tau)d\tau}\Big[a(t_2+s)u(t_2+s,x) + u_0(t_2+s,x)\Big]\Big| < \frac{\varepsilon}{\omega P^+}, \\
&\Big| e^{\int_{t_1}^{t_1+s} a_j(\tau)d\tau}\Big[a_j(t_1+s)u_j(t_1+s,x) + \frac{1}{g_0(x)}\frac{\partial g_0(x)}{\partial x_j}u_0(t_1+s,x)\Big] \\
&\quad - e^{\int_{t_2}^{t_2+s} a_j(\tau)d\tau}\Big[a_j(t_2+s)u_j(t_2+s,x) + \frac{1}{g_0(x)}\frac{\partial g_0(x)}{\partial x_j}u_0(t_2+s,x)\Big]\Big| < \frac{\varepsilon}{\omega P^+}, \\
&\Big| e^{\int_{t_1}^{t_1+s} a_0(\tau)d\tau}\Big[a_0(t_1+s)u_0(t_1+s,x) + \sum_{j=1}^n \frac{1}{g_j(x)}\frac{\partial g_j(x)}{\partial x_j}u_j(t_1+s,x) \\
&\quad + f(t_1+s, x, u(t_1+s,x), \frac{v'(t_1+s)}{v(t_1+s)}u(t_1+s,x), \frac{\partial g}{\partial x_1}\frac{1}{g(x)}u(t_1+s,x), \cdots, \\
&\qquad \frac{\partial g}{\partial x_n}\frac{1}{g(x)}u(t_1+s,x))\Big] \\
&\quad - e^{\int_{t_2}^{t_2+s} a_0(\tau)d\tau}\Big[a_0(t_2+s)u_0(t_2+s,x) + \sum_{j=1}^n \frac{1}{g_j(x)}\frac{\partial g_j(x)}{\partial x_j}u_j(t_2+s,x) \\
&\quad + f(t_2+s, x, u(t_2+s,x), \frac{v'(t_2+s)}{v(t_2+s)}u(t_2+s,x), \frac{\partial g}{\partial x_1}\frac{1}{g(x)}u(t_2+s,x), \cdots, \\
&\qquad \frac{\partial g}{\partial x_n}\frac{1}{g(x)}u(t_2+s,x))\Big]\Big| < \frac{\varepsilon}{\omega P^+}.
\end{aligned}$$

Then

$$\begin{aligned}
&|\chi(\bar{u})(t_1) - \chi(\bar{u})(t_2)| < \varepsilon, \\
&|\chi_j(\bar{u})(t_1) - \chi_j(\bar{u})(t_2)| < \varepsilon, \quad j = 0, 1, \cdots, n,
\end{aligned}$$

for $|t_1 - t_2| < \delta$ and for every fixed $x \in \mathbb{R}^n$. Then $\chi(\bar{u})$, $\chi_j(\bar{u})$, $j = 0, 1, \cdots, n$ are equicontinuous for every fixed $x \in \mathbb{R}^n$. From the Arzela-Ascoli theorem follows that the set $\{\chi(\bar{u}), \chi_1(\bar{u}), \cdots, \chi_n(\bar{u}), \chi_0(\bar{u})\}$ is compact subset of the space $\left[\mathscr{C}(\omega)\right]^{n+2}$ for every fixed $x \in \mathbb{R}^n$. From here and from uniformly bounded of the functions $\chi(\bar{u}), \chi_1(\bar{u}), \cdots, \chi_n(\bar{u}), \chi_0(\bar{u})$ follows that the operators $\chi(\bar{u}), \chi_1(\bar{u}), \cdots, \chi_n(\bar{u})$, $\chi_0(\bar{u})$ are completely continuous in $\left[\mathscr{C}(\omega)\right]^{n+2}$ for every fixed $x \in \mathbb{R}^n$. $\square$

3. Existence of positive periodic solutions

The proof for existence of nontrivial solution to the equation (1.3), which is positive continuous ω-periodic with respect to the variable t and positive continuous with respect to the variable x is based on the theory of completely continuous vector field presented by M. Krasnosel'skii and P. Zabrejko in [6]. More precisely we will prove that the equation (1.3) has nontrivial solution, which is positive continuous ω-periodic with respect to the variable t and positive continuous with respect to the variable x after we use the following theorem which is extracted from [6].

THEOREM 3.1. [6] *Let Y be a real Banach space with a cone Q and $L : Y \longrightarrow Y$ be a completely continuous and positive with respect to Q. Then the following propositions are valid.*

i) Let $L(0) = 0$. Let also for every sufficiently small $r > 0$ there is no $y \in Q$, $||y||_Y = r$, with $y \overset{\circ}{\leqslant} L(y)$. Then there exists $ind(0, L; Q) = 1$.

ii) Let for every sufficiently large $R > 0$ there is no $y \in Q$ with $||y||_Y = R$ and $y \overset{\circ}{\geqslant} L(y)$. Then there exists $ind(\infty, L; Q) = 0$.

iii) Let $L(0) = 0$ and let there exist $ind(0, L; Q) \neq ind(\infty, L; Q)$. Then L has nontrivial fixed point in Q.

Here $ind(\cdot, L; Q)$ denotes an index of a point with respect to L and Q. The sign $\overset{\circ}{\leqslant}$ denotes the semiordering generated by Q.

THEOREM 3.2. *Let $n \geqslant 2$ be fixed and $\omega > 0$ be fixed; the constants $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, $m_3 > 0$ are taken so that the conditions (1.4) hold. Let $f(t, x, u, u_t, u_x) \in \mathscr{C}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$ be ω-periodic function with respect to the time variable t, $0 \leqslant f(t, x, u, u_t, u_x) \leqslant M_2$ for every $(t, x, u, u_t, u_x) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, $f(t, x, 0, u_t, u_x) = 0$ for every $(t, x, u_t, u_x) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $g(x)$, $g_j(x)$, $j = 0, 1, \cdots, n$ be fixed functions for which $g(x) \neq 0$, $g_j(x) \neq 0$, $j = 0, 1, \cdots, n$ for every $x \in \mathbb{R}^n$, $g(x) \in \mathscr{C}^2(\mathbb{R}^n)$, $g_j(x) \in \mathscr{C}^1(\mathbb{R}^n)$, $j = 0, 1, \cdots, n$, and the conditions (1.5) hold. Let also $a(t)$, $a_j(t)$ be fixed positive continuous ω-periodic functions for which $m_3 \leqslant a(t) \leqslant M_3$, $m_3 \leqslant a_j(t) \leqslant M_3$, $j = 0, 1, \cdots, n$, for every $t \in \mathbb{R}$. Then the system (1.3) has a nontrivial solution in the form (1.6) which is positive continuous ω-periodic with respect to the time variable t and positive continuous with respect to the variable x.*

Proof. First we note that $\mathscr{G}(0) = 0$. Let $x \in \mathbb{R}^n$ be fixed. From the conditions

(1.4) we have

$$\frac{P^-}{{P^+}^2(M_3+1)\omega}>1, \qquad \frac{P^-}{{P^+}^2(M_3+M_1)\omega}>1,$$
$$\frac{P^-}{{P^+}^2(M_3+nM_1+M_2)\omega}>1, \qquad P^-m_3\omega>1. \tag{3.1}$$

Let us suppose that for every sufficiently small $r>0$ there is $\bar{u}\in\mathscr{C}_+^\circ(\omega)$ such that $||\bar{u}||=r$, $\bar{u}(t,x)\leqslant\mathscr{G}(\bar{u})(t,x)$ for every $t\in[0,\omega]$.

We suppose that $r>0$ is sufficiently small so that $f(t,x,u,u_t,u_x)\leqslant M_2r$ (this is possible because $f(t,x,0,u_t,u_x)=0$ and f is a continuous function).

If $\max_{t\in[0,\omega]}u(t,x)=r$ then, after we use (2.4),

$$u(t,x)\leqslant P^+(M_3+1)r\omega. \tag{3.2}$$

From the definition of the cone $\mathscr{C}_+^\circ(\omega)$ we have that

$$\min_{t\in[0,\omega]}u(t,x)\geqslant\frac{P^-}{P^+}\max_{t\in[0,\omega]}u(t,x)=\frac{P^-}{P^+}r.$$

From here and from (3.2) we obtain

$$\frac{P^-}{P^+}r\leqslant P^+(M_3+1)r\omega,$$

from where

$$\frac{P^-}{{P^+}^2(M_3+1)\omega}\leqslant 1,$$

which is a contradiction with (3.1).

If there exists $j\in\{1,2,\cdots,n\}$ such that $\max_{t\in[0,\omega]}u_j(t,x)=r$ then we have, after we use (2.5),

$$u_j(t,x)\leqslant P^+(M_3+M_1)r\omega,$$

from here

$$\max_{t\in[0,\omega]}u_j(t,x)\leqslant P^+(M_3+M_1)r\omega. \tag{3.3}$$

Also, from the definition of the cone $\mathscr{C}_+^\circ(\omega)$ we have

$$u_j(t,x)\geqslant\frac{P^-}{P^+}r.$$

From here and from (3.3) we obtain

$$\frac{P^-}{P^+}r\leqslant P^+(M_3+M_1)r\omega$$

or

$$\frac{P^-}{{P^+}^2(M_3+M_1)\omega}\leqslant 1$$

which is a contradiction with (3.1).

If $\max_{t\in[0,\omega]} u_0(t,x) = r$ then we have, after we use (2.6),

$$u_0(t,x) \leqslant P^+\omega(M_3r + nM_1r + M_2r).$$

From the last inequalities we get

$$\max_{t\in[0,\omega]} u_0(t,x) \leqslant P^+\omega(M_3 + nM_1 + M_2)r. \tag{3.4}$$

From the definition of the cone $\mathscr{C}_+^\circ(\omega)$ we have

$$\min_{t\in[0,\omega]} u_0(t,x) \geqslant \frac{P^-}{P^+}r.$$

From here and from (3.4) we get

$$\frac{P^-}{P^+}r \leqslant P^+\omega(M_3 + nM_1 + M_2)r$$

or

$$\frac{P^-}{{P^+}^2\omega(M_3 + nM_1 + M_2)} \leqslant 1$$

which is a contradiction with (3.1).

Therefore from the Theorem 3.1 i) we conclude that there exists $ind(0,\mathscr{G};\mathscr{C}_+^\circ(\omega))$ and

$$ind(0,\mathscr{G};\mathscr{C}_+^\circ(\omega)) = 1. \tag{3.5}$$

Let $R > 0$ be sufficiently large. We suppose that there exists $\bar{u}(t,x) \in \mathscr{C}_+^\circ(\omega)$ such that $||\bar{u}|| = R$, $\bar{u}(t,x) \geqslant \mathscr{G}(\bar{u})(t,x)$.

If $\int_0^\omega u(t,x)dt > 0$ then

$$\begin{aligned}
u(t,x) &\geqslant \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \int_0^\omega e^{\int_t^{t+s} a(\tau)d\tau}[a(t+s)u(t+s,x) + u_0(t+s,x)]ds \\
&\geqslant \frac{e^{-[a]\omega}}{1-e^{-[a]\omega}} \int_0^\omega e^{\int_t^{t+s} a(\tau)d\tau} a(t+s)u(t+s,x)ds \\
&\geqslant P^-m_3 \int_0^\omega u(s,x)ds,
\end{aligned}$$

from here

$$\int_0^\omega u(t,x)dt \geqslant P^-m_3\omega \int_0^\omega u(t,x)dt,$$

which is a contradiction with (3.1).

If there exists $j \in \{1,2,\cdots,n\}$ such that $\int_0^\omega u_j(t,x)dt > 0$ we have

$$\begin{aligned}
u_j(t,x) &\geqslant \frac{e^{-[a_j]\omega}}{1-e^{-[a_j]\omega}} \int_0^\omega e^{\int_t^{t+s} a_j(\tau)d\tau} a_j(t+s)u_j(t+s,x)ds \\
&\geqslant P^-m_3 \int_0^\omega u_j(s,x)ds,
\end{aligned}$$

from here

$$\int_0^\omega u_j(t,x)dt \geqslant P^- m_3 \omega \int_0^\omega u_j(s,x)ds,$$

which is a contradiction with (3.1).

If $\int_0^\omega u_0(t,x)dt > 0$ then

$$\begin{aligned} u_0(t,x) &\geqslant \frac{e^{-[a_0]\omega}}{1-e^{-[a_0]\omega}} \int_0^\omega e^{\int_t^{t+s} a_0(\tau)d\tau} a_0(t+s)u_0(t+s,x)ds \\ &\geqslant P^- m_3 \int_0^\omega u_0(t,x)dt, \end{aligned}$$

from here

$$\int_0^\omega u_0(t,x)dt \geqslant P^- m_3 \omega \int_0^\omega u_0(t,x)dt,$$

which is a contradiction with (3.1).

Consequently, from Theorem 3.1 ii) we conclude that there exists $ind(\infty,\mathscr{G};\mathscr{C}_+^\circ(\omega))$ and

$$ind(\infty,\mathscr{G};\mathscr{C}_+^\circ(\omega)) = 0.$$

From here and from (3.5) and Theorem 3.1 iii) we conclude that the operator $\mathscr{G}$ has a nontrivial fixed point in $\mathscr{C}_+^\circ(\omega)$. From Lemma 2.1 follows that the system (1.3) has a nontrivial solution $\bar{u}(t,x)$ which is positive continuous ω-periodic with respect to the time variable t and positive continuous with respect to the variable x. □

4. Uniqueness of the positive periodic solutions

Here we use the following theorem.

THEOREM 4.1. [6] *Let Q be a cone in the Banach space Y and the operator $A : Y \longrightarrow Y$ be k_0-monotonous ($k_0 \in Q$). Then the equation $x = Ax$ has in the cone Q no more than one nontrivial solution.*

When we say that the operator $A : Y \longrightarrow Y$, where Y is a Banach space with a cone Q, is k_0*-monotonous* ($k_0 \in Q$) we understand: *if for every $x \in Q$, $x \neq 0$ we have*

$$\alpha(x)k_0 \leqslant Ax \leqslant \beta(x)k_0, \tag{4.1}$$

where $\alpha(x) > 0$, $\beta(x) > 0$;

$$A(\lambda x) \geqslant \lambda Ax, \quad \lambda \in [0,1], \quad x \in Q; \tag{4.2}$$

$$A(\lambda x) \neq \lambda Ax \tag{4.3}$$

for $0 < \lambda < 1$ and for every $x \geqslant \gamma(x)k_0$ ($\gamma(x) > 0$);

$$Ax \geqslant Ay + \varepsilon_0 k_0 \tag{4.4}$$

for $x \geqslant y$, $x \neq y$, where $\varepsilon_0 = \varepsilon_0(x,y) > 0$.

THEOREM 4.2. *Let $n \geqslant 2$ be fixed and $\omega > 0$ be fixed; the constants $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, $m_3 > 0$ are taken so that the condition (1.4) hold. Let $f(t,x,u,u_t,u_x) \in \mathscr{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n)$ be ω - periodic function with respect to the time variable t, $f(t,x,u_1,u_{1t},u_{1x}) \geqslant f(t,x,u_2,u_{2t},u_{2x})$ for every $t \in [0,\omega]$, $x \in \mathbb{R}^n$, $u_1 \geqslant u_2$, $0 \leqslant f(t,x,u,u_t,u_x) \leqslant M_2$, $f(t,x,\lambda u,\lambda u_t,\lambda u_x) \geqslant \sqrt{\lambda} f(t,x,u,u_t,u_x)$ for every $(t,x,u,u_t,u_x) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ and $\lambda \in [0,1]$, $f(t,x,0,u_t,u_x) = 0$ for every $(t,x,u_t,u_x) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, $g(x)$, $g_j(x)$, $j = 0,1,\cdots,n$ be fixed functions for which $g(x) \neq 0$, $g_j(x) \neq 0$, $j = 0,1,\cdots,n$ for every $x \in \mathbb{R}^n$, $g(x) \in \mathscr{C}^2(\mathbb{R}^n)$, $g_j(x) \in \mathscr{C}^1(\mathbb{R}^n)$, $j = 0,1,\cdots,n$, and the conditions (1.5) hold. Let also $a(t)$, $a_j(t)$ be fixed positive continuous ω-periodic functions for which $m_3 \leqslant a(t) \leqslant M_3$, $m_3 \leqslant a_j(t) \leqslant M_3$, $j = 0,1,\cdots,n$ for every $t \in \mathbb{R}$. Then the system (1.3) has exactly one nontrivial solution in the form (1.6) which is positive continuous ω-periodic with respect to the time variable t and positive continuous with respect to the variable x.*

Proof. From Theorem 3.2 we have that the system (1.3) has a solution in the form (1.6) which is continuous positive ω-periodic solution with respect to the time variable t and positive continuous with respect to the variable x. Let $k_0 = 1$. Evidently $\chi(\bar{u})$, $\chi_j(\bar{u})$, $j = 1,2,\cdots,n$, satisfy the conditions (4.1), (4.2), (4.3), (4.4). We note that $\chi(\lambda\bar{u}) = \lambda\chi(\bar{u})$, $\chi_j(\lambda\bar{u}) = \lambda\chi_j(\bar{u})$, $j = 1,2,\cdots,n$. Also, for $\lambda \in [0,1]$, $\bar{u} \in \mathscr{C}_+^\circ(\omega)$ we have

$$
\begin{aligned}
\chi_0(\lambda\bar{u}) = \frac{e^{-[a_0]\omega}}{1-e^{-[a_0]\omega}} \int_0^\omega e^{\int_t^{t+s} a_0(\tau)d\tau} \Big[& \lambda a_0(t+s)u_0(t+s,x) \\
& + \lambda \sum_{j=1}^n \frac{1}{g_j(x)} \frac{\partial g_j(x)}{\partial x_j} u_j(t+s,x) \\
& + f\big(t+s,x,\lambda u(t+s,x), \lambda \frac{v'(t+s)}{v(t+s)} u(t+s,x), \lambda \frac{\partial g}{\partial x_1}\frac{1}{g(x)} u(t+s,x), \\
& \quad \cdots, \lambda \frac{\partial g}{\partial x_n}\frac{1}{g(x)} u(t+s,x)\big)\Big] ds \\
\geqslant \frac{e^{-[a_0]\omega}}{1-e^{-[a_0]\omega}} \int_0^\omega e^{\int_t^{t+s} a_0(\tau)d\tau} \Big[& \lambda a_0(t+s)u_0(t+s,x) \\
& + \lambda \sum_{j=1}^n \frac{1}{g_j(x)} \frac{\partial g_j(x)}{\partial x_j} u_j(t+s,x) \\
& + \sqrt{\lambda} f\big(t+s,x,u(t+s,x), \frac{v'(t+s)}{v(t+s)} u(t+s,x), \frac{\partial g}{\partial x_1}\frac{1}{g(x)} u(t+s,x), \\
& \quad \cdots, \frac{\partial g}{\partial x_n}\frac{1}{g(x)} u(t+s,x)\big)\Big] ds \\
\geqslant \lambda \frac{e^{-[a_0]\omega}}{1-e^{-[a_0]\omega}} \int_0^\omega e^{\int_t^{t+s} a_0(\tau)d\tau} \Big[& a_0(t+s)u_0(t+s,x)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\frac{1}{g_j(x)}\frac{\partial g_j(x)}{\partial x_j}u_j(t+s,x)\\
&+f\Big(t+s,x,u(t+s,x),\frac{v'(t+s)}{v(t+s)}u(t+s,x),\frac{\partial g}{\partial x_1}\frac{1}{g(x)}u(t+s,x),\\
&\quad\cdots,\frac{\partial g}{\partial x_n}\frac{1}{g(x)}u(t+s,x)\Big)\Big]ds\\
=\lambda\chi_0(\bar u),&\qquad \chi_0(\lambda\bar u)\neq\lambda\chi_0(\bar u)\quad\text{for}\quad\lambda\in(0,1).
\end{aligned}
$$

From the last inequality we conclude that $\mathcal{G}(\lambda\bar u)\neq\lambda\mathcal{G}(\bar u)$ for every $\lambda\in(0,1)$. From the condition $f(t,x,u_1,u_{1t},u_{1x})\geqslant f(t,x,u_2,u_{2t},u_{2x})$ if $u_1\geqslant u_2$, follows that $\chi_0(\bar u_1)\geqslant\chi_0(\bar u_2)$ if $\bar u_1\geqslant\bar u_2$. Also, $\chi_0(\bar u_1)\geqslant\chi_0(\bar u_2)+\varepsilon_0$ if $\bar u_1\geqslant\bar u_2$, $\bar u_1\neq\bar u_2$. Consequently the operator $\mathcal{G}$ is 1- monotonous operator. From here and from Theorem 3.2, Theorem 4.1 follows that the system (1.3) has exactly one solution in the form (1.6) which is positive continous ω-periodic solution with respect to the time variable t and positive continuous with respect to the variable x. □

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(Received July 23, 2008)

Department of Differential Equations
Faculty of Mathematics and Informatics
University of Sofia
Bulgaria
e-mail: `sgg2000bg@yahoo.com`