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ATTRACTIVITY AND POSITIVITY RESULTS FOR NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS VIA MEASURE OF NONCOMPACTNESS

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Abstract. Using the techniques of some new measures of noncompactness we prove in this paper some existence theorems concerning the global attractivity and ultimate positivity of the solutions for a nonlinear functional integral equation. Our investigations are placed in the Banach space of real-valued functions defined, continuous and bounded on unbounded intervals together with the applications of a recent measure theoretic fixed point theorem of Dhage [7]. On one hand, our results generalize the attractivity results of Dhage [9] with a different method and the results of Banas and Rzepka [4] and Banas and Dhage [5] with similar method but under weaker conditions and on the other hand they are new to the literature as regards ultimate positivity of the solutions for nonlinear functional integral equations. A few realizations of the obtained results are also indicated.

1. Introduction

Nonlinear integral equations with bounded intervals have been studied extensively in the literature as regard various aspects of the solutions. This includes existence, uniqueness, stability and extremality of solutions. But the study of nonlinear integral equations with unbounded intervals is relatively new and exploited for the new characteristics of attractivity and asymptotic attractivity of solutions. There are two approaches for dealing with these characteristics of solutions, namely, classical fixed point theorems involving the hypotheses from analysis and topology and the fixed point theorems involving the use of measure of noncompactness. Each one of these approaches has some advantages and disadvantages over the others (cf. Dhage [8, 9]).

In this paper we are going to prove some theorems on the existence and global attractivity and positivity of solutions for a functional integral equation by using fixed point theorem involving the use of measures of noncompactness. Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on the right hand real half axis $\mathbb{R}_+$. The mentioned equation has rather general form and contains as particular cases a lot of functional equations and nonlinear integral

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equations of Volterra type. The main tool used in our considerations is the technique of measures of noncompactness and a fixed point theorem of Dhage [7].

The measures of noncompactness used in the paper allows us not only to obtain the existence of solutions of the mentioned functional integral equation but also to characterize those solutions in terms of global attractivity and positivity on unbounded intervals. This assertion means that all possible solutions of the functional integral equation in question are globally uniformly attractive and positive in the sense which will be defined further on. The assumptions imposed in our main existence theorems admit several natural realizations. These realizations are constructed with help of a certain class of subadditive functions.

The results obtained in this paper generalize and extend several ones obtained earlier in a lot of papers concerning asymptotic stability of solutions for some functional integral equations (cf. [4,5,6,10,13]). It is worthwhile mentioning that the novelty of our approach consists mainly in the possibility of obtaining the global attractivity, asymptotic attractivity and positivity of solutions for considered functional integral equations.

2. Notations, definitions and auxiliary results

This section is devoted to collect some definitions and auxiliary results which will be needed in the further considerations of this paper. At the beginning we present some basic facts concerning the measures of noncompactness. We accept the following definition of the concept of a measure of noncompactness given in Dhage [7]. The details of the different types of measures of noncompatness appear in Akhmerov et al. [1], Appell [2], Banas and Goebel [3], and the references given therein.

Let E be a Banach space, $\mathscr{P}(E)$, a class of subsets of E and let $\mathscr{P}_p(E)$ denote the class of all non-empty subsets of E with property p. Here p may be $p =$ closed (in short cl), $p =$ bounded (in short bd), $p =$ relatively compact (in short rcp) etc. Thus, $\mathscr{P}_{cl}(E), \mathscr{P}_{bd}(E), \mathscr{P}_{cl,bd}(E)$ and $\mathscr{P}_{rcp}(E)$ denote respectively the classes of closed, bounded, closed and bounded and relatively compact subsets of E. A function $d_H : \mathscr{P}(E) \times \mathscr{P}(E) \to \mathbb{R}^+$ defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A} D(a,B)\ ,\ \sup_{b\in B} D(b,A)\right\} \tag{2.1}$$

satisfies all the conditions of a metric on $\mathscr{P}(E)$ and is called a Hausdorff-Pompeiu metric on E, where $D(a,B) = \inf\{\|a-b\| : b \in B\}$. It is known that the hyperspace $(\mathscr{P}_{cl}(E), d_H)$ is a complete metric space.

The axiomatic way of defining the measures of noncompactness has been adopted in several papers in the literature. See Akhmerov et al. [1], Appell [2], Banas and Goebel [3], Väth [12] and the references therein. In this paper, we adopt the following axiomatic definition of the measure of noncompactness in a Banach space given by Dhage [7]. The other useful forms appear in Banas and Goebel [3] and the references therein. We need the following definitions in the sequel.

DEFINITION 2.1. A sequence $\{A_n\}$ of non-empty sets in $\mathscr{P}_p(E)$ is said to converge to a set A, called the *limiting set* if $d_H(A_n,A) \to 0$ as $n \to \infty$. A mapping $\mu : \mathscr{P}_p(E) \to \mathbb{R}^+$ is called continuous if for any sequence $\{A_n\}$ in $\mathscr{P}_p(E)$ we have that

$$d_H(A_n,A) \to 0 \Longrightarrow |\mu(A_n) - \mu(A)| \to 0 \quad \text{as} \quad n \to \infty.$$

DEFINITION 2.2. A mapping $\mu : \mathscr{P}_p(E) \to \mathbb{R}^+$ is called nondecreasing if $A,B \in \mathscr{P}_p(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$, where $\subseteq$ is a order relation by inclusion in $\mathscr{P}_p(E)$.

Now we are equipped with the necessary details to define the measures of noncompactness for a bounded subset of the Banach space E.

DEFINITION 2.3. A function $\mu : \mathscr{P}_{bd}(E) \to \mathbb{R}^+$ is called a *measure of noncompactness* if it satisfies:
1^o $\emptyset \neq \mu^{-1}(0) \subset \mathscr{P}_{rcp}(E)$,
2^o $\mu(A) = \mu(\overline{A})$, where $\overline{A}$ is the closure of A,
3^o $\mu(A) = \mu(\mathrm{Conv}(A))$, where $\mathrm{Conv}(A)$ is the convex hull of A,
4^o μ is nondecreasing, and
5^o if $\{A_n\}$ is a decreasing sequence of sets in $\mathscr{P}_{bd}(E)$ such that $\lim_{n\to\infty} \mu(A_n) = 0$, then the limiting set $A_\infty = \lim_{n\to\infty} \overline{A}_n = \cap_{n=0}^{\infty} \overline{A}_n$ is non-empty.

The family $\ker \mu$ described in 1^o is said to be the *kernel of the measure of noncompactness* μ and

$$\ker \mu = \{A \in \mathscr{P}_{bd}(E) \mid \mu(A) = 0\} \subset \mathscr{P}_{rcp}(E).$$

A measure μ is called *complete* or *full* if the kernel $\ker \mu$ of μ consists of all possible relatively compact subsets of E. Next, a measure μ is called *sublinear* if it satisfies:
6^o $\mu(\lambda A) = |\lambda| \mu(A)$ for $\lambda \in \mathbb{R}$, and
7^o $\mu(A+B) \leqslant \mu(A) + \mu(B)$.

There do exist the sublinear measures of noncompactness on Banach spaces E. Indeed, the Kuratowskii and Hausdorff measures of noncompactness are sublinear in E. A good collection of different types of measures of noncompactness appears in Appell [2].

Observe that the limiting set A_∞ from 6^o is a member of the family $\ker \mu$. In fact, since $\mu(A_\infty) \leqslant \mu(\overline{A}_n) = \mu(A_n)$ for any n, we infer that $\mu(A_\infty) = 0$. This yields that $A_\infty \in \ker \mu$. This simple observation will be essential in our further investigations.

Now we state a key fixed point theorem of Dhage [7] which will be used in the sequel. Before stating this fixed point result, we give a useful definition.

DEFINITION 2.4. A mapping $Q : E \to E$ is called $\mathscr{D}$*-set-Lipschitz* if there exists a continuous nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \leqslant \phi(\mu(A))$ for all $A \in \mathscr{P}_{bd}(E)$ with $Q(A) \in \mathscr{P}_{bd}(E)$, where $\phi(0) = 0$. Sometimes we call the function

ϕ to be a $\mathscr{D}$*-function* of Q on E. In the special case, when $\phi(r) = kr, k > 0$, Q is called a k*-set-Lipschitz* mapping and if $k < 1$, then Q is called a k*-set-contraction* on E. Further, if $\phi(r) < r$ for $r > 0$, then Q is called a *nonlinear* $\mathscr{D}$*-set-contraction* on E.

THEOREM 2.1. (Dhage [7]) *Let* C *be a non-empty, closed, convex and bounded subset of a Banach space* E *and let* $Q : C \to C$ *be a continuous and nonlinear* $\mathscr{D}$*-set-contraction. Then* Q *has a fixed point.*

REMARK 2.1. Let us denote by $\text{Fix}(Q)$ the set of all fixed points of the operator Q which belong to C. It can be shown that the set $\text{Fix}(Q)$ existing in Theorem 2.1 belongs to the family $\ker \mu$. In fact if $\text{Fix}(Q) \notin \ker \mu$, then $\mu(\text{Fix}(Q)) > 0$ and $Q(\text{Fix}(Q)) = \text{Fix}(Q)$. Now from nonlinear $\mathscr{D}$-set-contraction it follows that $\mu(Q(\text{Fix}(Q))) \leqslant \phi(\mu(\text{Fix}(Q)))$ which is a contradiction since $\phi(r) < r$ for $r > 0$. Hence $\text{Fix}\,(Q) \in \ker \mu$.

Our further considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$||x|| = \sup\{|x(t)| : t \in \mathbb{R}_+\} .$$

For our purposes we will use the Hausdorff or ball measure of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$. A handy formula for Hausdorff measure of noncompactness useful in application is defined as follows. Let us fix a nonempty and bounded subset X of the space $BC(\mathbb{R}_+, \mathbb{R})$ and a positive number T. For $x \in X$ and $\varepsilon \geqslant 0$ denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the closed and bounded interval $[0, T]$ defined by

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leqslant \varepsilon\} .$$

Next, let us put

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\} ,$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon) .$$

It is known that ω_0^T is a measure of noncompactness in the Banach space $C([0.T], \mathbb{R})$ of continuous and real-valued functions defined on a closed and bounded interval $[0, T]$ in $\mathbb{R}$ which is equivalent to Hausdorff or ball measure χ of noncompactness in it. In fact, one has

$$\chi(X) = \frac{1}{2}\omega_0^T(X)$$

for any bounded subset X of $C([0, T], \mathbb{R})$ (See Banas and Goebel [3] and the references therein). Finally, we define

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X) .$$

Now, for a fixed number $t \in \mathbb{R}_+$ let us denote

$$X(t) = \{x(t) : x \in X\},$$

$$\|X(t)\| = \sup\{|x(t| \ : \ x \in X\} \ ,$$

and

$$\|X(t) - c\| = \sup\{|x(t) - c| : \ x \in X\} \ .$$

Finally, let us consider the functions $\mu' s$ defined on the family $\mathscr{P}_{cl,bd}(X)$ by the formulas

$$\mu_a(X) = \max\left\{\omega_0(X), \limsup_{t\to\infty} \operatorname{diam} X(t)\right\} \ , \tag{2.2}$$

$$\mu_b(X) = \max\left\{\omega_0(X), \limsup_{t\to\infty} \|X(t)\|\right\} \ , \tag{2.3}$$

and

$$\mu_c(X) = \max\left\{\omega_0(X), \limsup_{t\to\infty} \|X(t) - c\|\right\} \ . \tag{2.4}$$

Let $T > 0$ be fixed. Then for any $x \in BC(\mathbb{R}_+, \mathbb{R})$ define

$$\delta_T(x) = \sup\{\big|\,|x(t)| - x(t)\big| \ : \ t \geqslant T\}.$$

Similarly, for any bounded subset X of $BC(\mathbb{R}_+, \mathbb{R})$ define

$$\delta_T(X) = \sup\{\delta_T(x) \ : \ x \in X\},$$

and

$$\delta(X) = \lim_{T\to\infty} \delta_T(X).$$

Define the functions $\mu_{ad}, \mu_{bd}, \mu_{cd} \ : \ \mathscr{P}_{bd}(E) \to \mathbb{R}_+$ by by the formulas

$$\mu_{ad}(X) = \max\{\mu_a(X), \delta(X)\}, \tag{2.5}$$

$$\mu_{bd}(X) = \max\{\mu_b(X), \delta(X)\}, \tag{2.6}$$

and

$$\mu_{cd}(X) = \max\{\mu_c(X), \delta(X)\} \tag{2.7}$$

for all $X \in \mathscr{P}_{cl,bd}(E)$.

REMARK 2.2. It can be shown as in Banas and Goebel [3] that the functions μ_a, μ_b, μ_c, μ_{ad}, μ_{bd} and μ_{cd} are measures of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. The kernels $\ker \mu_a$, $\ker \mu_b$ and $\ker \mu_c$ of the measures μ_a, μ_b and μ_c consist of nonempty and bounded subsets X of $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity. Moreover, the functions from $\ker \mu_c$ come closer along a line $y(t) = c$ and the functions from $\ker \mu_b$ come closer along the line $y(t) = 0$ as t increases to ∞ through $\mathbb{R}_+$. A similar situation is also true for the kernels $\ker \mu_{ad}$, $\ker \mu_{bd}$, and $\ker \mu_{cd}$ of the measures of noncompactness μ_{ad}, μ_{bd} and μ_{cd}. Moreover, these measures μ_{ad}, μ_{bd} and μ_{cd} characterize the ultimate positivity of the functions belonging to the kernels of $\ker \mu_{ad}$, $\ker \mu_{bd}$, and $\ker \mu_{cd}$.

The above expressed property of ker μ_a, ker μ_b, ker μ_c and ker μ_{ad}, ker μ_{bd}, ker μ_{cd} permits us to characterize solutions of the integral equations considered in the sequel.

In order to introduce further concepts used in this paper, let us assume that $E = BC(\mathbb{R}_+,\mathbb{R})$ and let Ω be a subset of X. Let $Q : E \to E$ be an operator and consider the following operator equation in E,

$$Qx(t) = x(t) \tag{2.8}$$

for all $t \in \mathbb{R}_+$. Below we give different characterizations of the solutions for the operator equation (2.8) on $\mathbb{R}_+$.

DEFINITION 2.5. We say that solutions of the equation (2.8) are *locally attractive* if there exists a closed ball $\overline{\mathscr{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2.8) belonging to $\overline{\mathscr{B}}_r(x_0) \cap \Omega$ we have that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \tag{2.9}$$

In the case when the limit (2.9) is uniform with respect to the set $B(x_0,r) \cap \Omega$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leqslant \varepsilon \tag{2.10}$$

for all $x,y \in \overline{\mathscr{B}}_r(x_0) \cap \Omega$ being solutions of (2.1) and for $t \geqslant T$, we will say that solutions of equation (2.8) are *uniformly locally attractive* on $\mathbb{R}_+$.

DEFINITION 2.6. The solution $x = x(t)$ of equation (2.8) is said to be *globally attractive* if (2.9) holds for each solution $y = y(t)$ of (2.8) on Ω. In other words, we may say that solutions of the equation (2.8) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (2.8) on Ω, the condition (2.9) is satisfied. In the case when the condition (2.9) is satisfied uniformly with respect to the set Ω, i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (2.10) is satisfied for all $x,y \in \Omega$ being the solutions of (2.8) and for $t \geqslant T$, we will say that solutions of the equation (2.8) are *uniformly globally attractive* on $\mathbb{R}_+$.

The following definitions appear in Dhage [8].

DEFINITION 2.7. A line $y(t) = c$, where c a real number, is called an *attractor* for a solution $x \in BC(\mathbb{R}_+,\mathbb{R})$ to the equation (2.8) if $\lim_{t\to\infty}[x(t) - c] = 0$. In this case the solution x to the equation (2.8) is also called to be asymptotic to the line $y(t) = c$ and the line is an asymptote for the solution x on $\mathbb{R}_+$.

Now we introduce the following definition useful in the sequel.

DEFINITION 2.8. The solutions of the equation (2.8) are said to be *globally asymptotically attractive* if for any two solutions $x = x(t)$ and $y = y(t)$ of the equation (2.8), the condition (2.9) is satisfied and there is a line which is a common attractor to them on $\mathbb{R}_+$. In the case when condition (2.9) is satisfied uniformly, that is, if for every $\varepsilon > 0$

there exists $T > 0$ such that the inequality (2.10) is satisfied for $t \geqslant T$ and for all x, y being the solutions of (2.8) and having a line as a common attractor, we will say that solutions of the equation (2.8) are *uniformly globally asymptotically attractive* on $\mathbb{R}_+$.

REMARK 2.3. Let us mention that the concepts of global attractivity of solutions are recently introduced in Hu and Yan [11] while the concepts of local and global asymptotic attractivity have been presented in Dhage [8]. Similarly, the concepts of uniform local and global attractivity (in the above sense) were introduced in Banas and Rzepka [4].

Next, we introduce the new concept of local and global asymptotic positivity of the solutions for the operator equation (2.8) in $BC(\mathbb{R}_+, \mathbb{R})$.

DEFINITION 2.9. A solution x of the equation (2.8) is called *locally ultimately positive* if there exists a closed ball $\overline{\mathscr{B}}_r(x_0)$ in $BC(\mathbb{R}_+, \mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+, \mathbb{R})$ such that $x \in \overline{\mathscr{B}}_r(x_0)$ and

$$\lim_{t \to \infty} \big[|x(t)| - x(t) \big] = 0. \tag{2.11}$$

In the case when the limit (2.11) is uniform with respect to the solution set of the operator equation (2.8), i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\big| |x(t)| - x(t) \big| \leqslant \varepsilon \tag{2.12}$$

for all x being solutions of (2.8) and for $t \geqslant T$, we will say that solutions of equation (2.8) are *uniformly locally ultimately positive* on $\mathbb{R}_+$.

DEFINITION 2.10. A solution $x \in C(\mathbb{R}_+, \mathbb{R})$ of the equation (2.8) is called *globally ultimately positive* if (2.11) is satisfied. In the case when the limit (2.11) is uniform with respect to the solution set of the operator equation (2.8) in $C(\mathbb{R}_+, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (2.12) is satisfied for all x being solutions of (2.8) and for $t \geqslant T$, we will say that solutions of equation (2.8) are *uniformly globally ultimately positive* on $\mathbb{R}_+$.

REMARK 2.4. We note that the global attractivity and global asymptotic attractivity implies respectively the local attractivity and local asymptotic attractivity of the solutions for the operator equation (2.8) on $\mathbb{R}_+$. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (2.8) on unbounded intervals. However, the converse of the above two statements may not be true. A few details of ultimate positivity are given Dhage [10].

In the following section we prove the main results of this paper.

3. Attractivity and positivity results

In this section we will investigate the following functional integral equation (in short FIE)

$$x(t) = q(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), x(\gamma_2(s)))\, ds\,, \tag{3.1}$$

for all $t \in \mathbb{R}_+$, where $q : \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha_1, \alpha_1, \beta, \gamma_1, \gamma_2 \,:\, \mathbb{R}_+ \to \mathbb{R}_+$.

By a *solution* of the FIE (3.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (3.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions defined on $\mathbb{R}_+$.

When $\alpha_1(t) = t = \gamma_1(t)$ for $t \in \mathbb{R}_+$, the FIE (3.1) reduces to the functional integral equation

$$x(t) = q(t) + f(t, x(t), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t, s, x(s), x(\gamma_2(s)))\, ds\,, \tag{3.2}$$

for $t \in \mathbb{R}_+$. The integral equation (3.2) has been studied in Dhage [9] for the global attractivity and global asymptotic attractivity of solutions via classical hybrid fixed point theory due to the present author (see Dhage [9] and the references given therein). Observe that the above integral equation (3.2) includes several classes of functional, integral and functional integral equations considered in the literature (cf. [4,5,6,10,13] and references therein). Let us also mention that the following functional integral equation considered in Banas and Dhage [5],

$$x(t) = f(t, x(\alpha(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\,, \tag{3.3}$$

is also a spacial case of the equation (3.2) which further includes the functional integral equation considered in Banas and Rzepka [4], where $\alpha(t) = \beta(t) = \gamma(t) = t$, $t \in \mathbb{R}_+$. Therefore, our FIE (3.1) is more general and so, the attractivity and positivity results of this paper include the attractivity and positivity results for all the above mentioned integral equations which are also new to the literature.

The equation (3.1) will be considered under the following assumptions.

(H_0) The functions $\alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy $t \leqslant \alpha_1(t)$, $t \leqslant \alpha_2(t)$ for all $t \in \mathbb{R}_+$.

(H_1) The function $q : \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded.

(H_2) The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a bounded function $\ell : \mathbb{R}_+ \to \mathbb{R}$ with bound L and a positive constant M such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leqslant \frac{\ell(t) \max\{|x_1 - y_1|, |x_2 - y_2|\}}{M + \max\{|x_1 - y_1|, |x_2 - y_2|\}}$$

for $t \in \mathbb{R}_+$ and for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, we assume that $L \leqslant M$.

(H_3) The function $t \to f(t,0,0)$ is bounded on $\mathbb{R}_+$ with

$$F_0 = \sup\{|f(t,0,0)| : \; t \in \mathbb{R}_+\}.$$

(H_4) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $b : \; \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,s,x,y)| \leqslant b(t,s)$$

for $t,s \in \mathbb{R}_+$. Moreover, we assume that

$$\lim_{t\to\infty} \int_0^{\beta(t)} b(t,s)\,ds = 0 \,.$$

REMARK 3.1. Hypothesis (H_2) is satisfied if in particular, the function f satisfies the condition,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leqslant \frac{\ell(t)\left[|x_1-y_1|+|x_2-y_2|\right]}{2M+\left[|x_1-y_1|+|x_2-y_2|\right]} \tag{3.4}$$

for all $t \in \mathbb{R}_+$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$, where $L \leqslant M$, and the function ℓ is defined as in hypothesis (H_2) which further yields the usual Lipschitz condition on the function f,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leqslant \frac{\ell(t)}{2M}\left[|x_1-y_1|+|x_2-y_2|\right] \tag{3.5}$$

for all $t \in \mathbb{R}_+$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$ provided $L < M$. As mentioned in Dhage [9], our hypothesis (H_2) is more general than that existing in the literature.

Then we can formulate our main results of this paper.

THEOREM 3.1. *Under the above assumptions* (H_0) - (H_4), *the FIE* (3.1) *has at least one solution in the space* $BC(\mathbb{R}_+,\mathbb{R})$. *Moreover, solutions of the equation* (3.1) *are globally uniformly attractive on* $\mathbb{R}_+$.

Proof. Consider the operator Q defined on the space $BC(\mathbb{R}_+,\mathbb{R})$ be the formula

$$Qx(t) = q(t) + f(t,x(\alpha_1(t)),x(\alpha_2(t))) + \int_0^{\beta(t)} g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))\,ds \,. \tag{3.6}$$

Observe that in view of our assumptions, for any function $x \in BC(\mathbb{R}_+,\mathbb{R})$ the function Qx is continuous on $\mathbb{R}_+$. Moreover, for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned}
|(Qx)(t)| &\leqslant |q(t)| + |f(t,x(\alpha_1(t)),x(\alpha_2(t)))| + \int_0^{\beta(t)} |g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))|\,ds \\
&\leqslant \|q\| + |f(t,x(\alpha_1(t)),x(\alpha_2(t))) - f(t,0,0)| + |f(t,0,0)| + \int_0^{\beta(t)} b(t,s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|q\| + \frac{L\max\{|x(\alpha_1(t))|,|x(\alpha_2(t))|\}}{M+\max\{|x(\alpha_1(t))|,|x(\alpha_2(t))|\}} + |f(t,0,0)| + \int_0^{\beta(t)} b(t,s)\,ds \\
&\leqslant \|q\| + \frac{L||x||}{M+||x||} + F_0 + v(t) \leqslant \|q\| + \frac{L||x||}{M+||x||} + F_0 + V \\
&\leqslant \|q\| + L + F_0 + V \ ,
\end{aligned}
$$

where we denoted

$$
v(t) = \int_0^{\beta(t)} b(t,s)\,ds \ , \ \ V = \sup\{v(t):\ t \in \mathbb{R}_+\} \ .
$$

Obviously in view of assumption (H_4), we infer that V is finite.

From the above estimate we deduce that

$$
||Qx|| \leqslant \|q\| + L + F_0 + V \tag{3.7}
$$

for all $x \in BC(\mathbb{R}_+,\mathbb{R})$. This means that the operator Q transforms the space $BC(\mathbb{R}_+,\mathbb{R})$ into itself. More precisely, from (3.7) we obtain that the operator Q transforms continuously the space $BC(\mathbb{R}_+,\mathbb{R})$ into the closed ball $\overline{\mathscr{B}}_r(0)$, where $r = \|q\| + L + F_0 + V$. Because of this fact, the existence of solutions for the FIE (3.1) is global in nature.

In what follows we will consider the operator Q as a mapping from $\overline{\mathscr{B}}_r(0)$ into itself. Now we show that the operator Q is continuous on the ball $\overline{\mathscr{B}}_r(0)$. To do this let us fix arbitrarily $\varepsilon > 0$ and take $x,y \in \overline{\mathscr{B}}_r(0)$ such that $||x-y|| \leqslant \varepsilon$. Then we get:

$$
\begin{aligned}
|(Qx)(t)-(Qy)(t)| &\leqslant |f(t,x(\alpha_1(t)),x(\alpha_2(t))) - f(t,y(\alpha_1(t)),y(\alpha_2(t)))| \\
&\quad + \int_0^{\beta(t)} |g(t,s,x(\gamma_1(s)),x(\gamma_2(s))) - g(t,s,y(s),y(\gamma_2(s)))|\,ds \\
&\leqslant \frac{L\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}}{M+\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}} \\
&\quad + \int_0^{\beta(t)} [|g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))| + |g(t,s,y(s),y(\gamma_2(s)))|]\,ds \\
&\leqslant \frac{L||x-y||}{M+||x-y||} + 2\int_0^{\beta(t)} b(t,s)\,ds \\
&\leqslant \varepsilon + 2v(t).
\end{aligned}
$$

Hence, in virtue of assumption (H_4), we infer that there exists $T>0$ such that $v(t) \leqslant \varepsilon$ for $t \geqslant T$. Thus, for $t \geqslant T$ from above estimate, we derive that

$$
|(Qx)(t)-(Qy)(t)| \leqslant 3\varepsilon \ . \tag{3.8}
$$

Further, let us assume that $t \in [0,T]$. Then, evaluating similarly as above we get:

$$
|(Qx)(t)-(Qy)(t)| \leqslant \varepsilon + \int_0^{\beta(t)} |g(t,s,x(\gamma_1(s)),x(\gamma_2(s))) - g(t,s,y(s),y(\gamma_2(s)))|\,ds
$$

$$\leqslant \varepsilon + \int_0^{\beta(t)} \omega_r^T(g,\varepsilon)\,ds \leqslant \varepsilon + \beta_T \omega_r^T(g,\varepsilon), \tag{3.9}$$

where we denoted

$$\beta_T = \sup\{\beta(t):\ t \in [0,T]\}\ ,$$

and

$$\omega_r^T(g,\varepsilon) = \sup\{|g(t,s,x_1,x_2) - g(t,s,y_1,y_2)| :\ t \in [0,T], s \in [0,\beta_T]$$
$$x_1,x_2,y_1,y_2 \in [-r,r],\ |x_1-y_1| \leqslant \varepsilon,\ \ |x_2-y_2| \leqslant \varepsilon\}\ . \tag{3.10}$$

Obviously we have that $\beta_T < \infty$. Moreover, from the uniform continuity of the function $g(t,s,x,y)$ on the set $[0,T]\times[0,\beta_T]\times[-r,r]\times[-r,r]$ we derive that $\omega_r^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, linking (3.9), (3.10) and the above established facts we conclude that the operator Q maps continuously the closed ball $\overline{\mathscr{B}}_r(0)$ into itself.

Further, let us take a nonempty subset X of the ball $\overline{\mathscr{B}}_r(0)$. Next, fix arbitrarily $T>0$ and $\varepsilon>0$. Let us choose $x \in X$ and $t_1,t_2 \in [0,T]$ with $|t_2-t_1| \leqslant \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, taking into account our assumptions (H_2) and (H_4), we get:

$$|(Qx)(t_2)-(Qx)(t_1)| \tag{3.11}$$
$$\leqslant |q(t_2)-q(t_1)| + |f(t_2,x(\alpha_1(t_2)),x(\alpha_2(t_2))) - f(t_2,x(\alpha_1(t_1)),x(\alpha_2(t_1)))|$$
$$+ |f(t_2,x(\alpha_1(t_1)),x(\alpha_2(t_1)) - f(t_1,x(\alpha_1(t_1)),x(\alpha_2(t_1)))|$$
$$+ \left| \int_0^{\beta(t_2)} g(t_2,s,x(\gamma_1(s)),x(\gamma_2(s)))\,ds - \int_0^{\beta(t_2)} g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))\,ds \right|$$
$$+ \left| \int_0^{\beta(t_2)} g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))\,ds - \int_0^{\beta(t_1)} g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))\,ds \right|$$
$$\leqslant \omega^T(q,\varepsilon) + \frac{L\max\{|x(\alpha_1(t_2)) - x(\alpha_1(t_1))|, |x(\alpha_2(t_2)) - x(\alpha_2(t_1))|\}}{M+\max\{|x(\alpha_1(t_2)) - x(\alpha_1(t_1))|, |x(\alpha_2(t_2)) - x(\alpha_2(t_1))|\}}$$
$$+ \omega_r^T(f,\varepsilon) + \int_0^{\beta(t_2)} |g(t_2,s,x(\gamma_1(s)),x(\gamma_2(s))) - g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))|\,ds$$
$$+ \left| \int_{\beta(t_1)}^{\beta(t_2)} |g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))|\,ds \right|$$
$$\leqslant \omega^T(q,\varepsilon) + \frac{L\max\{\omega^T(x,\omega^T(\alpha_1,\varepsilon)),\omega^T(x,\omega^T(\alpha_2,\varepsilon))\}}{M+\max\{\omega^T(x,\omega^T(\alpha_1,\varepsilon)),\omega^T(x,\omega^T(\alpha_2,\varepsilon))\}} + \omega_r^T(f,\varepsilon)$$
$$+ \int_0^{\beta_T} \omega_r^T(g,\varepsilon)\,ds\ + \omega^T(\beta,\varepsilon)G_T^r\ , \tag{3.12}$$

where we denoted

$$\omega^T(q,\varepsilon)=\sup\{|q(t_2)-q(t_1)|:\ t_1,t_2\in[0,T],\ |t_2-t_1|\leqslant\varepsilon\}\ ,$$

$$\omega_r^T(f,\varepsilon)=\sup\{|f(t_2,x,y)-f(t_1,x,y)|:\ t_1,t_2\in[0,T],\ |t_2-t_1|\leqslant\varepsilon,\ x,y\in[-r,r]\}\ ,$$

$$\omega_r^T(g,\varepsilon)=\sup\{|g(t_2,s,x,y)-g(t_1,s,x,y)|:\ t_1,t_2\in[0,T],\\ |t_2-t_1|\leqslant\varepsilon,\ s\in[0,\beta_T],\ x,y\in[-r,r]\}\ ,$$

$$G_T^r=\sup\{|g(t,s,x,y)|:\ t\in[0,T],\ s\in[0,\beta_T],\ x\in[-r,r]\}\ .$$

From the above estimate we derive the following inequality:

$$\omega^T(QX,\varepsilon)\leqslant\omega^T(q,\varepsilon)+\frac{L\max\{\omega^T(X,\omega^T(\alpha_1,\varepsilon)),\omega^T(X,\omega^T(\alpha_2,\varepsilon))\}}{M+\max\{\omega^T(X,\omega^T(\alpha_1,\varepsilon)),\omega^T(X,\omega^T(\alpha_2,\varepsilon))\}}\\ +\omega_r^T(f,\varepsilon)+\int_0^{\beta_T}\omega_r^T(g,\varepsilon)\,ds+\omega^T(\beta,\varepsilon)G_T^r\ . \tag{3.13}$$

Observe that $\omega^T(q,\varepsilon)\to0$, $\omega_r^T(f,\varepsilon)\to0$ and $\omega_r^T(g,\varepsilon)\to0$ as $\varepsilon\to0$, which is a simple consequence of the uniform continuity of the functions q, f and g on the sets $[0,T]$, $[0,T]\times[-r,r]\times[-r,r]$ and $[0,T]\times[0,\beta_T]\times[-r,r]\times[-r,r]$ respectively. Moreover, it is obvious that the constant G_T^r is finite and $\omega^T(\alpha_1,\varepsilon)\to0$, $\omega^T(\alpha_2,\varepsilon)\to0$, $\omega^T(\beta,\varepsilon)\to0$ as $\varepsilon\to0$. Thus, linking the established facts with the estimate (3.13) we get

$$\omega_0^T(QX)\leqslant\frac{L\omega_0^T(X)}{M+\omega_0^T(X)}\ . \tag{3.14}$$

Now, taking into account our assumptions, for arbitrarily fixed $t\in\mathbb{R}_+$ as well as for $x_1,x_2,y_1,y_2\in X$ we deduce the following estimate (cf. the estimate (3.8)-(3.9)):

$$\begin{aligned}|(Qx)(t)-(Qy)(t)|&\leqslant\frac{L\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}}{M+\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}}\\&\quad+\int_0^{\beta(t)}[|g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))|+|g(t,s,y(s),y(\gamma_2(s)))|]\,ds\\&\leqslant\frac{L\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}}{M+\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,|x(\alpha_2(t))-y(\alpha_2(t))|\}}+2v(t)\\&\leqslant\frac{L\max\{\operatorname{diam}X(\alpha_1(t)),\operatorname{diam}X(\alpha_2(t))\}}{M+\max\{\operatorname{diam}X(\alpha_1(t)),\operatorname{diam}X(\alpha_2(t))\}}+2v(t)\ .\end{aligned}$$

Hence we obtain

$$\operatorname{diam}(QX)(t)\leqslant\frac{L\max\{\operatorname{diam}X(\alpha_1(t)),\operatorname{diam}X(\alpha_2(t))\}}{M+\max\{\operatorname{diam}X(\alpha_1(t)),\operatorname{diam}X(\alpha_2(t))\}}+2v(t)\ .$$

In view of assumptions (H_0) and (H_4) this yields

$$\limsup_{t\to\infty}\operatorname{diam}(QX)(t)\leqslant\frac{L\limsup\limits_{t\to\infty}\max\{\operatorname{diam}X(\alpha_1(t)),\operatorname{diam}X(\alpha_2(t))\}}{M+\limsup\limits_{t\to\infty}\max\{\operatorname{diam}X(\alpha_1(t)),\operatorname{diam}X(\alpha_2(t))\}}$$

$$\leqslant \frac{L \limsup_{t\to\infty} \operatorname{diam} X(t)}{M + \limsup_{t\to\infty} \operatorname{diam} X(t)} . \tag{3.15}$$

Further, using the measure of noncompactness μ_a defined by the formula (2.2) and keeping in mind the estimates (3.14) and (3.15), we obtain

$$\begin{aligned} \mu_a(QX) &= \max\left\{ \omega_0(QX), \limsup_{t\to\infty} \operatorname{diam} QX(t) \right\} \\ &\leqslant \max\left\{ \frac{L\omega_0(X)}{M+\omega_0(X)}, \frac{L \limsup_{t\to\infty} \operatorname{diam} X(t)}{M + \limsup_{t\to\infty} \operatorname{diam} X(t)} \right\} \\ &\leqslant \frac{L \max\left\{ \omega_0(X), \limsup_{t\to\infty} \operatorname{diam} X(t) \right\}}{M + \max\left\{ \omega_0(X), \limsup_{t\to\infty} \operatorname{diam} X(t) \right\}} \\ &= \frac{L\mu_a(X)}{M+\mu_a(X)}. \end{aligned} \tag{3.16}$$

Since $L \leqslant M$ in view of assumption (H_2), from the above estimate we infer that $\mu_a(QX) \leqslant \phi(\mu_a(X))$, where $\phi(r) = \frac{Lr}{M+r} < r$ for $r > 0$. Hence we apply Theorem 2.1 to deduce that the operator Q has a fixed point x in the ball $\overline{\mathscr{B}_r}(0)$. Obviously x is a solution of the FIE (3.1). Moreover, taking into account that the image of the space $BC(\mathbb{R}_+,\mathbb{R})$ under the operator Q is contained in the ball $\overline{\mathscr{B}_r}(0)$ we infer that the set $\mathrm{Fix}(Q)$ of all fixed points of Q is contained in $\overline{\mathscr{B}_r}(0)$. Obviously, the set $\mathrm{Fix}(Q)$ contains all solutions of the FIE (3.1). On the other hand, from Remark 2.1 we conclude that the set $\mathrm{Fix}(Q)$ belongs to the family $\ker \mu_a$. Now, taking into account the description of sets belonging to $\ker \mu_a$ (given in Section 2) we deduce that all solutions for the FIE (3.1) are globally uniformly attractive on $\mathbb{R}_+$. This completes the proof.

REMARK 3.2. When $q \equiv 0$, $f(t,x,y) = f(t,x)$ and $g(t,s,x,y) = g(t,s,x)$ in our Theorem 3.1, we obtain the global attractivity result for the FIE (3.3). Note that the global attractivity result for the FIE (3.3) is also proved in Banas and Dhage [5] under the same hypotheses, but under the stronger hypothesis of (H_2) that $L < M$. Therefore, our Theorem 3.1 generalize and improve the existence results of Dhage [9] and Banas and Dhage [5] and thereby the results of Banas and Rzepka [4] under weaker conditions with a new measure of noncompactness in the Banach space $BC(\mathbb{R}_+,\mathbb{R})$.

To prove next result concerning the asymptotic positivity of the attractive solutions, we need the following hypothesis in the sequel.

(H_5) The functions q and f satisfy

$$\lim_{t\to\infty} \big[|q(t)| - q(t) \big] = 0 \quad \text{and} \quad \lim_{t\to\infty} \big[|f(t,x,y)| - f(t,x,y) \big] = 0$$

for all $x, y \in \mathbb{R}$.

THEOREM 3.2. *Under the hypotheses of Theorem* 3.1 *and* (H_5), *the FIE* (3.1) *has at least one solution on* $\mathbb{R}_+$. *Moreover, solutions of the FIE* (3.1) *are uniformly globally attractive and ultimately positive on* $\mathbb{R}_+$.

Proof. Consider the closed ball $\overline{\mathscr{B}}_r(0)$ in the Banach space $BC(\mathbb{R}_+,\mathbb{R})$, where the real number r is given as in the proof of Theorem 3.1 and define a mapping $Q: BC(\mathbb{R}_+,\mathbb{R}) \to BC(\mathbb{R}_+,\mathbb{R})$ by (3.7). Then it is shown as in the proof of Theorem 3.1 that Q defines a continuous mapping from the space $BC(\mathbb{R}_+,\mathbb{R})$ into $\overline{\mathscr{B}}_r(0)$. In particular, Q maps $\overline{\mathscr{B}}_r(0)$ into itself. Next we show that Q is a nonlinear-set-contraction with respect to the measure μ_{ad} of noncompactness in $BC(\mathbb{R}_+,\mathbb{R})$. We know that for any $x,y \in \mathbb{R}$, one has the inequality, $|x|+|y| \geqslant |x+y| \geqslant x+y$, and therefore,

$$\big||x+y|-(x+y)\big| \leqslant \big||x|+|y|-(x+y)\big| \leqslant \big||x|-x\big| + \big||y|-y\big|$$

for all $x,y \in \mathbb{R}$. Now for any $x \in \overline{\mathscr{B}}_r(0)$, one has

$$\begin{aligned}
&\big||Qx(t)|-Qx(t)\big| \\
&\quad \leqslant \big||q(t)|-q(t)\big| + \big||f(t,x(\alpha_1(t)),x(\alpha_2(t)))|-f(t,x(\alpha_1(t)),x(\alpha_2(t)))\big| \\
&\qquad + \int_0^{\beta(t)} \big[|g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))|-g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))\big]\,ds \\
&\quad \leqslant \big||q(t)|-q(t)\big| + \big||f(t,x(\alpha_1(t)),x(\alpha_2(t)))|-f(t,x(\alpha_1(t)),x(\alpha_2(t)))\big| \\
&\qquad + 2v(t) \\
&\quad \leqslant \delta_T(q)+\delta_T(f)+2V_T,
\end{aligned}$$

where $V_T = \sup_{t\geqslant T} v(t)$. From the above inequality, it follows that

$$\delta_T(X) \leqslant \delta_T(q)+\delta_T(f)+2V_T$$

for all closed $X \subset \overline{\mathscr{B}}_r(0)$. Taking the limit superior as $T \to \infty$, we obtain,

$$\limsup_{T\to\infty} \delta_T(X) \leqslant \limsup_{T\to\infty} \delta_T(q) + \limsup_{T\to\infty} \delta_T(f) + 2\limsup_{T\to\infty} V_T = 0 \tag{3.17}$$

for all closed $X \subset \overline{\mathscr{B}}_r(0)$.

Hence,

$$\delta(QX) = \lim_{T\to\infty} \delta_T(X) = 0$$

for all closed subsets X of $\overline{\mathscr{B}}_r(0)$. Further, using the measure of noncompactness μ_a defined by the formula (2.2) and keeping in mind the estimates (3.14) and (3.15), we obtain:

$$\begin{aligned}
\mu_{ad}(QX) &= \max\{\mu_{ad}(QX),\, \delta(QX)\} \leqslant \max\left\{\frac{L\mu_a(X)}{M+\mu_a(X)},\, 0\right\} \\
&= \frac{L\mu_a(X)}{M+\mu_a(X)} \leqslant \frac{L\mu_{ad}(X)}{M+\mu_{ad}(X)}
\end{aligned} \tag{3.18}$$

Since $L \leqslant M$ in view of assumption (H_2), from the above estimate we infer that $\mu_{ad}(QX) \leqslant \phi(\mu_{ad}(X))$, where $\phi(r) = \frac{Lr}{M+r} < r$ for $r > 0$. Hence we apply Theorem 2.1 to deduce that the operator Q has a fixed point x in the ball $\overline{\mathscr{B}}_r(0)$. Obviously x is a solution of the FIE (3.1). Moreover, taking into account that the image of the space $BC(\mathbb{R}_+,\mathbb{R})$ under the operator Q is contained in the ball $\overline{\mathscr{B}}_r(0)$ we infer that the set Fix(Q) of all fixed points of Q is contained in $\overline{\mathscr{B}}_r(0)$. Obviously, the set Fix$(Q)$ contains all solutions of the equation (3.1). On the other hand, from Remark 2.1 we conclude that the set Fix(Q) belongs to the family ker μ_{ad}. Now, taking into account the description of sets belonging to ker μ_{ad} (given in Section 2) we deduce that all solutions of the equation (3.1) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$. This completes the proof.

Next we prove the global asymptotic attractivity results for the FIE (3.1). We need the following hypotheses in the sequel.

(H_6) The function $q : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $\lim_{t\to\infty} q(t) = c$.
(H_7) $f(t,0,0) = 0$ for all $t \in \mathbb{R}_+$, and
(H_8) $\lim_{t\to\infty} \ell(t) = 0$, where the function ℓ is defined as in hypothesis (H_2).

THEOREM 3.3. *Assume that the hypotheses (H_0), (H_2)-(H_4) and (H_6)-(H_8) hold. Then the FIE (3.1) has at least one solution in the space $BC(\mathbb{R}_+,\mathbb{R})$. Moreover, solutions are uniformly globally asymptotically attractive on $\mathbb{R}_+$.*

Proof. Consider the closed ball $\overline{\mathscr{B}}_r(0)$ in the Banach space $BC(\mathbb{R}_+,\mathbb{R})$, where the real number r is given as in the proof of Theorem 3.1 and define a mapping $Q : \overline{\mathscr{B}}_r(0) \to \overline{\mathscr{B}}_r(0)$ by (3.6). Then Q is continuous and maps the space $BC(\mathbb{R}_+,\mathbb{R})$ and in particular, $\overline{\mathscr{B}}_r(0)$ into $\overline{\mathscr{B}}_r(0)$. We show that Q is a nonlinear $\mathscr{D}$-set-contraction with respect to the measure μ_c of noncompatness in $BC(\mathbb{R}_+,\mathbb{R})$. Let $x \in \overline{\mathscr{B}}_r(0)$ be arbitrary. Then we have

$$\begin{aligned}
|Qx(t) - c| &\leqslant |q(t) - c| + |f(t,x(\alpha_1(t)),x(\alpha_2(t)))| \\
&\qquad + \int_0^{\beta(t)} |g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))|\,ds \\
&\leqslant |q(t)-c| + \frac{\ell(t)\max\{|x(\alpha_1(t))|,|x(\alpha_2(t))|\}}{M+\max\{|x(\alpha_1(t))|,|x(\alpha_2(t))|\}} + v(t) \\
&\leqslant |q(t)-c| + \frac{\ell(t)\,\|x\|}{M+\|x\|} + v(t) \\
&\leqslant |q(t)-c| + \frac{\ell(t)\,r}{M+r} + v(t) \\
&\leqslant |q(t)-c| + \ell(t) + v(t)
\end{aligned}$$

for all $t \in \mathbb{R}_+$. This further implies that

$$\|QX(t) - c\| \leqslant |q(t) - c| + \ell(t) + v(t).$$

Taking the limit superior in the above inequality, we obtain

$$\limsup_{t\to\infty} \|QX(t)-c\| \leqslant \limsup_{t\to\infty} |q(t)-c| + \limsup_{t\to\infty} \ell(t) + \limsup_{t\to\infty} v(t) = 0. \tag{3.19}$$

Further, using the measure of noncompactness μ_c defined by the formula (2.4) and keeping in mind the estimates (3.15) and (3.19), we obtain

$$\begin{aligned}\mu_c(QX) &= \max\left\{\omega_0(QX), \limsup_{t\to\infty} \|QX(t)-c\|\right\} \\ &\leqslant \max\left\{\frac{L\omega_0(X)}{M+\omega_0(X)}, 0\right\} \leqslant \frac{L\max\{\omega_0(X), 0\}}{M+\max\{\omega_0(X),0\}} = \frac{L\mu_c(X)}{M+\mu_c(X)}.\end{aligned} \tag{3.20}$$

Since $L \leqslant M$ in view of assumption (H_2), from the above estimate we infer that $\mu_c(QX) \leqslant \phi(\mu_c(X))$, where $\phi(r) = \frac{Lr}{M+r} < r$ for $r > 0$. Hence we apply Theorem 2.1 to deduce that the operator Q has a fixed point x in the ball $\overline{\mathscr{B}}_r(0)$. Obviously x is a solution of the functional integral equation (3.1). Moreover, taking into account that the image of the space $BC(\mathbb{R}_+,\mathbb{R})$ under the operator Q is contained in the ball $\overline{\mathscr{B}}_r(0)$ we infer that the set $\mathrm{Fix}(Q)$ of all fixed points of Q is contained in $\overline{\mathscr{B}}_r(0)$. Obviously, the set $\mathrm{Fix}(Q)$ contains all solutions of the equation (3.1). On the other hand, from Remark 2.1 we conclude that the set $\mathrm{Fix}(Q)$ belongs to the family $\ker \mu_c$. Now, taking into account the description of sets belonging to $\ker \mu_c$ (given in Section 2) we deduce that all solutions of the equation (3.1) are uniformly globally asymptotically attractive on $\mathbb{R}_+$. This completes the proof.

THEOREM 3.4. *Under the hypotheses of Theorem* 3.3 *and* (H_5), *the FIE* (3.1) *has at least one solution on* $\mathbb{R}_+$. *Moreover, solutions of the FIE* (3.1) *are uniformly globally asymptotically attractive and ultimately positive on* $\mathbb{R}_+$.

Proof. The proof is similar to Theorem 3.2 with appropriate modifications. Now the desired conclusion follows by an application of the measure of noncompactness μ_{cd} in $BC(\mathbb{R}_+,\mathbb{R})$. This completes the proof.

4. The examples

In what follows, we show that the assumptions imposed in Theorems 3.1 and 3.2 admit some natural realizations. First, we indicate some possible forms for expressing the function f that satisfies the hypothesis (H_2). Define a class Φ of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:

(i) ϕ is continuous,

(ii) ϕ is nondecreasing, and

(iii) ϕ is subadditive, i.e., $\phi(x+y) \leqslant \phi(x)+\phi(y)$ for all $x,y \in \mathbb{R}_+$.

Notice that if $\phi \in \Phi$, then after simple computation it can be shown that

$$|\phi(x)-\phi(y)| \leqslant \phi(|x-y|)$$

for all $x,y \in \mathbb{R}_+$.

Now consider the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(t,x,y) = l(t)\frac{\phi_1(|x|)+\phi_2(|y|)}{2M+\phi_1(|x|)+\phi_2(|y|)} + m(t)\,, \tag{4.1}$$

where the functions $l,m : \mathbb{R}_+ \to \mathbb{R}$ are continuous and bounded on $\mathbb{R}_+$, i.e., $l,m \in BC(\mathbb{R}_+,\mathbb{R})$ with $\sup_{t\geqslant 0} l(t) = L$, $\phi_1,\phi_2 \in \Phi$ satisfying $\phi_1(r) \leqslant r$, $\phi_2(r) \leqslant r$, and M is a positive constant such that $L \leqslant M$. It is shown as in Dhage [9] that the function f satisfies the condition (3.4) and consequently the hypothesis (H_2). There do exist functions ϕ given in the expression (4.1). Indeed, the following functions $\phi(r) = r$, $\phi(r) = \ln(1+r)$, $\phi(r) = \operatorname{arctg} r$, and $\phi(r) = 2(\sqrt{1+r}-1)$ satisfy all the requirements of the functions ϕ_1 and ϕ_2 given in (4.1) (cf. Banas and Dhage [5]).

Finally, we provide two examples of the nonlinear functional integral equations of the form (3.1) for which there are global attractive and ultimate positive solutions.

EXAMPLE 4.1. Consider the following nonlinear functional integral equation

$$x(t) = \frac{t}{t+1} + \frac{t^2+2}{t^2+1}\cdot\frac{\operatorname{arctg}(|x(t)|)+\operatorname{arctg}(|x(2t)|)}{2+\operatorname{arctg}(|x(t)|)+\operatorname{arctg}(|x(2t)|)} + \int_0^{t^{3/2}} \frac{\ln(1+s[|x(\gamma_1(s))|+|x(\sqrt{s})|])}{(1+t^4)(1+x^2(s)+x^2(\sqrt{s}))}\,ds \tag{4.2}$$

where $t \in \mathbb{R}_+$.

Observe that the equation (4.2) is a special case of the equation (3.1), where we have

$$\alpha_1(t) = t,\ \alpha_2(t) = 2t,\ \beta(t) = t^{3/2},\ \gamma_1(t) = t,\ \gamma_2(t) = \sqrt{t},\ q(t) = \frac{t}{t+1}\,,$$

$$f(t,x,y) = \frac{t^2+2}{t^2+1}\cdot\frac{\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)}{2+\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)}\,,$$

and

$$g(t,s,x,y) = \frac{\ln(1+s[|x|+|y|])}{(1+t^4)(1+x^2+y^2)}\,.$$

Obviously the functions α_1,α_2,β and γ_1,γ_2 satisfy hypothesis (H_0). Further notice that the function $q(t) = \dfrac{t}{t+1}$ is continuous and bounded on $\mathbb{R}_+$ with $\|q\| = 1$ and the function $f(t,x,y)$ has the form (4.1) with $\ell(t) = \dfrac{t^2+2}{t^2+1}$. Moreover, $\phi(r) = \operatorname{arctg} r$ and $M = 2$. Since $\|\ell\| = L = 2$ we have that $L \leqslant M$. Additionally we have that the function ϕ satisfies above discussed requirements of the class of functions Φ, so the function $f(t,x,y)$ satisfies assumption (H_2).

Finally, we observe that the function $g(t,s,x,y)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ and

$$|g(t,s,x,y)| \leqslant \frac{\ln(1+s[|x|+|y|])}{(1+t^4)(1+x^2+y^2)} \leqslant \frac{s}{1+t^4}\,.$$

Thus we can put $b(t,s) = \dfrac{s}{1+t^4}$. Indeed, we have

$$\int_0^{\beta(t)} b(t,s)\,ds = \frac{1}{1+t^4}\int_0^{t^{3/2}} s\,ds = \frac{t^3}{1+t^4}\,.$$

Therefore,

$$\lim_{t\to\infty}\int_0^{\beta(t)} b(t,s)\,ds = \lim_{t\to\infty}\frac{t^3}{1+t^4} = 0.$$

This yields that there is satisfied hypothesis (H_4). Now, based on Theorem 3.1 we conclude that the functional integral equation (4.2) has solutions in the space $BC(\mathbb{R}_+,\mathbb{R})$ and all solutions of this equation are uniformly globally attractive on $\mathbb{R}_+$. Furthermore,

$$|f(t,x,y)| = \frac{t^2+2}{t^2+1}\cdot\frac{\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)}{2+\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)} = f(t,x,y)$$

for all $t\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$. Hence the functions q and $f(t,x,y)$ satisfy the hypothesis (H_5). Hence by Theorem 3.2, solutions of the FIE (4.2) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$.

EXAMPLE 4.2. Consider the following nonlinear functional integral equation

$$x(t) = \frac{t^2+2}{t^2+1} + e^{-t}\cdot\frac{\operatorname{arctg}(|x(t)|)+\operatorname{arctg}(|x(2t)|)}{2+\operatorname{arctg}(|x(t)|)+\operatorname{arctg}(|x(2t)|)} + \int_0^{t^{3/2}} \frac{\ln(1+s[|x(s)|+|x(\sqrt{s})|])}{(1+t^4)(1+x^2(s)+x^2(\sqrt{s}))}\,ds\,, \tag{4.3}$$

where $t\in\mathbb{R}_+$.

Observe that the equation (4.3) is a special case of the equation (3.1), where we have

$$\alpha_1(t)=t,\ \alpha_2(t)=2t,\ \beta(t)=t^{3/2},\ \gamma_1(t)=t,\ \gamma_2(t)=\sqrt{t},\ q(t)=\frac{t^2+2}{t^2+1}\,,$$

$$f(t,x,y) = e^{-t}\cdot\frac{\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)}{2+\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)}$$

and

$$g(t,s,x,y) = \frac{\ln(1+s[|x|+|y|])}{(1+t^4)(1+x^2+y^2)}\,.$$

Obviously, the functions α_2,β and γ_2 satisfy assumption (H_0). Further, notice that the function $f(t,x,y)$ has the form (4.1) with $\ell(t)=e^{-t}$ and $\lim_{t\to\infty}\ell(t)=\lim_{t\to\infty}e^{-t}=0$. Moreover, $\phi(r)=\operatorname{acrtg}r$, $M=1$. Since $\|\ell\|=1$ we have that $L\leqslant M$. Additionally we have that $\phi\in\Phi$, so the function $f(t,x,y)$ satisfies assumption (H_2).

Finally, it is shown as in Example 4.1 that the function $g(t,s,x,y)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ and satisfies hypothesis (H_4). Now, based on Theorem 3.3 we conclude that the functional integral equation (4.1) has solutions in the space $BC(\mathbb{R}_+,\mathbb{R})$

and all solutions of this equation are uniformly globally asymptotically attractive on $\mathbb{R}_+$. Furthermore,

$$|f(t,x,y)| = e^{-t} \cdot \frac{\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)}{2+\operatorname{arctg}(|x|)+\operatorname{arctg}(|y|)} = f(t,x,y)$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Hence the functions q and $f(t,x,y)$ satisfy the hypotheses (H_5)-(H_8). Hence by Theorem 3.4, solutions of the FIE (4.1) are uniformly globally asymptotically attractive and ultimately positive on $\mathbb{R}_+$.

REMARK 4.1. We remark that the global existence as well as attractivity and positivity results of the FIE (3.1) can be extended to the FIE,

$$x(t) = q(t) + f(t,x(\alpha_1(t)),\ldots,x(\alpha_n(t))) + \int_0^{\beta(t)} g(t,s,x(\gamma_1(s)),\ldots,x(\gamma_n(s)))\,ds \quad (4.4)$$

with similar method under appropriate modifications. Then so obtained results are useful in determining the global attractivity and positivity and global asymptotic attractivity and positivity of solutions for the nonlinear functional integral equations defined respectively by

$$x(t) = \frac{t}{t+1} + \frac{t^2+2}{t^2+1} \cdot \frac{\sum_{i=1}^n \operatorname{arctg}(|x(it)|)}{n+\sum_{i=1}^n \operatorname{arctg}(|x(it)|)} + \int_0^{t^{3/2}} \frac{\ln\Big(1+s\big[\sum_{i=1}^n |x(s^i)|\big]\Big)}{(1+t^4)\Big(1+\sum_{i=1}^n x^2(s^i)\Big)}\,ds \quad (4.5)$$

and

$$x(t) = \frac{t^2+2}{t^2+1} + e^{-t} \cdot \frac{\sum_{i=1}^n \operatorname{arctg}(|x(it)|)}{n+\sum_{i=1}^n \operatorname{arctg}(|x(it)|)} + \int_0^{t^{3/2}} \frac{\ln\Big(1+s\big[\sum_{i=1}^n |x(s^i)|\big]\Big)}{(1+t^4)\Big(1+\sum_{i=1}^n x^2(s^i)\Big)}\,ds. \quad (4.6)$$

5. The conclusion

As mentioned earlier, the fixed point theorems involving the measures of noncompactness automatically yield the characterizations of the solutions for the nonlinear integral equations on bounded as well as unbounded intervals. This technology depends upon the clever selection of the measures of noncompactness suitable for the characterizations of solutions. Some of the useful measures of noncompacctness in the applications to nonlinear integral equations have been discussed in a recent paper of Appell [2]. In this paper, by using the measures of noncomactness μ_{ad} and μ_{cd} defined by (2.5) and (2.7), we have been able to prove the global existence as well as global attractivity and ultimate positivity of the solutions for the FIE (3.1) under certain Lipschitz and growth conditions on the functions involved in it. However, other characterizations of the solutions such as monotonic attractivity and positivity can also be obtained by choosing the suitable measures of noncompactness in the Banach space of continuous and bounded real-valued functions on $\mathbb{R}_+$. Similarly, here all the nonlinearities involved in FIE (3.1) are assumed to be continuous on the domains of their definitions, however, it is conjectured that the results of this paper are also true if we replace the continuous function g by a Carathéodory one. Thus, all of these and other problems form the further scope for the research work in the theory of nonlinear integral equations. Some of the results on the lines of monotonic attractivity and ultimate positivity will be reported elsewhere.

REFERENCES

[1] R. R. AKHMEROV, M. I. KAMENSKII, A. S. POTAPOV, A. E. RODHINA AND B. N. SADOVSKII, *Measures of noncompactness and condensing operators*, Birkhauser Verlag, 1992.

[2] J. APPELL, *Measures of noncompactness, condensing operators and fixed points: an application-oriented survey*, Fixed Point Theory, **6** (2005), 157–229.

[3] J. BANAS, K. GOEBEL, *Measures of Noncompactness in Banach Space*, in: Lecture Notes in Pure and Applied Mathematics, Vol. 60, Dekker, New York, 1980.

[4] J. BANAS, B. RZEPKA, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Letters, **16** (2003), 1–6.

[5] J. BANAS, B. C. DHAGE, *Global asymptotic stability of solutions of a functional integral equations*, Nonlinear Analysis, **69** (2008), 1945–1952.

[6] T. A. BURTON, B. ZHANG, *Fixed points and stability of an integral equation: nonuniqueness*, Appl. Math. Letters, **17** (2004), 839–846.

[7] B. C. DHAGE, *Asymptotic stability of nonlinear functional integral equations via measures of noncompactness*, Comm. Appl. Nonlinear Anal., **15**, 2 (2008), 89–101.

[8] B. C. DHAGE, *Local asymptotic attractivity for nonlinear quadratic functional integral equations*, Nonlinear Analysis, **70**, 5 (2009), 1912–1922.

[9] B. C. DHAGE, *Global attractivity results for nonlinear functional integral equations via a Krasnoselskii type fixed point theorem*, Nonlinear Analysis, **70** (2009), 2485–2493.

[10] B. C. DHAGE, *Applicable fixed point theory in functional differential equations on unbounded intervals*, Dynamic Systems & Appl., **18** (2009), 701–724.

[11] X. HU, J. YAN, *The global attractivity and asymptotic stability of solution of a nonlinear integral equation*, J. Math. Anal. Appl., **321** (2006), 147–156.

[12] M. VÄTH, *Volterra and Integral Equations of Vector Functions*, Pure and Applied Mathematics, Marcel Dekker, New York, 2000.

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