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ON THE OSCILLATION OF SECOND ORDER LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. For the second order linear impulsive differential equation with oscillatory coefficient

$$\begin{cases} (r(t)x'(t))' + h(t)x(t) = 0, \quad t \neq t_k,\ t_k \geqslant t_0,\ k = 1,2,\cdots, \\ x(t_k^+) = a_k x(t_k),\ x'(t_k^+) = b_k x'(t_k), \quad k = 1,2,\cdots, \\ x(t_0^+) = x_0,\ x'(t_0^+) = x'_0, \end{cases} \tag{E}$$

where h can be changed sign on $[t_0,\infty)$, by using the equivalence transformation, we establish an associated impulsive differential equation with damping and give oscillation criteria for the equation. As applications, we obtain oscillation theorems for Eq.(E). Moveover, an example is also given to illustrate the relevance of the results.

1. Introduction

In this paper, we are concerned with the oscillation of the following second order impulsive linear differential equation with oscillatory coefficient

$$\begin{cases} (r(t)x'(t))' + h(t)x(t) = 0, \quad t \neq t_k,\ t_k \geqslant t_0,\ k = 1,2,\cdots, \\ x(t_k^+) = a_k x(t_k),\ x'(t_k^+) = b_k x'(t_k), \quad k = 1,2,\cdots, \\ x(t_0^+) = x_0,\ x'(t_0^+) = x'_0, \end{cases} \tag{1.1}$$

where $0 \leqslant t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = +\infty$, and

$$x'(t_k) = \lim_{t\to t_k^-} \frac{x(t) - x(t_k)}{t - t_k}, \quad x'(t_k^+) = \lim_{t\to t_k^+} \frac{x(t) - x(t_k^+)}{t - t_k}.$$

Here we assume that the following conditions hold.

(A1) $a_k > 0, b_k > 0$, $k = 1,2,\cdots$, are constants, and $r(t) : [t_0,\infty) \to (0,\infty)$ is a continuous differentiable function;

(A2) $h(t) : [t_0,+\infty) \to \mathbb{R}$ is a continuous function, and there exists a continuous differentiable function $p(t) : [t_0,\infty) \to [0,\infty)$ such that

$$q(t) := h(t) + p'(t) + \frac{p^2(t)}{r(t)} \geqslant 0 \quad \text{and} \quad p(t_k) = 0,\ k = 1,2,\cdots.$$

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Note that $h(t)$ can be changed sign in $[t_0,\infty)$. Let $J \subset \mathbb{R}$ be an interval, define

$$\mathrm{PC}(J,\mathbb{R}) := \{x : J \to \mathbb{R} : x(t) \text{ is continuous everywhere except some } t_k \text{ at which } x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k^-) = x(t_k)\}.$$

DEFINITION 1.1. A function $x \in \mathrm{PC}([t_0,\infty),\mathbb{R})$ is called a solution of Eq.(1.1) if
(a) $x(t_0^+) = x_0$, $x'(t_0^+) = x_0'$;
(b) $x(t)$ satisfies $(r(t)x'(t))' + h(t)x(t) = 0$ for $t \in [t_0,\infty)$, $t \neq t_k$;
(c) $x(t_k^+) = a_k x(t_k)$, $x'(t_k^+) = b_k x'(t_k)$ for such t_k, and assume that both $x(t)$ and $x'(t)$ are left continuous.

DEFINITION 1.2. A solution of Eq.(1.1) is said to be non-oscillatory if this solution is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

We note that the impulsive differential equations are an adequate mathematical apparatus for simulation of process and phenomena observed on control theory, physics, chemistry, population dynamics, ecology, biological systems, industrial, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equations, see [5]. Compared to equations without impulses, there is little known about the oscillation of solutions of impulsive differential equations due to the difficulties caused by impulsive perturbations. It seems that the first article studying the oscillation of second order impulsive differential equations was given by Chen and Feng [2]. Since then, there has been an increasing interest in finding the oscillation criteria for such equations; see [1,3,4,6-12] and the reference therein. It should be noted that all almost impulsive differential equations considered in the literature so far have been imposed the restrictive condition which the coefficient function is nonnegative. To develop the qualitative theory of impulsive differential equations, in this paper, by using the equivalence transformation, we establish an associated impulsive differential equation with damping (cf, Eq.(2.2)), and give oscillation criteria for Eq.(2.2). As applications, we obtain oscillation theorems for Eq.(1.1). Moveover, an example is also given to illustrate the relevance of the results. Here we would like to point out that the obtained oscillation theorems in present paper are essentially new even for Eq.(2.2).

2. Main results

We are interested in the oscillation of solutions of Eq.(1.1). To this end, we introduce an equivalence transform (2.1) to establish an associated equation (2.2). We now begin with the following two lemmas.

LEMMA 2.1. *Let $x(t)$ be a solution of Eq.* (1.1) *and let*

$$y(t) = x(t)\exp\Big(-\int_{t_0}^{t} \frac{p(s)}{r(s)}ds\Big). \tag{2.1}$$

Then $y(t)$ is a solution of the following impulsive differential equation with damping

$$\begin{cases} (r(t)y'(t))' + 2p(t)y'(t) + q(t)y(t) = 0,\ t \neq t_k,\ t_k \geqslant t_0,\ k = 1,2,\cdots, \\ y(t_k^+) = a_k y(t_k),\ y'(t_k^+) = b_k y'(t_k), \qquad k = 1,2,\cdots, \\ y(t_0^+) = y_0,\ y'(t_0^+) = y_0'. \end{cases} \tag{2.2}$$

Proof. Clearly,

$$x(t) = y(t)\exp\Big(\int_{t_0}^{t} \frac{p(s)}{r(s)} ds\Big).$$

Consequently,

$$(r(t)x'(t))' = \Big[(r(t)y'(t))' + 2p(t)y'(t) + \Big(p'(t) + \frac{p^2(t)}{r(t)}\Big)y(t)\Big]\exp\Big(\int_{t_0}^{t} \frac{p(s)}{r(s)} ds\Big). \tag{2.3}$$

Noting that $x(t)$ is the solution of Eq.(1.1), and by (2.3), we have

$$(r(t)y'(t))' + 2p(t)y'(t) + q(t)y(t) = 0, \quad t \neq t_k. \tag{2.4}$$

On the other hand, by (2.1), we get

$$\begin{aligned} y(t_k^+) &= x(t_k^+)\exp\Big(-\int_{t_0}^{t_k} \frac{p(s)}{r(s)} ds\Big) \\ &= a_k x(t_k)\exp\Big(-\int_{t_0}^{t_k} \frac{p(s)}{r(s)} ds\Big) = a_k y(t_k), \end{aligned} \tag{2.5}$$

and, by $p(t_k) = 0$,

$$\begin{aligned} y'(t_k^+) &= \Big[x'(t_k^+) - \frac{p(t_k)}{r(t_k)}x(t_k^+)\Big]\exp\Big(-\int_{t_0}^{t_k} \frac{p(s)}{r(s)} ds\Big) \\ &= b_k\Big[x'(t_k) - \frac{p(t_k)}{r(t_k)}x(t_k)\Big]\exp\Big(-\int_{t_0}^{t_k} \frac{p(s)}{r(s)} ds\Big) \\ &\quad + (b_k - a_k)\frac{p(t_k)}{r(t_k)}x(t_k)\exp\Big(-\int_{t_0}^{t_k} \frac{p(s)}{r(s)} ds\Big) \\ &= b_k y'(t_k). \end{aligned} \tag{2.6}$$

It follows from (2.4)-(2.6) that $y(t)$ is a solution of Eq.(2.2). This completes the proof. □

LEMMA 2.2. *Let $y(t)$ be a non-oscillatory solution of Eq.(2.2). If*

$$\limsup_{n\to\infty}\Big\{\int_{t_0}^{t_1} q(t)dt + \frac{a_1}{b_1}\int_{t_1}^{t_2} q(t)dt + \frac{a_1a_2}{b_1b_2}\int_{t_2}^{t_3} q(t)dt + \cdots + \prod_{i=1}^{n-1}\frac{a_i}{b_i}\int_{t_{n-1}}^{t_n} q(t)dt\Big\} = +\infty, \tag{2.7}$$

then there exists a $T \geqslant t_0$ such that $y(t)y'(t) < 0$ for $t \geqslant T$ and $t \neq t_k$.

Proof. Without loss of generality we may assume that there exists a $T \in [t_0,\infty)$ such that $y(t)>0$ for $t \geqslant T$. Next we show that $y'(t)<0$ for $t \geqslant T$ by three steps.

Step 1. We prove that $y'(t)$ is non-oscillatory in $(t_k,t_{k+1}]$ for all sufficiently large k.

If not, then there exist a positive integer j and some $t' \in (t_j,t_{j+1}]$ such that $y'(t')=0$. By (2.2), we have

$$r(t')y''(t)|_{t=t'} = -q(t')y(t') < 0.$$

Hence, there exists some $\delta>0$ such that $y'(t)$ is decreasing in $(t',t'+\delta] \subset (t_j,t_{j+1}]$, consequently,

$$y'(t)<0,\ t \in (t',t'+\delta]. \tag{2.8}$$

We now claim that there does not exist any $t'' \in (t',t_{j+1})$ such that $y'(t'')=0$. Otherwise, we let $t'' \in (t',t_{j+1}] \subset (t_j,t_{j+1}]$ be the first point such that $y'(t'')=0$. By (2.8), we see $y''(t'') \geqslant 0$. But, from (2.2), we know $r(t'')y''(t'')<0$, so $y''(t'')<0$, which is a contradiction. Thus we have $y'(t)<0$ for $t \in (t',t_{j+1}]$. Moreover, $y'(t_{j+1})<0$ and $y'(t_{j+1}^+) = b_{j+1}y'(t_{j+1})<0$.

Similarly, applying the above method, we get there does not exist $\overline{t''} \in (t_{j+1},t_{j+2}]$ such that $y'(\overline{t''})=0$ and $y'(t)<0$ for $t \in (t_{j+1},t_{j+2}]$. Similar procedures have been demonstrated repeatedly, we can obtain that $y'(t)$ is non-oscillatory for $t \in (t_{j+i},t_{j+i+1}]$, $i=0,1,2,\cdots$.

Step 2. We prove that there exists some $T' \geqslant T$ such that $y'(t)$ is non-oscillatory for $t \geqslant T'$.

Without loss of generality we can let $t_j \geqslant T$, by Step 1, we know that $y'(t)$ is non-oscillatory in $(t_{j+i},t_{j+i+1}]$, $i=0,1,2,\cdots$. Now we consider two case as follows.

Case 1. If $y'(t)<0$ for $t \in (t_j,t_{j+1}]$, it is easy to see that $y'(t_{j+1}^+) = b_{j+1}y'(t_{j+1})<0$. Noting that $y'(t)$ is negative in every interval $(t_{j+i},t_{j+i+1}]$, $i=0,1,2,\cdots$, we have that there is some $T' \geqslant T$ such that $y'(t)<0$ for $t \geqslant T'$.

Case 2. If $y'(t)>0$ for $t \in (t_k,t_{k+1}]$, we easily see that $y'(t_{k+1}^+) = b_{k+1}y'(t_{k+1})>0$. Since $y'(t)$ is positive in every interval $(t_{k+i},t_{k+i+1}]$, $i=0,1,2,\cdots$. Hence, there exists some $T' \geqslant T$ such that $y'(t)>0$ for $t \geqslant T'$.

Step 3. We will prove that there exists some $T' \geqslant T$ such that $y'(t)<0$ for $T' \geqslant T$.

First, we prove that $y'(t_k)<0$. If not, there exists some $t_j \geqslant T'$ such that $y'(t_j)>0$, using $y'(t_j^+) = b_j y'(t_j) > 0$ and Step 2, we have

$$y'(t)>0, \quad t \geqslant t_j, \tag{2.9}$$

which follows from (2.2) that

$$(r(t)y'(t))' = -2p(t)y'(t) - q(t)y(t) < -q(t)y(t) \leqslant 0. \tag{2.10}$$

Hence, $r(t)y'(t)$ is nonincreasing in $(t_{j+i-1},t_{j+i}]$, $i=1,2,\cdots$.

We next claim that, for any $n \geqslant 2$,

$$r(t_{j+n})y'(t_{j+n}) \leqslant b_{j+1}\cdots b_{j+n-1}\Big\{ r(t_j)y'(t_j^+) - y(t_j^+)\Big[\int_{t_j}^{t_{j+1}} q(t)dt$$

$$+\frac{a_{j+1}}{b_{j+1}}\int_{t_{j+1}}^{t_{j+2}} q(t)dt+\cdots+\frac{a_{j+1}\cdots a_{j+n-1}}{b_{j+1}\cdots b_{j+n-1}}\int_{t_{j+n-1}}^{t_{j+n}} q(t)dt\Big]\Big\}. \quad (2.11)$$

Indeed, integrating (2.10) from t_j to t_{j+1}, we have

$$\begin{aligned} r(t_{j+1})y'(t_{j+1}) &\leqslant r(t_j^+)y'(t_j^+)-\int_{t_j}^{t_{j+1}} q(t)y(t)dt \\ &\leqslant r(t_j)y'(t_j^+)-y(t_j^+)\int_{t_j}^{t_{j+1}} q(t)dt. \end{aligned} \quad (2.12)$$

Similar to the proof of (2.12), we get

$$\begin{aligned} &r(t_{j+2})y'(t_{j+2}) \\ &\quad\leqslant r(t_{j+1}^+)y'(t_{j+1}^+)-\int_{t_{j+1}}^{t_{j+2}} q(t)y(t)dt \\ &\quad= b_{j+1}r(t_{j+1})y'(t_{j+1})-\int_{t_{j+1}}^{t_{j+2}} q(t)y(t)dt \\ &\quad\leqslant b_{j+1}\Big[r(t_j)y'(t_j^+)-y(t_j^+)\int_{t_j}^{t_{j+1}} q(t)dt\Big]-y(t_{j+1}^+)\int_{t_{j+1}}^{t_{j+2}} q(t)dt \\ &\quad\leqslant b_{j+1}\Big\{r(t_j)y'(t_j^+)-y(t_j^+)\Big[\int_{t_j}^{t_{j+1}} q(t)dt+\frac{a_{j+1}}{b_{j+1}}\int_{t_{j+1}}^{t_{j+2}} q(t)dt\Big]\Big\}. \end{aligned} \quad (2.13)$$

Thus, (2.11) holds for $n=2$. Next we suppose that (2.11) holds for $n=N$, i.e.,

$$\begin{aligned} r(t_{j+N})y'(t_{j+N}) \leqslant b_{j+1}\cdots b_{j+N-1}\Big\{&r(t_j)y'(t_j^+)-y(t_j^+)\Big[\int_{t_j}^{t_{j+1}} q(t)dt \\ &+\frac{a_{j+1}}{b_{j+1}}\int_{t_{j+1}}^{t_{j+2}} q(t)dt+\cdots+\frac{a_{j+1}\cdots a_{j+N-1}}{b_{j+1}\cdots b_{j+N-1}}\int_{t_{j+N-1}}^{t_{j+N}} q(t)dt\Big]\Big\}. \end{aligned} \quad (2.14)$$

From (2.9), (2.10) and (2.14), we have, for $t\in(t_{j+N},t_{j+N+1}]$,

$$\begin{aligned} &y'(t_{j+N+1}) \\ &\quad\leqslant \frac{1}{r(t_{j+N+1})}\Big\{r(t_{j+N}^+)y'(t_{j+N}^+)-y(t_{j+N}^+)\int_{t_{j+N}}^{t_{j+N+1}} q(t)dt\Big\} \\ &\quad\leqslant \frac{b_{j+1}\cdots b_{j+N}}{r(t_{j+N+1})}\Big\{r(t_j)y'(t_j^+)-y(t_j^+)\Big[\int_{t_j}^{t_{j+1}} q(t)dt+\frac{a_{j+1}}{b_{j+1}}\int_{t_{j+1}}^{t_{j+2}} q(t)dt \\ &\qquad+\cdots+\frac{a_{j+1}\cdots a_{j+N-1}}{b_{j+1}\cdots b_{j+N-1}}\int_{t_{j+N-1}}^{t_{j+N}} q(t)dt\Big]\Big\}-\frac{a_{j+1}\cdots a_{j+N}}{r(t_{j+N+1})}y(t_j^+)\int_{t_{j+N}}^{t_{j+N+1}} q(t)dt \\ &\quad= \frac{b_{j+1}\cdots b_{j+N}}{r(t_{j+N+1})}\Big\{r(t_j)y'(t_j^+)-y(t_j^+)\Big[\int_{t_j}^{t_{j+1}} q(t)dt+\frac{a_{j+1}}{b_{j+1}}\int_{t_{j+1}}^{t_{j+2}} q(t)dt \\ &\qquad+\cdots+\frac{a_{j+1}\cdots a_{j+N-1}a_{j+N}}{b_{j+1}\cdots b_{j+N-1}b_{j+N}}\int_{t_{j+N}}^{t_{j+N+1}} q(t)dt\Big]\Big\}. \end{aligned}$$

Hence (2.11) holds for $n = N+1$. By induction, (2.11) holds for any $n \geqslant 2$. By (2.11), it follows from (2.7) that $y'(t_{j+n}) < 0$ as $n \to +\infty$ which contradicts $y'(t) > 0$ for $t \geqslant t_j$. Therefore, $y'(t_k) < 0$ for $t_k \geqslant T'$. It follows from Step 2 that $y'(t) < 0$ for $t \neq t_{j+n}$ and $t \geqslant T'$. The proof is complete. □

We are now in a position to establish oscillation criteria for Eq.(2.2).

THEOREM 2.1. *Let* (2.7) *hold. If*

$$\limsup_{n\to\infty}\Big\{\int_{t_0}^{t_1}\Theta(t_0,s)ds + \frac{b_1}{a_1}\int_{t_1}^{t_2}\Theta(t_0,s)ds + \cdots + \prod_{i=1}^{n-1}\frac{b_i}{a_i}\int_{t_{n-1}}^{t_n}\Theta(t_0,s)ds\Big\} = +\infty \qquad (2.15)$$

holds, where

$$\Theta(s,t) = \frac{1}{r(t)}\exp\Big(-2\int_s^t \frac{p(\tau)}{r(\tau)}d\tau\Big),\ t > s \geqslant t_0,$$

then Eq. (2.2) *is oscillatory.*

Proof. Suppose to the contrary that $y(t)$ is a non-oscillatory solution of Eq.(2.2). Without loss of generality we may assume $y(t) > 0$ for $t \geqslant t_0$. By Lemma 2.2, there exists a $T \geqslant t_0$ such that

$$y'(t) < 0, \quad t \geqslant T,\ t \neq t_k. \qquad (2.16)$$

By (2.2), we have

$$(r(t)y'(t))' + 2p(t)y'(t) < 0, \quad t \geqslant T,\ t \neq t_k,$$

which follows from (2.16) that

$$\frac{(r(t)y'(t))'}{r(t)y'(t)} + \frac{2p(t)}{r(t)} > 0. \qquad (2.17)$$

Let $t_k = \min\{t_k : t_k \geqslant T, k \in \mathbb{N}\}$, integrating (2.17) from t_k to t, where $t \in (t_k, t_{k+1}]$, we have

$$\ln\Big(\frac{r(t)y'(t)}{r(t_k^+)y'(t_k^+)}\Big) > -2\int_{t_k}^t \frac{p(\tau)}{r(\tau)}d\tau,$$

or,

$$y'(t) < r(t_k^+)y'(t_k^+)\Theta(t_k,t), \qquad (2.18)$$

and

$$y'(t_{k+1}) < r(t_k)y'(t_k^+)\Theta(t_k,t_{k+1}). \qquad (2.19)$$

Integrating (2.18) from t_k to t_{k+1}, we have

$$y(t_{k+1}) < y(t_k^+) + r(t_k)y'(t_k^+)\int_{t_k}^{t_{k+1}}\Theta(t_k,s)ds. \qquad (2.20)$$

Similar to the proof of (2.18), we have, for $t \in (t_{k+1}, t_{k+2}]$,

$$y'(t) < r(t_{k+1}^+)y'(t_{k+1}^+)\Theta(t_{k+1}, t). \tag{2.21}$$

Noting that (2.19) and (2.20), and integrating (2.21) from t_{k+1} to t_{k+2}, we have

$$\begin{aligned}
y(t_{k+2}) <& y(t_{k+1}^+) + r(t_{k+1}^+)y'(t_{k+1}^+)\int_{t_{k+1}}^{t_{k+2}} \Theta(t_{k+1}, s)ds \\
<& a_{k+1}\left[y(t_k^+) + r(t_k)y'(t_k^+)\int_{t_k}^{t_{k+1}} \Theta(t_k, s)ds\right] \\
& + b_{k+1}r(t_k)y'(t_k^+)\int_{t_{k+1}}^{t_{k+2}} \Theta(t_k, s)ds,
\end{aligned}$$

or,

$$y(t_{k+2}) < a_{k+1}\left\{y(t_k^+) + r(t_k)y'(t_k^+)\left[\int_{t_k}^{t_{k+1}} \Theta(t_k, s)ds + \frac{b_{k+1}}{a_{k+1}}\int_{t_{k+1}}^{t_{k+2}} \Theta(t_k, s)ds\right]\right\}.$$

By induction, it can be proved that for any $n \geqslant k+1$,

$$\begin{aligned}
y(t_n) <& y(t_{n-1}^+) + r(t_{n-1}^+)y'(t_{n-1}^+)\int_{t_{n-1}}^{t_n} \Theta(t_{n-1}, s)ds \\
<& a_{k+1}\cdots a_{n-1}\left\{y(t_k^+) + r(t_k)y'(t_k^+)\left[\int_{t_k}^{t_{k+1}} \Theta(t_k, s)ds\right.\right. \\
& \left.\left. + \frac{b_{k+1}}{a_{k+1}}\int_{t_{k+1}}^{t_{k+2}} \Theta(t_k, s)ds + \cdots + \frac{b_{k+1}\cdots b_{n-1}}{a_{k+1}\cdots a_{n-1}}\int_{t_{n-1}}^{t_n} \Theta(t_k, s)ds\right]\right\} \\
<& 0 \quad \text{as } n \to \infty, \ (\text{by } (2.15)),
\end{aligned}$$

which contradicts $y(t) > 0$ for $t \geqslant T$. The proof is complete. □

THEOREM 2.2. *Let* (2.7) *hold. If exists a continuous differentiable function* $\eta(t)$: $[t_0, \infty) \to (0, \infty)$ *with* $r(t)\eta'(t) \geqslant 2\eta(t)p(t)$ *such that*

$$\begin{aligned}
\limsup_{n\to\infty} \Bigg\{ & \prod_{i=1}^{n-1} \frac{b_i}{a_i}\int_{t_0}^{t_1} \eta(s)q(s)ds \\
& + \prod_{i=2}^{n-1} \frac{b_i}{a_i}\int_{t_1}^{t_2} \eta(s)q(s)ds + \cdots + \int_{t_{n-1}}^{t_n} \eta(s)q(s)ds\Bigg\} = +\infty,
\end{aligned} \tag{2.22}$$

and

$$\limsup_{k\to\infty} \int_{t_k}^{t_{k+1}} \frac{dt}{r(t)\eta(t)} \geqslant 1, \tag{2.23}$$

then Eq. (2.2) *is oscillatory.*

Proof. Suppose that there exists a non-oscillatory solution $y(t)$ of Eq.(2.2) so that $y(t) \neq 0$ for $t \geqslant t_0$. Without loss of generality we may assume that $y(t) > 0$ for $t \geqslant t_0$.

By Lemma 2.2, there exists a $T \geqslant t_0$ such that $y'(t) < 0$ for $t \geqslant T$. It follows from (2.2) that

$$(r(t)y'(t))' + 2p(t)y'(t) + q(t)y(t) = 0, \quad t \neq t_k. \tag{2.24}$$

Let $t_k = \min\{t_k : t_k \geqslant T, k \in N\}$, multiplying (2.24) by $\eta(t)/y(t)$, and integrating from t_k to t, we get, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned}\frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} =& \frac{r(t)\eta(t)y'(t)}{y(t)} - \int_{t_k}^{t} [r(s)\eta'(s) - 2\eta(s)p(s)]\frac{y'(s)}{y(s)}ds \\ &+ \int_{t_k}^{t} r(s)\eta(s)\Big(\frac{y'(s)}{y(s)}\Big)^2 ds + \int_{t_k}^{t} \eta(s)q(s)ds.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{r(t)\eta(t)y'(t)}{y(t)} <& \frac{r(t)\eta(t)y'(t)}{y(t)} + \int_{t_k}^{t} r(s)\eta(s)\Big(\frac{y'(s)}{y(s)}\Big)^2 ds \\ =& \frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} \\ & - \int_{t_k}^{t} \eta(s)q(s)ds + \int_{t_k}^{t} [r(s)\eta'(s) - 2\eta(s)p(s)]\frac{y'(s)}{y(s)}ds \\ <& \frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} - \int_{t_k}^{t} \eta(s)q(s)ds.\end{aligned} \tag{2.25}$$

Then,

$$\frac{r(t_{k+1})\eta(t_{k+1})y'(t_{k+1})}{y(t_{k+1})} < \frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} - \int_{t_k}^{t_{k+1}} \eta(s)q(s)ds. \tag{2.26}$$

Similar to the proof of (2.25) and (2.26), we get, for $t \in (t_{k+1}, t_{k+2}]$,

$$\begin{aligned}\frac{r(t)\eta(t)y'(t)}{y(t)} &< \frac{r(t_{k+1})\eta(t_{k+1})y'(t_{k+1}^+)}{y(t_{k+1}^+)} - \int_{t_{k+1}}^{t} \eta(s)q(s)ds \\ &= \frac{b_{k+1}}{a_{k+1}}\frac{r(t_{k+1})\eta(t_{k+1})y'(t_{k+1})}{y(t_{k+1})} - \int_{t_{k+1}}^{t} \eta(s)q(s)ds \\ &< \frac{b_{k+1}}{a_{k+1}}\Big[\frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} - \int_{t_k}^{t_{k+1}} \eta(s)q(s)ds\Big] - \int_{t_{k+1}}^{t} \eta(s)q(s)ds,\end{aligned}$$

so,

$$\begin{aligned}&\frac{r(t_{k+2})\eta(t_{k+2})y'(t_{k+2})}{y(t_{k+2})} \\ &\qquad < \frac{b_{k+1}}{a_{k+1}}\Big[\frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} - \int_{t_k}^{t_{k+1}} \eta(s)q(s)ds\Big] - \int_{t_{k+1}}^{t_{k+2}} \eta(s)q(s)ds.\end{aligned}$$

Since $y'(t_k^+)y(t_k^+) < 0$, by induction, it is easy to prove that, for any $n \geqslant k+1$,

$$\begin{aligned}
&\frac{r(t_n)\eta(t_n)y'(t_n)}{y(t_n)}\\
&\quad < \frac{b_{k+1}\cdots b_{n-1}}{a_{k+1}\cdots a_{n-1}}\Big[\frac{r(t_k)\eta(t_k)y'(t_k^+)}{y(t_k^+)} - \int_{t_k}^{t_{k+1}} \eta(s)q(s)ds\\
&\qquad - \frac{a_{k+1}}{b_{k+1}}\int_{t_{k+1}}^{t_{k+2}} \eta(s)q(s)ds - \frac{a_{k+1}\cdots a_{n-1}}{b_{k+1}\cdots b_{n-1}}\int_{t_{n-1}}^{t_n} \eta(s)q(s)ds\Big]\\
&\quad < -\Big\{\frac{b_{k+1}\cdots b_{n-1}}{a_{k+1}\cdots a_{n-1}}\int_{t_k}^{t_{k+1}} \eta(s)q(s)ds + \frac{b_{k+2}\cdots b_{n-1}}{a_{k+2}\cdots a_{n-1}}\int_{t_{k+1}}^{t_{k+2}} \eta(s)q(s)ds\\
&\qquad + \cdots + \int_{t_{n-1}}^{t_n} \eta(s)q(s)ds\Big\},
\end{aligned}$$

which follows from (2.22) that

$$\lim_{n\to\infty} \frac{r(t_n)\eta(t_n)y'(t_n)}{y(t_n)} = -\infty. \tag{2.27}$$

By (2.25) and (2.27), there exists some j such that

$$\frac{r(t)\eta(t)y'(t)}{y(t)} + \int_{t_j}^{t} r(s)\eta(s)\Big(\frac{y'(s)}{y(s)}\Big)^2 ds \leqslant -1, \quad t \geqslant t_j.$$

Consequently,

$$\frac{r(t)\eta(t)y'(t)}{y(t)} \leqslant -1 - \int_{t_j}^{t} r(s)\eta(s)\Big(\frac{y'(s)}{y(s)}\Big)^2 ds. \tag{2.28}$$

Noting that $y'(t) < 0$, multiplying (2.28) by

$$-\frac{y'(t)}{y(t)}\Big(1 + \int_{t_j}^{t} r(s)\eta(s)\Big(\frac{y'(s)}{y(s)}\Big)^2 ds\Big)^{-1},$$

we have

$$r(t)\eta(t)\Big(\frac{y'(t)}{y(t)}\Big)^2\Big(1 + \int_{t_j}^{t} \Big(\frac{y'(s)}{y(s)}\Big)^2 ds\Big)^{-1} \geqslant -\frac{y'(t)}{y(t)}. \tag{2.29}$$

Integrating (2.29) from t_j to t, we get

$$\ln\Big(1 + \int_{t_j}^{t} r(s)\eta(s)\Big(\frac{y'(s)}{y(s)}\Big)^2 ds\Big) \geqslant \ln\Big(\frac{y(t_j^+)}{y(t)}\Big). \tag{2.30}$$

From (2.28) and (2.30), it follows that

$$-r(t)\eta(t)\frac{y'(t)}{y(t)} \geqslant \frac{y(t_j^+)}{y(t)},$$

which can be rewritten as

$$y'(t) \leqslant -\frac{y(t_j^+)}{r(t)\eta(t)}. \tag{2.31}$$

Integrating (2.31) from t_j to t_{j+1}, we have

$$y(t_{j+1}) \leqslant y(t_j^+)\Big(1-\int_{t_j}^{t_{j+1}} \frac{dt}{r(t)\eta(t)}dt\Big).$$

By (2.23), we have $\limsup_{j\to\infty} y(t_{j+1}) \leqslant 0$, which contradicts $y(t)>0$ for $t \geqslant T$. The proof is complete. $\square$

As the immediate consequences of Theorems 2.1 and 2.2, we obtain the following oscillation theorems for Eq.(1.1).

THEOREM 2.3. *If the conditions of Theorem 2.1 hold, then Eq.* (1.1) *is oscillatory.*

THEOREM 2.4. *If the conditions of Theorem 2.2 hold, then Eq.* (1.1) *is oscillatory.*

Finally, we provide an example to illustrate the applications of Theorems 2.3 and 2.4.

EXAMPLE 2.1. Consider

$$\begin{cases} x''(t)+(\frac{\sin t}{4t}+\frac{1}{8t}+\frac{3}{16t^2}+\frac{\cos t}{8t^2})x(t)=0,\ t\geqslant t_0,\ t\neq 3^k\pi,\ k=1,2,\cdots, \\ x(t_k^+)=x(t_k),\ x'(t_k^+)=\frac{k+1}{k}x'(t_k), \qquad k=1,2,\cdots, \\ x(t_0^+)=x_0,\ x'(t_0^+)=x_0', \end{cases} \tag{2.32}$$

where

$$r(t)=1,\ h(t)=(\tfrac{\sin t}{4t}+\tfrac{1}{8t}+\tfrac{3}{16t^2}+\tfrac{\cos t}{8t^2}), \text{ and } a_k=1,\ t_k=3^k\pi,\ b_k=\tfrac{k+1}{k}.$$

Let

$$p(t)=\frac{1}{4t}(1+\cos t)\geqslant 0.$$

Then

$$p(t_k)=0 \quad \text{and} \quad q(t)=\frac{2t+\cos^2 t}{16t^2}>\frac{1}{8t}>0.$$

A straightforward calculation shows

$$\begin{aligned} &\lim_{n\to\infty}\int_{t_0}^{t_1} q(t)dt+\frac{a_1}{b_1}\int_{t_1}^{t_2} q(t)dt+\frac{a_1a_2}{b_1b_2}\int_{t_2}^{t_3} q(t)dt+\cdots+\prod_{i=1}^{n-1}\frac{a_i}{b_i}\int_{t_{n-1}}^{t_n} q(t)dt \\ &\qquad \geqslant \lim_{n\to\infty}\Big(\frac{1}{2}\int_{\pi}^{3\pi}\frac{1}{8t}dt+\frac{1}{3}\int_{3\pi}^{9\pi}\frac{1}{8t}dt+\cdots+\frac{1}{n+1}\int_{3^{n-1}\pi}^{3^n\pi}\frac{1}{8t}dt\Big) \\ &\qquad \geqslant \frac{1}{8}\lim_{n\to\infty}\Big(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\Big)=+\infty, \end{aligned}$$

Hence the conditions (2.7) is satisfied. Note that $\frac{-1}{t} \leqslant -2p(t) \leqslant 0$ and

$$\begin{aligned}
&\lim_{n\to\infty}\Big\{\int_{t_0}^{t_1}\Theta(t_0,s)ds+\frac{b_1}{a_1}\int_{t_1}^{t_2}\Theta(t_0,s)ds+\cdots+\prod_{i=1}^{n-1}\frac{b_i}{a_i}\int_{t_{n-1}}^{t_n}\Theta(t_0,s)ds\Big\}\\
&\qquad\geqslant \lim_{n\to\infty}\int_{\pi}^{3\pi}\frac{1}{s}ds+\int_{3\pi}^{3^2\pi}\frac{1}{s}ds+\cdots+\int_{3^{n-1}\pi}^{3^n\pi}\frac{1}{s}ds\\
&\qquad=\ln 3\lim_{n\to\infty}(n-1)=+\infty.
\end{aligned}$$

Then (2.15) holds. By Theorem 2.3, Eq.(2.32) is oscillatory.

At fact, we can also show that Eq.(2.32) is oscillatory by Theorem 2.4. Indeed, let $\eta(t)=t$. Then

$$\begin{aligned}
&\lim_{n\to\infty}\Big\{\prod_{i=1}^{n-1}\frac{b_i}{a_i}\int_{t_0}^{t_1}tq(t)dt+\prod_{i=2}^{n-1}\frac{b_i}{a_i}\int_{t_1}^{t_2}tq(t)dt+\cdots+\int_{t_{n-1}}^{t_n}tq(t)dt\}\\
&\qquad\geqslant \lim_{n\to\infty}\Big(n\int_{\pi}^{3\pi}\frac{1}{8}dt+\frac{n}{2}\int_{3\pi}^{9\pi}\frac{1}{8}dt+\cdots+\int_{3^{n-1}\pi}^{3^n\pi}\frac{1}{8}dt\Big)=+\infty,
\end{aligned}$$

and

$$\int_{3^{k-1}\pi}^{3^k\pi}\frac{dt}{r(t)\eta(t)}=\int_{3^{k-1}\pi}^{3^k\pi}\frac{1}{t}dt=\ln 3>1,$$

hence, (2.22) and (2.23) hold. Then, by Theorem 2.4, Eq.(2.32) is oscillatory.

REMARK 2.1. Note that $h(t)$ can be changed sign in $[t_0,\infty)$. Therefore, the oscillation criteria in [1-4,6-8,11,12] can not apply to Example 2.1.

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