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QUASI–PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS VIA THE FLOQUET–LIN THEORY

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Abstract. We use a Floquet theory for quasi-periodic linear ordinary differential equations due to Zhensheng Lin to obtain results on the quasi-periodic solutions of quasi-periodic nonlinear ordinary differential equations. First we obtain an existence result, secondly we obtain a result on the continuous dependence by using a parametrized fixed point theorem, and thirdly we obtain a local result on the differentiable dependence by using an implicit function theorem in function spaces.

1. Introduction

Our aim is to study quasi-periodic solutions of ordinary differential equations in the following forms :

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \tag{1.1}$$

$$\dot{x}(t) = A(t)x(t) + g(t, x(t), u(t)), \tag{1.2}$$

$$\dot{x}(t) = g(t, x(t), u(t)), \tag{1.3}$$

where A is a quasi-periodic matrix, u is a quasi-periodic function (a forcing term or a control term), f and g are quasi-periodic with respect to t.

To treat these problems we use the properties of the following forced linear ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + b(t), \tag{1.4}$$

where b is a quasi-periodic function. To study equation (1.4) we use a Floquet theory of quasi-periodic equations due to Zhensheng Lin [8], [9], [10], and several tools of Nonlinear Functional Analysis.

In Section 2 we fix our notation on the quasi-periodic function spaces.

In Section 3 we recall results of Lin and we use them to study equation (1.4), notably to obtain a generalization to (1.4) of a classical theorem of Bohr and Neugebauer on the constant coefficients systems.

In Section 4, by using results of Section 3, we build a Fixed Point approach to obtain an existence result on quasi-periodic solutions of equation (1.1).

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In Section 5, by using results of Section 3, we build a Parametrized Fixed Point approach to obtain an existence result and a continuous dependence results on quasi-periodic solutions of equation (1.2).

In Section 6, by using results of Section 3, we build an Implicit Function Theorem approach to obtain a differentiable perturbation result on the quasi-periodic solutions of equation (1.3).

2. Notation

$AP^0(\mathbb{R}^n)$ denotes the space of the almost periodic functions from $\mathbb{R}$ into $\mathbb{R}^n$ in the sense of H. Bohr, [6], [7], [5]. Endowed with the norm $\|\varphi\|_\infty = \sup_{t\in\mathbb{R}} \|\varphi(t)\|$, it is a Banach space.

When $k \in \mathbb{N}_* = \mathbb{N}\setminus\{0\}$, $C^k(\mathbb{R},\mathbb{R}^n)$ denotes the space of the functions from $\mathbb{R}$ into $\mathbb{R}^n$ which are of class C^k. $AP^k(\mathbb{R}^n)$ denotes the space of the functions $\varphi \in AP^k(\mathbb{R}^n) \cap C^k(\mathbb{R},\mathbb{R}^n)$ such that the derivatives $\frac{d^j\varphi}{dt^j}$ belong to $AP^0(\mathbb{R}^n)$ for all $j = 1,...,k$. Endowed with the norm:

$$\|\varphi\|_{C^k} = \|\varphi\|_\infty + \sum_{j=1}^{k} \|\frac{d^j\varphi}{dt^j}\|_\infty,$$

$AP^k(\mathbb{R}^n)$ is a Banach space.

When $\varphi \in AP^0(\mathbb{R}^n)$ and when $\lambda \in \mathbb{R}$, we consider the Fourier-Bohr coefficient

$$a(\varphi,\lambda) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(t)e^{-i\lambda t}dt.$$

We set

$$\Lambda(\varphi) = \{\lambda \in \mathbb{R} : a(\varphi,\lambda) \neq 0\}$$

and $\mathrm{Mod}(\varphi)$ is the $\mathbb{Z}$-module generated by $\Lambda(\varphi)$ in $\mathbb{R}$.

When $\omega = (\omega_1,...,\omega_N)$ is a list of N real numbers which are $\mathbb{Z}$-linearly independent, we set

$$\langle\omega\rangle = \left\{\sum_{j=1}^{k} l_j\omega_j : (l_1,...,l_N) \in \mathbb{Z}^N\right\}.$$

We set

$$QP^0_\omega(\mathbb{R}^n) = \left\{\varphi \in AP^0(\mathbb{R}^n) : \mathrm{Mod}(\varphi) \subset \langle\omega\rangle\right\}.$$

The functions which belong to $QP^0_\omega(\mathbb{R}^n)$ are so-called ω-quasi-periodic functions. We also set

$$QP^k_\omega(\mathbb{R}^n) = AP^k(\mathbb{R}^n) \cap QP^0_\omega(\mathbb{R}^n),$$

when $k \in \mathbb{N}$. It is a Banach subspace of $AP^k(\mathbb{R}^n)$.

When $\mathbb{T}^N$ denotes the usual N-dimensional torus, if $\varphi \in QP^k_\omega(\mathbb{R}^n)$ then there exists a unique $\phi \in C^k(\mathbb{T}^N,\mathbb{R}^n)$ such that $\varphi(t) = \phi(t\omega)$ for all $t \in \mathbb{R}$, [3].

By $W^{k,2}(\mathbb{T}^N,\mathbb{R}^n)$ we denote the space of Sobolev defined as follows:

$$W^{k,2}(\mathbb{T}^N,\mathbb{R}^n)=\{\phi\in L^2(\mathbb{T}^N,\mathbb{R}^n)|\ \forall\alpha=(\alpha_1,...,\alpha_N)\in\mathbb{N}^N \text{ such that } |\alpha|\leqslant k,\ D^\alpha\phi\in L^2(\mathbb{T}^N,\mathbb{R}^n)\},$$

where $D^\alpha\phi$ is the derivative of ϕ in the sense of Schwartz distributions, and $|\alpha|=\sum_{j=1}^N\alpha_j$.

Following [13, Definition 2.1, p.5,6], a function $g:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}^n$, $(t,x,u)\mapsto g(t,x,u)$, is called almost periodic in t uniformly for $(x,u)\in\mathbb{R}^n\times\mathbb{R}^p$ when g is continuous and satisfies the following property:

$$\begin{gathered}\forall\varepsilon>0,\forall\, K \text{ compact subset of } \mathbb{R}^n\times\mathbb{R}^p,\exists\, l_\varepsilon>0,\\ \forall\alpha\in\mathbb{R},\exists\tau\in[\alpha,\alpha+l_\varepsilon],\forall t\in\mathbb{R},\forall(x,u)\in K,\\ \|f(t+\tau,x,u)-f(t,x,u)\|\leqslant\varepsilon.\end{gathered}$$

We denote by $APU(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ the space of such functions as in [1], [2]. Ever following [13, Definition 2.2, p.6], when $g\in APU(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$, we define

$$\Lambda(g)=\{\lambda\in\mathbb{R}:\ \exists(x,u)\in\mathbb{R}^n\times\mathbb{R}^p,\ \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t,x,u)e^{-i\lambda t}dt\neq 0\}$$

and $\mathrm{Mod}(g)$ is the $\mathbb{Z}$-module generated by $\Lambda(g)$ in $\mathbb{R}$.

When $\omega=(\omega_1,...,\omega_N)$ is a list of N real numbers which are $\mathbb{Z}$-linearly independent, we define $QPU_\omega(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ as the set of the functions $g\in APU(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ such that $\mathrm{Mod}(g)\subset\langle\omega\rangle$. When $g\in QPU_\omega(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ there exists a unique $G\in C^0(\mathbb{T}^N\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ such that $g(t,x,u)=G(t\omega,x,u)$ for all $(t,x,u)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p$; see [3, Remark p.101].

We denote by $M_{p,n}(\mathbb{R})$ the space of the $p\times n$ real matrices, by $M_n(\mathbb{R})$ the space of the $n\times n$ real matrices, and we denote by $GL(n,\mathbb{R})$ the so-called general linear group of the $n\times n$ real invertible matrices.

3. The linear case

First we recall elements of the Floquet theory for quasi-periodic systems due to Z. Lin [9]. We consider the following homogeneous linear ordinary differential equation

$$\dot y(t)=A(t)y(t), \tag{3.1}$$

where

$$A\in QP^0_\omega(M_n(\mathbb{R})) \text{ and } A(t)=F(t\omega) \text{ for all } t\in\mathbb{R}, \tag{3.2}$$

where $F\in W^{\tau,2}(\mathbb{T}^N,M_n(\mathbb{R}))$ is such that $\int_{\mathbb{T}^N}F(u)du=0$, $\tau=2(N+1)\left(\frac{n(n+1)}{2}+1\right)$, and $\omega=(\omega_1,...,\omega_N)$ satisfies the following condition.

$$\begin{cases}\text{There exists } K(\omega)\in\ (0,\infty) \text{ such that, for all } (l_1,...,l_N)\in\mathbb{Z}^N_*,\\ |\sum_{j=1}^N l_j\omega_j|\geqslant K(\omega)(\sum_{j=1}^N|l_j|)^{-(N+1)}.\end{cases} \tag{3.3}$$

We can find some properties of the condition (3.3) in [9] and [10]. Note also that a condition of this kind is used in [12, p.24]. Under conditions (3.2) and (3.3), if

$Y(t) = col[y_1(t),...,y_n(t)]$ is a fundamental matrix of (3.1), where the notation means that the $y_j(t)$ are the columns of $Y(t)$, Z. Lin defines the following real numbers, for $j = 1,...,n$,

$$\beta_j = \lim_{k\to\infty} \frac{1}{t_k} \ln \|y_j(t_k)\| \text{ when } \lim_{k\to\infty} t_k\omega = 0 \text{ modulo } 2\pi,$$

see [9], [10].

Lin proves that these numbers are independent of the choice of the fundamental matrix $Y(t)$, and he calls them the FL-CER of A, where FL-CER is an abbreviation of Floquet and Characteristics Exponential Roots.

In the following lemma we improve a result of Z. Lin in [9, lemma 1, p.202] by weakening the assumption of differentiability. Precisely we replace the strong differentiability by the distributional differentiability.

LEMMA 3.1. *Let $f \in QP^0_\omega(\mathbb{R}^n)$ such that $f(t) = F(t\omega)$ for all $t \in \mathbb{R}$, where:*

$$F \in W^{\tau,2}(\mathbb{T}^N,\mathbb{R}^n),\ \tau = 2(N+1)\left(\tfrac{n(n+1)}{2}+1\right), \textit{ and } \omega \textit{ satisfy } (3.3).$$

Assume that $\int_{\mathbb{T}^N} F(u)du = 0$. Then the function $t \mapsto G(t) = \int_0^t f(s)ds$ belongs to $QP^0_\omega(\mathbb{R}^n)$.

Proof. $F(u)$ can be expressed as follows:

$$F(u) = \sum_{k\neq 0} a_k e^{i\langle k,u\rangle}.$$

Then we have:

$$\frac{\partial^\tau F(u)}{\partial u_j^\tau} = \sum_{k\neq 0} (ik_j)^\tau a_k e^{i\langle k,u\rangle}, \text{ for all } j = 1,...,N$$

and, for all $k = (k_1,...,k_N) \in \mathbb{Z}^N_*$,

$$(ik_j)^\tau a_k = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} ... \int_0^{2\pi} \left(\frac{\partial^\tau F(u)}{\partial u_j^\tau}\right) e^{-i\langle k,u\rangle} du \ .$$

Let $\|k\|_\infty = max|k_1|,...,|k_N|$, by taking j such that $\|k\|_\infty = |k_j|$, and by using the Cauchy-Schwarz inequality, we have:

$$\|k\|^\tau_\infty |a_k| \leqslant \left\|\frac{\partial^\tau F(u)}{\partial u_j^\tau}\right\|_{L^2(\mathbb{T}^N)} \left(\frac{1}{(2\pi)^N}\int_{Q_N} |e^{-i\langle k,u\rangle}|^2 du\right)^{\frac{1}{2}} = \left\|\frac{\partial^\tau F(u)}{\partial u_j^\tau}\right\|_{L^2(\mathbb{T}^N)},$$

where $Q_N = (0,2\pi)^N$, thus:

$$|a_k| \leqslant \|k\|^{-\tau}_\infty \left\|\frac{\partial^\tau F(u)}{\partial u_j^\tau}\right\|_{L^2(\mathbb{T}^N)}.$$

Let
$$M=\max_{1\leqslant j\leqslant N}\left\|\frac{\partial^{\tau}F(u)}{\partial u_j^{\tau}}\right\|_{L^2(\mathbb{T}^N)}.$$
Then we obtain:
$$|a_k|\leqslant M\|k\|_\infty^{-\tau}. \tag{3.4}$$
Now, for all integer $r\in\mathbb{N}_*$, we define:
$$C(r,N)=\sum_{\|k\|_\infty=r}1=2N(2r+1)^{N-1}=2N\sum_{j=1}^{N-1}C_j^{N-1}(2r)^j.$$
Therefore, there is a constant $C(N)$ such that:
$$C(r,N)\leqslant C(N)r^{N-1}. \tag{3.5}$$
Let $\|k\|_1=|k_1|+...+|k_N|$. Since $\|k\|_1\leqslant N\|k\|_\infty$, we have $\|k\|_\infty^{-\tau}\leqslant N^\tau\|k\|_1^{-\tau}$. Now combining this one with (3.3) and (3.4), we obtain
$$\begin{aligned}\sum_{\|k\|_\infty=r}\left|\frac{a_k}{\langle k,\omega\rangle}\right| &\leqslant \frac{1}{K(\omega)}\sum_{\|k\|_\infty=r}|a_k|.\|k\|_1^{N+1}\\ &\leqslant \frac{M}{K(\omega)}\sum_{\|k\|_\infty=r}\|k\|_\infty^{-\tau}\|k\|_1^{N+1}\\ &\leqslant \frac{M}{K(\omega)}\sum_{\|k\|_\infty=r}\|k\|_\infty^{-\tau}N^{N+1}\|k\|_\infty^{N+1}\\ &= \frac{M}{K(\omega)}N^{N+1}\sum_{\|k\|_\infty=r}\|k\|_\infty^{-\tau+(N+1)}\\ &= \frac{M}{K(\omega)}N^{N+1}\left(\sum_{\|k\|_\infty=r}1\right)r^{-\tau+(N+1)}\\ &= \frac{M}{K(\omega)}N^{N+1}C(r,N)r^{-\tau+(N+1)}\end{aligned}$$
and by using (3.5), we get
$$\sum_{\|k\|_\infty=r}\left|\frac{a_k}{\langle k,\omega\rangle}\right|\leqslant\frac{M}{K(\omega)}N^{N+1}C(N)r^{N-1}r^{-\tau+(N+1)}.$$
Thus we have proved that for all $r\in\mathbb{N}_*$,
$$\sum_{\|k\|_\infty=r}\left|\frac{a_k}{\langle k,\omega\rangle}\right|\leqslant C_0(N)r^{-\tau+2N}$$
where $C_0(N)=\frac{M}{K(\omega)}N^{N+1}C(N)$. Hence
$$\sum_{r=1}\sum_{\|k\|_\infty=r}\left|\frac{a_k}{\langle k,\omega\rangle}\right|\leqslant\sum_{r=1}C_0(N)r^{-\tau+2N}$$

and if $\tau = s + 2(N+1)$, where $s \in \mathbb{N}_*$, we have

$$\sum_{r=1} \sum_{\|k\|_\infty = r} \left| \frac{a_k}{\langle k, \omega \rangle} \right| \leqslant C_0(N) \sum_{r=1} r^{-(s+2)} < \infty.$$

Hence the series $\sum_{k \neq 0} \frac{a_k}{\langle k, \omega \rangle} e^{i\langle k, u \rangle}$ converges absolutely and G is a quasi-periodic function.

REMARK 3.2. Using this lemma, in Theorem 3 in [9, p.210], we can replace the assumption $A \in C^\tau(M_n(\mathbb{R}))$ by (3.2) to get the following theorem.

THEOREM 3.3. *Under* (3.2) *and* (3.3)*, there exists $C \in M_n(\mathbb{R})$ such that the FL-CER of A are the real parts of the eigenvalues of C, there exists $S \in QP^1_\omega(GL(n,\mathbb{R}))$ such that if z is a solution of the equation*

$$\dot{z}(t) = Cz(t), \tag{3.6}$$

then $t \mapsto y(t) = S(t)z(t)$ is a solution of (3.1)*, and conversely if y is a solution of* (3.1) *then $t \mapsto z(t) = S(t)^{-1}y(t)$ is a solution of* (3.6)*.*

It is not difficult to verify that the transformation S satisfies the following relation for all $t \in \mathbb{R}$,

$$\dot{S}(t) = A(t)S(t) - S(t)C. \tag{3.7}$$

We also recall a classical result, due to Bohr and Neugebauer, on the constant coefficients linear systems [11].

THEOREM 3.4. *Let $\Omega \in M_n(\mathbb{R})$ be such that the real parts of all the eigenvalues of Ω are non zero. Then for all $d \in AP^0(\mathbb{R}^n)$ there exists a unique $z_d \in AP^1(\mathbb{R}^n)$ which is a solution of the following equation*

$$\dot{z}(t) = \Omega z(t) + d(t). \tag{3.8}$$

Moreover there exists a constant $\alpha \in (0,\infty)$ such that $\|z_d\|_\infty \leqslant \alpha.\|d\|_\infty$ for all $d \in AP^0(\mathbb{R}^n)$.

DEFINITION 3.5. We so-call the Bohr-Neugebauer constant the least constant α which satisfies the last assertion of the Bohr-Neugebauer theorem.

LEMMA 3.6. *Let $A \in QP^0_\omega(M_n(\mathbb{R}))$ which satisfies* (3.2) *and* (3.3) *and the following condition:*

$$\text{the FL-CER } \beta_1, \ldots, \beta_n \text{ of } A \text{ are non zero}. \tag{3.9}$$

Then for all $b \in QP^0_\omega(\mathbb{R}^n)$ there exists a unique $y_b \in QP^1_\omega(\mathbb{R}^n)$ which is a solution of (1.4)*. Moreover there exists a constant $\gamma \in (0,\infty)$ such that $\|y_b\|_\infty \leqslant \gamma \|b\|_\infty$ for all $b \in QP^0_\omega(\mathbb{R}^n)$.*

Proof. We consider C and S provided by Theorem 3.3. Let $b \in QP^0_\omega(\mathbb{R}^n)$ be arbitrarily chosen. We set $d(t) = S(t)^{-1}b(t)$, and then we have $d \in QP^0_\omega(\mathbb{R}^n)$. Since $\beta_1,...,\beta_n$ are the real parts of the eigenvalues of C, condition (3.9) permits us to use the Bohr-Neugebauer theorem with $\Omega = C$, and so we can assert that there exists a unique $z_d \in QP^1_\omega(\mathbb{R}^n)$ such that $\dot{z}_d(t) = Cz_d(t) + d(t)$ for all $t \in \mathbb{R}$. Now we set $y_b(t) = S(t).z_d(t)$, then we have $y_b \in QP^1_\omega(\mathbb{R}^n)$ and by using (3.7), we obtain, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\dot{y}_b(t) &= \dot{S}(t)z_d(t) + S(t)\dot{z}_d(t) \\
&= [A(t)S(t) - S(t)C]z_d(t) + S(t)[Cz_d(t) + d(t)] \\
&= A(t)y_b(t) + 0 + S(t)d(t) \\
&= A(t)y_b(t) + b(t).
\end{aligned}$$

That proves the existence.

If $y \in QP^1_\omega(\mathbb{R}^n)$ also satisfies $\dot{y}(t) = A(t)y(t) + b(t)$, for all $t \in \mathbb{R}$, by setting $z(t) = S(t)^{-1}y(t)$, we verify that $\dot{z}(t) = Cz(t) + d(t)$ and the uniqueness provided by the Bohr-Neugebauer theorem implies $z = z_d$ which implies $y = y_b$. And the uniqueness is proven.

We denote by α the Bohr-Neugebauer constant of C. Since S and $S^{-1} = [t \mapsto S(t)^{-1}]$ are quasi-periodic, they are bounded on $\mathbb{R}$, and consequently we have:

$$\begin{aligned}
\|y_b\|_\infty = \|Sz_d\|_\infty &\leqslant \|S\|_\infty.\|z_d\|_\infty \\
&\leqslant \|S\|_\infty \alpha \|d\|_\infty = \|S\|_\infty \alpha \|S^{-1}b\|_\infty \\
&\leqslant \|S\|_\infty \alpha \|S^{-1}\|_\infty \|b\|_\infty,
\end{aligned}$$

and so it suffices to take $\gamma = \|S\|_\infty \alpha \|S^{-1}\|_\infty$.

DEFINITION 3.7. We call the Bohr-Neugebauer constant of A the least constant γ which satisfies the last assertion of Lemma 3.6.

4. An existence result

In this section we obtain an existence result by using the Z. Lin theorem and the Picard-Banach fixed point theorem.

THEOREM 4.1. *Let $A \in QP^0_\omega(M_n(\mathbb{R}))$ and $f \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. We assume that* (3.2)*,* (3.3) *and* (3.9) *are fulfilled. Let γ denote the Bohr-Neugebauer constant of A. We also assume that the following condition is fulfilled:*

$$\begin{cases} \text{there exists } c \in (0, (\|A\|\gamma + 1 + \gamma)^{-1}) \text{ such that} \\ \|f(t,x) - f(t,y)\| \leqslant c\|x - y\| \\ \text{for all } t \in \mathbb{R} \text{ and for all } x, y \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

Then equation (1.1) *possesses a unique solution in $QP^1_\omega(\mathbb{R}^n)$.*

Proof. We consider the following linear operator $L : QP^1_\omega(\mathbb{R}^n) \longrightarrow QP^0_\omega(\mathbb{R}^n)$ defined by $Lx = [t \mapsto \dot{x}(t) - A(t)x(t)]$. By using Lemma 3.6 we know that L is invertible, and for all $b \in QP^0_\omega(\mathbb{R}^n)$, $L^{-1}(b) = x_b$ the unique solution of $\dot{x}(t) = A(t)x(t) + b(t)$ in $QP^1_\omega(\mathbb{R}^n)$.

By using the Bohr-Neugebauer constant we know that $\|x_b\|_\infty \leqslant \gamma\|b\|_\infty$, and moreover we have $\|\dot{x}_b\|_\infty \leqslant \|A\|_\infty\|x_b\|_\infty + \|b\|_\infty \leqslant (\|A\|_\infty\gamma + 1)\|b\|_\infty$. And so we obtain $\|L^{-1}(b)\|_{C^1} \leqslant (\|A\|_\infty\gamma + 1 + \gamma)\|b\|_\infty$, that implies the following inequality for the norm of the inverse operator:

$$\|L^{-1}\|_{\mathscr{L}} \leqslant \|A\|_\infty\gamma + 1 + \gamma. \tag{4.2}$$

We note that when $x \in QP^0_\omega(\mathbb{R}^n)$ there exists $\varphi \in C^0(\mathbb{T}^N, \mathbb{R}^n)$ such that $x(t) = \varphi(t\omega)$ for all $t \in \mathbb{R}$; [3, Theorem 2, p.97]. Since $f \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, by using Remark p.101 in [3], we know that there exists $F \in C^0(\mathbb{T}^N \times \mathbb{R}^n, \mathbb{R}^n)$ such that $f(t,x) = F(t\omega, x)$ for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$. It is clear that the function ψ, defined by $\psi(\theta) = F(\theta, \varphi(\theta))$ for all $\theta \in \mathbb{T}^N$, belongs to $C^0(\mathbb{T}^N, \mathbb{R}^n)$ as a composition of continuous periodic functions. Consequently, we have $[t \mapsto f(t, x(t)) = \psi(t\omega)] \in QP^0_\omega(\mathbb{R}^n)$. And so the superposition operator build on f, $N_f : QP^0_\omega(\mathbb{R}^n) \longrightarrow QP^0_\omega(\mathbb{R}^n)$, $N_f(x) = [t \mapsto f(t, x(t))]$, is well defined. From the assumption (4.1) it is easy to obtain the following inequality:

$$\|N_f(x) - N_f(y)\|_\infty \leqslant c\|x - y\|_\infty \tag{4.3}$$

for all $x, y \in QP^0_\omega(\mathbb{R}^n)$.

Consequently by setting $c_1 = c(\|A\|_\infty\gamma + 1 + \gamma)^{-1}$ we have $c_1 \in (0,1)$ and by using (4.2) and (4.3), the following inequality holds:

$$\|L^{-1} \circ N_f(x) - L^{-1} \circ N_f(y)\|_\infty \leqslant c_1\|x - y\|_\infty$$

for all $x, y \in QP^0_\omega(\mathbb{R}^n)$. And so the operator $L^{-1} \circ N_f : QP^0_\omega(\mathbb{R}^n) \longrightarrow QP^0_\omega(\mathbb{R}^n)$ is a contraction. Then by using the Picard-Banach Fixed Point Theorem, we obtain that there exists a unique $x \in QP^0_\omega(\mathbb{R}^n)$ such that $L^{-1} \circ N_f(x) = x$.

We note that, for $x \in QP^0_\omega(\mathbb{R}^n)$, $L^{-1} \circ N_f(x) = x$ is equivalent to say that x is a solution of (1.1) in $QP^1_\omega(\mathbb{R}^n)$, and so the theorem is proven.

5. A continuous dependence result

In this section, we establish the existence of quasi-periodic solutions of equation (1.2) and a continuous dependence result with respect to the parameters functions u.

First we recall a theorem on fixed points which is proven in [14, p.103].

THEOREM 5.1. (Parametrized fixed point) *Let E be a complete metric space , let Λ be a topological space and let $\phi : E \times \Lambda \longrightarrow E$ be a mapping which satisfies the two following properties:*

$$\textit{for all } x \in E,\ \lambda \mapsto \phi(x, \lambda) \textit{ is continuous from } \Lambda \textit{ into } E, \tag{5.1}$$

and

$$\begin{cases} \text{there exists } k \in (0,1) \text{ such that,} \\ \text{for all } \lambda \in \Lambda \text{ and for all } x,y \in E, \\ \text{the following inequality holds:} \\ d(\phi(x,\lambda),\phi(y,\lambda)) \leqslant k.d(x,y). \end{cases} \tag{5.2}$$

Then, for all $\lambda \in \Lambda$, denoting by $a[\lambda]$ the unique fixed point of the partial mapping $\phi(.,\lambda)$, the mapping $\lambda \mapsto a[\lambda]$ is continuous from Λ into E.

THEOREM 5.2. *Let $A \in QP^0_\omega(M_n(\mathbb{R}))$ and $g \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n)$. We assume that* (3.2), (3.3) *and* (3.9) *are fulfilled, and γ denotes the Bohr-Neugebauer constant. We also assume that the following condition is fulfilled.*

$$\begin{cases} \text{There exists } d \in (0, (\|A\|_\infty \gamma + 1 + \gamma)^{-1}) \\ \text{such that } \|g(t,x,u) - g(t,y,u)\| \leqslant d.\|x-y\| \\ \text{for all } t \in \mathbb{R}, \text{ for all } x,y \in \mathbb{R}^n \text{ and for all } u \in \mathbb{R}^p. \end{cases} \tag{5.3}$$

Then, for all $u \in QP^0_\omega(\mathbb{R}^p)$ there exists a unique solution $\mathfrak{X}[u] \in QP^1_\omega(\mathbb{R}^n)$ of (1.2)*, and moreover the mapping $u \mapsto \mathfrak{X}[u]$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^1_\omega(\mathbb{R}^n)$.*

Proof. We consider the operator L defined in the proof of Theorem 4.1. By using on g arguments similar to these ones used on f in the proof of Theorem 4.1, we obtain that the superposition operator $N_g : QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p) \longrightarrow QP^0_\omega(\mathbb{R}^n)$, $N_g(x,u) = [t \mapsto g(t,x(t),u(t))]$, is well defined. By using (5.3) we easily verify that the following property holds:

$$\|N_g(x,u) - N_g(y,u)\|_\infty \leqslant d.\|x-y\|_\infty \tag{5.4}$$

for all $x,y \in QP^0_\omega(\mathbb{R}^n)$ and for all $u \in QP^0_\omega(\mathbb{R}^n)$.

We define the nonlinear operator $\phi : QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p) \longrightarrow QP^0_\omega(\mathbb{R}^n)$ by setting:

$$\phi(x,u) = L^{-1} \circ N_g(x,u) \text{ for all } (x,u) \in QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p).$$

With

$$E = QP^0_\omega(\mathbb{R}^n) \text{ and } \Lambda = QP^0_\omega(\mathbb{R}^p),$$

by using (4.2) and (5.4) by setting $k = d.(\|A\|_\infty \gamma + 1 + \gamma) \in (0,1)$, we see that ϕ satisfies (5.2). By using [2, Theorem 3.5, p.47], we know that

$$N^1_g : AP^0(\mathbb{R}^n) \times AP^0(\mathbb{R}^p) \longrightarrow AP^0(\mathbb{R}^n), \; N^1_g(x,u) = [t \mapsto g(t,x(t),u(t))],$$

is continuous, and since N_g is a restriction of N^1_g, N_g is also continuous. Since L^{-1} is linear continuous, ϕ is continuous as a composition of continuous operators, and consequently the partial operator $u \mapsto \phi(x,u)$ is continuous for all $x \in QP^0_\omega(\mathbb{R}^n)$, and so ϕ satisfies (5.1).

Now we can use the theorem of parametrized fixed point, and we can assert that, for all $u \in QP^0_\omega(\mathbb{R}^p)$ there exists a unique $\mathfrak{X}[u] = L^{-1} \circ N_g(\mathfrak{X}[u],u)$, and moreover the mapping $u \mapsto \mathfrak{X}[u]$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n)$.

To say that $\mathfrak{X}[u]$ satisfies the equation $\mathfrak{X}[u] = L^{-1} \circ N_g(\mathfrak{X}[u], u)$ is equivalent to say that $\mathfrak{X}[u] \in QP^1_\omega(\mathbb{R}^n)$ and $\mathfrak{X}[u]$ is a solution of (1.2).

We note that

$$\dot{\mathfrak{X}}[u](t) = A(t)\mathfrak{X}[u](t) + g(t, \mathfrak{X}[u](t), u(t)).$$

Since $u \mapsto \mathfrak{X}[u]$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n)$ and since $v \mapsto Av = [t \mapsto A(t)v(t)]$ is linear continuous from $QP^0_\omega(\mathbb{R}^n)$ into $QP^0_\omega(\mathbb{R}^n)$, we obtain that $u \mapsto Ax[u]$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n)$. We have yet seen that the superposition operator N_g is continuous from $QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n)$, and it is clear that the operator $u \mapsto (\mathfrak{X}[u], u)$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p)$, and so $u \mapsto N_g(\mathfrak{X}[u], u)$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n)$ as a composition of continuous operators. Finally $u \mapsto \dot{\mathfrak{X}}[u] = A\mathfrak{X}[u] + N_g(\mathfrak{X}[u], u)$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^0_\omega(\mathbb{R}^n)$ as a sum of continuous operators. Therefore $u \mapsto \mathfrak{X}[u]$ is continuous from $QP^0_\omega(\mathbb{R}^p)$ into $QP^1_\omega(\mathbb{R}^n)$.

6. A differentiable perturbation result

We fix $\omega = (\omega_1, ..., \omega_N)$ a list of $\mathbb{Z}$-linearly independent real numbers. We consider, about the vector-field of the equation (1.3), the following condition:

$$\begin{cases} g \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \cap C^{\tau-1}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n), \\ D_x g \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_n(\mathbb{R})) \cap C^\tau(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_n(\mathbb{R})) \\ \text{with } \tau = 2(N+1)(\frac{n(n+1)}{2} + 1), \text{ and} \\ D_u g \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_{n,p}(\mathbb{R})). \end{cases} \tag{6.1}$$

In this condition, $D_x g$ denotes the partial differential of g with respect to the second vector variable and $D_u g$ denotes the partial differential of g with respect to the third vector variable.

THEOREM 6.1. *Let $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be a function which satisfies* (6.1) *where ω satisfies* (3.3)*. Let $u_* \in QP^\tau_\omega(\mathbb{R}^p)$ and let $x_* \in QP^1_\omega(\mathbb{R}^n)$ be a solution of* (1.3) *where $u = u_*$. We set $J(t) = D_x g(t, x_*(t), u_*(t))$ for all $t \in \mathbb{R}$ and we denote by $\beta_1, ..., \beta_n$ the FL-CER of J. Moreover we assume that the following condition is fulfilled.*

$$\textit{For all } j = 1, ..., n, \quad \beta_j \textit{ is non zero.} \tag{6.2}$$

Then there exists $r \in (0, \infty)$ such that, for all $u \in QP^0_\omega(\mathbb{R}^p)$ satisfying $\|u - u_*\|_\infty < r$, there exists $\mathfrak{X}[u] \in QP^1_\omega(\mathbb{R}^n)$ which is a solution of (1.3)*.*

Moreover the nonlinear operator $u \mapsto \mathfrak{X}[u]$ is of class C^1 from $\{u \in QP^0_\omega(\mathbb{R}^p) : \|u - u* \|_\infty < r\}$ into $QP^1_\omega(\mathbb{R}^n)$, and there exists a neighborhood $\mathcal{N}$ of x_* in $QP^1_\omega(\mathbb{R}^n)$ such that $\mathfrak{X}[u]$ is the unique solution of (1.3) *in $QP^1_\omega(\mathbb{R}^n)$ which belongs to $\mathcal{N}$.*

Before to do the proof of this theorem we need a lemma of Differential Calculus.

LEMMA 6.2. *When $g \in QPU_\omega(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ is such that its partial differentials with respect to the second and the third vector variables exist and satisfy*

$$D_x g \in QPU_\omega(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p, M_n(\mathbb{R})) \text{ and } D_u g \in QPU_\omega(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p, M_{n,p}(\mathbb{R})),$$

then the operator

$$\Gamma : QP^1_\omega(\mathbb{R}^n)\times QP^0_\omega(\mathbb{R}^p) \longrightarrow QP^0_\omega(\mathbb{R}^n),\ \Gamma(x,u) = [t \mapsto \dot{x}(t) - g(t,x(t),u(t))],$$

is well-defined and it is of class C^1.

The formula of its partial differential with respect to its first variable is the following one:

$$D_1\Gamma(x_*,u_*).y = [t \mapsto \dot{y}(t) - D_x g(t,x_*(t),u_*(t)).y(t)]$$

for all $y \in QP^1_\omega(\mathbb{R}^n)$.

Proof. When $x \in QP^0_\omega(\mathbb{R}^n)$ and $u \in QP^0_\omega(\mathbb{R}^p)$ there exist $\varphi \in C^0(\mathbb{T}^N,\mathbb{R}^n)$ and $\psi \in C^0(\mathbb{T}^N,\mathbb{R}^p)$ such that $x(t) = \varphi(t\omega)$ and $u(t) = \psi(t\omega)$ for all $t \in \mathbb{R}$, [3, Theorem 2, p.97]. By using Remark p.101 in [3], since $g \in QPU_\omega(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$, there exists $G \in C^0(\mathbb{T}^N\times\mathbb{R}^n\times\mathbb{R}^p,\mathbb{R}^n)$ such that $g(t,x,u) = G(t\omega,x,u)$ for all $(t,x,u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p$. We set $\chi(\theta) = G(\theta,\varphi(\theta),\psi(\theta))$, then $\chi \in C^0(\mathbb{T}^N,\mathbb{R}^n)$ as a composition of continuous periodic functions. Consequently we have:

$$[t \mapsto g(t,x(t),u(t)) = \chi(t\omega)] \in QP^0_\omega(\mathbb{R}^n).$$

And so the operator Γ is well-defined.

By using Theorem 5.1, p.54 in [2], we know that the superposition operator:

$$N^1_g : AP^0(\mathbb{R}^n)\times AP^0(\mathbb{R}^p) \longrightarrow AP^0(\mathbb{R}^n),\ N^1_g(x,u) = [t \mapsto g(t,x(t),u(t))],$$

is of class C^1. And so the superposition operator

$$N_g : QP^0_\omega(\mathbb{R}^n)\times QP^0_\omega(\mathbb{R}^p) \longrightarrow QP^0_\omega(\mathbb{R}^n),\ N_g(x,u) = [t \mapsto g(t,x(t),u(t))]$$

is of class C^1 as a restriction of N^1_g. And so the following assertion holds:

$$N_g \in C^1\left(QP^0_\omega(\mathbb{R}^n)\times QP^0_\omega(\mathbb{R}^p), QP^0_\omega(\mathbb{R}^n)\right). \tag{6.3}$$

The operator $\Pi_1 : QP^1_\omega(\mathbb{R}^n)\times QP^0_\omega(\mathbb{R}^p) \longrightarrow QP^1_\omega(\mathbb{R}^n)$, defined by $\Pi_1(x,u) = x$, is linear continuous, therefore the following assertion holds:

$$\Pi_1 \in C^1(QP^1_\omega(\mathbb{R}^n)\times QP^0_\omega(\mathbb{R}^p), QP^1_\omega(\mathbb{R}^n)). \tag{6.4}$$

The operator $\frac{d}{dt} : QP^1_\omega(\mathbb{R}^n) \longrightarrow QP^0_\omega(\mathbb{R}^n)$, $\frac{d}{dt}x = \dot{x}$, is linear continuous, therefore the following assertion holds:

$$\frac{d}{dt} \in C^1(QP^1_\omega(\mathbb{R}^n), QP^0_\omega(\mathbb{R}^n)). \tag{6.5}$$

The operator $in : QP^1_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p) \longrightarrow QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p)$, $in(x,u) = (x,u)$, is linear continuous, and so the following assertion holds:

$$in \in C^1(QP^1_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p), QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p)). \tag{6.6}$$

We note that $\Gamma = \frac{d}{dt} \circ \Pi_1 - N_g \circ in$, and so by using (6.3)-(6.6), Γ is of class C^1 as the difference of compositions of operators of class C^1.

Now, by using Theorem 5.1, p.54, in [2] and the chain rule of the differential calculus in Banach spaces, we obtain the following calculations:

$$\begin{aligned} D_1\Gamma(x_*,u_*).y &= D\Gamma(x_*,u_*).(y,0) \\ &= D\left(\frac{d}{dt} \circ \Pi_1\right)(x_*,u_*).(y,0) - D(N_g \circ in)(x_*,u_*).(y,0) \\ &= \frac{d}{dt} \circ \Pi_1(y,0) - DN_g(x_*,u_*).(y,0) \\ &= [t \longmapsto \dot{y}(t) - D_x g(t,x_*(t),u_*(t)).y(t)]. \end{aligned}$$

PROOF OF THEOREM 6.1. Since g is of class $C^{\tau-1}$, by using a bootstrapping argument we see that x_* is also of class C^τ. And so the matrix $J(t)$ satisfies the condition (3.2). The assumption (6.2) ensures that (3.9) is fulfilled for $A = J$. And so can use Lemma 3.6 to assert that for all $b \in QP^0_\omega(\mathbb{R}^n)$, there exists a unique $y \in QP^1_\omega(\mathbb{R}^n)$ such that $\dot{y}(t) = J(t)y(t) + b(t)$ for all $t \in \mathbb{R}$. And so, by using Lemma 6.2, we can translate this result in the following form:

$$D_1\Gamma(x_*,u_*) \text{ is a bijection from } QP^1_\omega(\mathbb{R}^n) \text{ onto } QP^0_\omega(\mathbb{R}^n). \tag{6.7}$$

Since $\dot{x}_*(t) = g(t,x_*(t),u_*(t))$ for all $t \in \mathbb{R}$, the following assertion holds:

$$\Gamma(x_*,u_*) = 0. \tag{6.8}$$

Since Γ is of class C^1, (6.7) and (6.8) permit to use the implicit function theorem of the differential calculus in Banach spaces, see [4, Theorem 4.7.1, p.61]. And so we can assert that there exist $\mathscr{V} = \left\{x \in QP^1_\omega(\mathbb{R}^n) : \|x - x_*\|_{C^1} < r\right\}$ with $r \in (0,\infty)$, a neighborhood $\mathscr{N}$ of u_* in $QP^0_\omega(\mathbb{R}^p)$, and a C^1-mapping $\mathfrak{X} : \mathscr{V} \longrightarrow \mathscr{N}$ such that, for all $(x,u) \in \mathscr{V} \times \mathscr{N}$, we have $\Gamma(x,u) = 0$ if and only if $x = \mathfrak{X}[u]$.

Notice that $\Gamma(x,u) = 0$ is equivalent to say that x is solution of (1.3) in $QP^1_\omega(\mathbb{R}^n)$. And so Theorem 6.1 is proven.

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