QUASI–PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS VIA THE FLOQUET–LIN THEORY

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Abstract. We use a Floquet theory for quasi-periodic linear ordinary differential equations due to Zhensheng Lin to obtain results on the quasi-periodic solutions of quasi-periodic nonlinear ordinary differential equations. First we obtain an existence result, secondly we obtain a result on the continuous dependence by using a parametrized fixed point theorem, and thirdly we obtain a local result on the differentiable dependence by using an implicit function theorem in function spaces.

1. Introduction

Our aim is to study quasi-periodic solutions of ordinary differential equations in the following forms:

\[ \dot{x}(t) = A(t)x(t) + f(t, x(t)), \]  
\[ \dot{x}(t) = A(t)x(t) + g(t, x(t), u(t)), \]  
\[ \dot{x}(t) = g(t, x(t), u(t)), \]

where \( A \) is a quasi-periodic matrix, \( u \) is a quasi-periodic function (a forcing term or a control term), \( f \) and \( g \) are quasi-periodic with respect to \( t \).

To treat these problems we use the properties of the following forced linear ordinary differential equation

\[ \dot{x}(t) = A(t)x(t) + b(t), \]

where \( b \) is a quasi-periodic function. To study equation (1.4) we use a Floquet theory of quasi-periodic equations due to Zhensheng Lin [8], [9], [10], and several tools of Nonlinear Functional Analysis.

In Section 2 we fix our notation on the quasi-periodic function spaces.

In Section 3 we recall results of Lin and we use them to study equation (1.4), notably to obtain a generalization to (1.4) of a classical theorem of Bohr and Neugebauer on the constant coefficients systems.

In Section 4, by using results of Section 3, we build a Fixed Point approach to obtain an existence result on quasi-periodic solutions of equation (1.1).


\textit{Keywords and phrases}: quasi-periodic solutions, Floquet theory, fixed-point theorem, implicit function theorem.
In Section 5, by using results of Section 3, we build a Parametrized Fixed Point approach to obtain an existence result and a continuous dependence results on quasi-periodic solutions of equation (1.2).

In Section 6, by using results of Section 3, we build an Implicit Function Theorem approach to obtain a differentiable perturbation result on the quasi-periodic solutions of equation (1.3).

2. Notation

$AP^0(\mathbb{R}^n)$ denotes the space of the almost periodic functions from $\mathbb{R}$ into $\mathbb{R}^n$ in the sense of H. Bohr, [6], [7], [5]. Endowed with the norm $\| \varphi \|_{\infty} = \sup_{t \in \mathbb{R}} \| \varphi(t) \|$, it is a Banach space.

When $k \in \mathbb{N}^\ast = \mathbb{N} \backslash \{0\}$, $C^k(\mathbb{R}, \mathbb{R}^n)$ denotes the space of the functions from $\mathbb{R}$ into $\mathbb{R}^n$ which are of class $C^k$. $AP^k(\mathbb{R}^n)$ denotes the space of the functions $\varphi \in AP^k(\mathbb{R}^n) \cap C^k(\mathbb{R}, \mathbb{R}^n)$ such that the derivatives $\frac{d^j \varphi}{dt^j}$ belong to $AP^0(\mathbb{R}^n)$ for all $j = 1, \ldots, k$. Endowed with the norm:

$$\| \varphi \|_{C^k} = \| \varphi \|_{\infty} + \sum_{j=1}^k \| \frac{d^j \varphi}{dt^j} \|_{\infty},$$

$AP^k(\mathbb{R}^n)$ is a Banach space.

When $\varphi \in AP^0(\mathbb{R}^n)$ and when $\lambda \in \mathbb{R}$, we consider the Fourier-Bohr coefficient

$$a(\varphi, \lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \varphi(t)e^{-it\lambda} dt.$$

We set

$$\Lambda(\varphi) = \{ \lambda \in \mathbb{R} : a(\varphi, \lambda) \neq 0 \}$$

and $\text{Mod}(\varphi)$ is the $\mathbb{Z}$-module generated by $\Lambda(\varphi)$ in $\mathbb{R}$.

When $\omega = (\omega_1, \ldots, \omega_N)$ is a list of $N$ real numbers which are $\mathbb{Z}$-linearly independent, we set

$$\langle \omega \rangle = \left\{ \sum_{j=1}^k l_j \omega_j : (l_1, \ldots, l_N) \in \mathbb{Z}^N \right\}.$$

We set

$$QP^0_\omega(\mathbb{R}^n) = \{ \varphi \in AP^0(\mathbb{R}^n) : \text{Mod}(\varphi) \subset \langle \omega \rangle \}.$$

The functions which belong to $QP^0_\omega(\mathbb{R}^n)$ are so-called $\omega$-quasi-periodic functions. We also set

$$QP^k_\omega(\mathbb{R}^n) = AP^k(\mathbb{R}^n) \cap QP^0_\omega(\mathbb{R}^n),$$

when $k \in \mathbb{N}$. It is a Banach subspace of $AP^k(\mathbb{R}^n)$.

When $\mathbb{T}^N$ denotes the usual $N$-dimensional torus, if $\varphi \in QP^k_\omega(\mathbb{R}^n)$ then there exists a unique $\phi \in C^k(\mathbb{T}^N, \mathbb{R}^n)$ such that $\varphi(t) = \phi(t\omega)$ for all $t \in \mathbb{R}$, [3].

By $W^{k,2}(\mathbb{T}^N, \mathbb{R}^n)$ we denote the space of Sobolev defined as follows:
We consider the following homogeneous linear ordinary differential equation:

\[ W^{k,2}(\mathbb{T}^N, \mathbb{R}^n) = \{ \phi \in L^2(\mathbb{T}^N, \mathbb{R}^n) \mid \forall \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N \] such that \(|\alpha| \leq k, \ D^\alpha \phi \in L^2(\mathbb{T}^N, \mathbb{R}^n) \}, \]

where \( D^\alpha \phi \) is the derivative of \( \phi \) in the sense of Schwartz distributions, and \(|\alpha| = \sum_{j=1}^N \alpha_j \).

Following [13, Definition 2.1, p.5,6], a function \( g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \), \((t, x, u) \mapsto g(t, x, u)\), is called almost periodic in \( t \) uniformly for \((x, u) \in \mathbb{R}^n \times \mathbb{R}^p \) when \( g \) is continuous and satisfies the following property:

\[ \forall \varepsilon > 0, \forall K \text{ compact subset of } \mathbb{R}^n \times \mathbb{R}^p, \exists l_\varepsilon > 0, \] 
\[ \forall \alpha \in \mathbb{R}, \exists \tau \in [\alpha, \alpha + l_\varepsilon], \forall t \in \mathbb{R}, \forall (x, u) \in K, \] 
\[ ||f(t + \tau, x, u) - f(t, x, u)|| \leq \varepsilon. \]

We denote by \( APU(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) the space of such functions as in [1], [2]. Ever following [13, Definition 2.2, p.6], when \( g \in APU(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \), we define

\[ \Lambda(g) = \{ \lambda \in \mathbb{R} : \exists (x, u) \in \mathbb{R}^n \times \mathbb{R}^p, \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T g(t, x, u)e^{-i\lambda t} dt \neq 0 \} \]

and \( \text{Mod}(g) \) is the \( \mathbb{Z} \)-module generated by \( \Lambda(g) \) in \( \mathbb{R} \).

When \( \omega = (\omega_1, \ldots, \omega_N) \) is a list of \( N \) real numbers which are \( \mathbb{Z} \)-linearly independent, we define \( \text{QPU}_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) as the set of the functions \( g \in APU(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) such that \( \text{Mod}(g) \subset \langle \omega \rangle \). When \( g \in \text{QPU}_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) there exists a unique \( G \in C^0(\mathbb{T}^N \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) such that \( g(t, x, u) = G(t\omega, x, u) \) for all \((t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \); see [3, Remark p.101].

We denote by \( M_{p,n}(\mathbb{R}) \) the space of the \( p \times n \) real matrices, by \( M_n(\mathbb{R}) \) the space of the \( n \times n \) real matrices, and we denote by \( GL(n, \mathbb{R}) \) the so-called general linear group of the \( n \times n \) real invertible matrices.

### 3. The linear case

First we recall elements of the Floquet theory for quasi-periodic systems due to Z. Lin [9]. We consider the following homogeneous linear ordinary differential equation

\[ \dot{y}(t) = A(t)y(t), \] (3.1)

where

\[ A \in QPU_\omega(M_n(\mathbb{R})) \text{ and } A(t) = F(t\omega) \text{ for all } t \in \mathbb{R}, \] (3.2)

where \( F \in W^{\tau,2}(\mathbb{T}^N, M_n(\mathbb{R})) \) is such that \( \int_{\mathbb{T}^N} F(u)du = 0, \tau = 2(N + 1) \left( \frac{n(n+1)}{2} + 1 \right) \), and \( \omega = (\omega_1, \ldots, \omega_N) \) satisfies the following condition.

\[ \begin{cases} \text{There exists } K(\omega) \in (0, \infty) \text{ such that, for all } (l_1, \ldots, l_N) \in \mathbb{Z}^N_+, \quad |\sum_{j=1}^N l_j \omega_j| \geq K(\omega)(\sum_{j=1}^N |l_j|)^{-(N+1)}. \end{cases} \] (3.3)

We can find some properties of the condition (3.3) in [9] and [10]. Note also that a condition of this kind is used in [12, p.24]. Under conditions (3.2) and (3.3), if
\[ Y(t) = \text{col}[y_1(t),...,y_n(t)] \] is a fundamental matrix of (3.1), where the notation means that the \( y_j(t) \) are the columns of \( Y(t) \), \( Z \). Lin defines the following real numbers, for \( j = 1,...,n \),

\[ \beta_j = \lim_{k \to \infty} \frac{1}{t_k} \ln \| y_j(t_k) \| \text{ when } \lim_{k \to \infty} t_k \omega = 0 \mod 2\pi, \]

see [9], [10].

Lin proves that these numbers are independent of the choice of the fundamental matrix \( Y(t) \), and he calls them the FL-CER of \( A \), where FL-CER is an abbreviation of Floquet and Characteristics Exponential Roots.

In the following lemma we improve a result of Z. Lin in [9, lemma 1, p.202] by weakening the assumption of differentiability. Precisely we replace the strong differentiability by the distributional differentiability.

**Lemma 3.1.** Let \( f \in QP^0_\omega(\mathbb{R}^n) \) such that \( f(t) = F(t)\omega \) for all \( t \in \mathbb{R} \), where:

\[ F \in W^{\tau,2}(\mathbb{T}^N,\mathbb{R}^n), \quad \tau = 2(N+1) \left( \frac{n(n+1)}{2} + 1 \right), \text{ and } \omega \text{ satisfy (3.3).} \]

Assume that \( \int_{\mathbb{T}^N} F(u) du = 0 \). Then the function \( t \mapsto G(t) = \int_0^t f(s) ds \) belongs to \( QP^0_\omega(\mathbb{R}^n) \).

**Proof.** \( F(u) \) can be expressed as follows:

\[ F(u) = \sum_{k \neq 0} a_k e^{i(k,u)}. \]

Then we have:

\[ \frac{\partial^\tau F(u)}{\partial u_j^\tau} = \sum_{k \neq 0} (ik_j)^\tau a_k e^{i(k,u)}, \text{ for all } j = 1,...,N \]

and, for all \( k = (k_1,...,k_N) \in \mathbb{Z}_+^N \),

\[ (ik_j)^\tau a_k = \left( \frac{1}{2\pi} \right)^N \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \frac{\partial^\tau F(u)}{\partial u_j^\tau} \right) e^{-i(k,u)} du. \]

Let \( \|k\|_\infty = \max |k_1|,...,|k_N| \), by taking \( j \) such that \( \|k\|_\infty = |k_j| \), and by using the Cauchy-Schwarz inequality, we have:

\[ \|k\|_\infty \|a_k\| \leq \left\| \frac{\partial^\tau F(u)}{\partial u_j^\tau} \right\|_{L^2(\mathbb{T}^N)} \left( \frac{1}{2\pi} \right)^N \int_{Q_N} |e^{-i(k,u)}|^2 du \right)^{\frac{1}{2}} = \left\| \frac{\partial^\tau F(u)}{\partial u_j^\tau} \right\|_{L^2(\mathbb{T}^N)}, \]

where \( Q_N = (0,2\pi)^N \), thus:

\[ |a_k| \leq \|k\|_\infty^{-\tau} \left\| \frac{\partial^\tau F(u)}{\partial u_j^\tau} \right\|_{L^2(\mathbb{T}^N)}. \]
Let
\[ M = \max_{1 \leq j \leq N} \left\| \frac{\partial^\tau F(u)}{\partial u_j^\tau} \right\|_{L^2(\mathbb{T}^N)}. \]
Then we obtain:
\[ |a_k| \leq M \| k \|_{\infty}^{-\tau}. \] (3.4)

Now, for all integer \( r \in \mathbb{N}_* \), we define:
\[ C(r, N) = \sum_{\| k \|_{\infty} = r} 1 = 2N(2r + 1)^{N-1} = 2N \sum_{j=1}^{N-1} C_j^{N-1}(2r)^j. \]

Therefore, there is a constant \( C(N) \) such that:
\[ C(r, N) \leq C(N) r^{N-1}. \] (3.5)

Let \( \| k \|_1 = |k_1| + \ldots + |k_N| \). Since \( \| k \|_1 \leq N \| k \|_{\infty} \), we have \( \| k \|_1^{-\tau} \leq N^\tau \| k \|_{\infty}^{-\tau} \). Now combining this one with (3.3) and (3.4), we obtain
\[
\sum_{\| k \|_{\infty} = r} \left| \frac{a_k}{\langle k, \omega \rangle} \right| \leq \frac{1}{K(\omega)} \sum_{\| k \|_{\infty} = r} |a_k| \cdot \| k \|_1^{N+1}
\leq \frac{M}{K(\omega)} \sum_{\| k \|_{\infty} = r} \| k \|_{\infty}^{-\tau} \cdot \| k \|_1^{N+1}
\leq \frac{M}{K(\omega)} \sum_{\| k \|_{\infty} = r} \| k \|_{\infty}^{-\tau} \cdot N^{N+1} \cdot \| k \|_1^{N+1}
= \frac{M}{K(\omega)} N^{N+1} \sum_{\| k \|_{\infty} = r} \| k \|_{\infty}^{-\tau} \cdot (N+1)\]
= \frac{M}{K(\omega)} N^{N+1} \left( \sum_{\| k \|_{\infty} = r} 1 \right) r^{N-\tau+(N+1)}
\leq \frac{M}{K(\omega)} N^{N+1} C(r, N) r^{N-\tau+(N+1)}
\]
and by using (3.5), we get
\[
\sum_{\| k \|_{\infty} = r} \left| \frac{a_k}{\langle k, \omega \rangle} \right| \leq \frac{M}{K(\omega)} N^{N+1} C(N) r^{N-1} r^{N-\tau+(N+1)}.
\]

Thus we have proved that for all \( r \in \mathbb{N}_* \),
\[
\sum_{\| k \|_{\infty} = r} \left| \frac{a_k}{\langle k, \omega \rangle} \right| \leq C_0(N) r^{-\tau+2N}
\]
where \( C_0(N) = \frac{M}{K(\omega)} N^{N+1} C(N) \). Hence
\[
\sum_{r=1}^N \sum_{\| k \|_{\infty} = r} \left| \frac{a_k}{\langle k, \omega \rangle} \right| \leq \sum_{r=1}^N C_0(N) r^{-\tau+2N}
\]
and if $\tau = s + 2(N + 1)$, where $s \in \mathbb{N}_*$, we have

$$\sum_{r=1}^{\infty} \sum_{\|k\| = r} \left| \frac{a_k}{\langle k, \omega \rangle} \right| \leq C_0(N) \sum_{r=1}^{\infty} r^{-(r+2)} < \infty.$$  

Hence the series $\sum_{k \neq 0} \frac{a_k}{\langle k, \omega \rangle} e^{i\langle k, u \rangle}$ converges absolutely and $G$ is a quasi-periodic function.

**Remark 3.2.** Using this lemma, in Theorem 3 in [9, p.210], we can replace the assumption $A \in C^\tau(M_n(\mathbb{R}))$ by (3.2) to get the following theorem.

**Theorem 3.3.** Under (3.2) and (3.3), there exists $C \in M_n(\mathbb{R})$ such that the FL-CER of $A$ are the real parts of the eigenvalues of $C$, there exists $S \in \text{QP}_\omega^1(GL(n, \mathbb{R}))$ such that if $z$ is a solution of the equation

$$\dot{z}(t) = Cz(t),$$  

then $t \mapsto y(t) = S(t)z(t)$ is a solution of (3.1), and conversely if $y$ is a solution of (3.1) then $t \mapsto z(t) = S(t)^{-1}y(t)$ is a solution of (3.6).

It is not difficult to verify that the transformation $S$ satisfies the following relation for all $t \in \mathbb{R}$,

$$\dot{S}(t) = A(t)S(t) - S(t)C. \quad (3.7)$$

We also recall a classical result, due to Bohr and Neugebauer, on the constant coefficients linear systems [11].

**Theorem 3.4.** Let $\Omega \in M_n(\mathbb{R})$ be such that the real parts of all the eigenvalues of $\Omega$ are non zero. Then for all $d \in \text{AP}_\omega^0(\mathbb{R}^n)$ there exists a unique $z_d \in \text{AP}_\omega^1(\mathbb{R}^n)$ which is a solution of the following equation

$$\dot{z}(t) = \Omega z(t) + d(t). \quad (3.8)$$

Moreover there exists a constant $\alpha \in (0, \infty)$ such that $\|z_d\|_\infty \leq \alpha \|d\|_\infty$ for all $d \in \text{AP}_\omega^0(\mathbb{R}^n)$.

**Definition 3.5.** We so-call the Bohr-Neugebauer constant the least constant $\alpha$ which satisfies the last assertion of the Bohr-Neugebauer theorem.

**Lemma 3.6.** Let $A \in \text{QP}_\omega^0(M_n(\mathbb{R}))$ which satisfies (3.2) and (3.3) and the following condition:

the FL-CER $\beta_1, \ldots, \beta_n$ of $A$ are non zero. \quad (3.9)  

Then for all $b \in \text{QP}_\omega^0(\mathbb{R}^n)$ there exists a unique $y_b \in \text{QP}_\omega^1(\mathbb{R}^n)$ which is a solution of (1.4). Moreover there exists a constant $\gamma \in (0, \infty)$ such that $\|y_b\|_\infty \leq \gamma \|b\|_\infty$ for all $b \in \text{QP}_\omega^0(\mathbb{R}^n)$.  

Proof. We consider $C$ and $S$ provided by Theorem 3.3. Let $b \in \mathcal{P}_{\omega}^{0}(\mathbb{R}^{n})$ be arbitrarily chosen. We set $d(t) = S(t)^{-1}b(t)$, and then we have $d \in \mathcal{P}_{\omega}^{0}(\mathbb{R}^{n})$. Since $\beta_{1},...,\beta_{n}$ are the real parts of the eigenvalues of $C$, condition (3.9) permits us to use the Bohr-Neugebauer theorem with $\Omega = C$, and so we can assert that there exists a unique $z_{d} \in \mathcal{P}_{\omega}^{0}(\mathbb{R}^{n})$ such that $\dot{z}_{d}(t) = Cz_{d}(t) + d(t)$ for all $t \in \mathbb{R}$. Now we set $y_{b}(t) = S(t)z_{d}(t)$, then we have $y_{b} \in \mathcal{P}_{\omega}^{1}(\mathbb{R}^{n})$ and by using (3.7), we obtain, for all $t \in \mathbb{R}$,

$$
\dot{y}_{b}(t) = S(t)z_{d}(t) + S(t)\dot{z}_{d}(t) = [A(t)S(t) - S(t)C]z_{d}(t) + S(t)[Cz_{d}(t) + d(t)] = A(t)y_{b}(t) + 0 + S(t)d(t) = A(t)y_{b}(t) + b(t).
$$

That proves the existence.

If $y \in \mathcal{P}_{\omega}^{1}(\mathbb{R}^{n})$ also satisfies $\dot{y}(t) = A(t)y(t) + b(t)$, for all $t \in \mathbb{R}$, by setting $z(t) = S(t)^{-1}y(t)$, we verify that $\dot{z}(t) = Cz(t) + d(t)$ and the uniqueness provided by the Bohr-Neugebauer theorem implies $z = z_{d}$ which implies $y = y_{b}$. And the uniqueness is proven.

We denote by $\alpha$ the Bohr-Neugebauer constant of $C$. Since $S$ and $S^{-1} = [t \mapsto S(t)^{-1}]$ are quasi-periodic, they are bounded on $\mathbb{R}$, and consequently we have:

$$
\|y_{b}\|_{\infty} = \|Sz_{d}\|_{\infty} \leq \|S\|_{\infty}\|z_{d}\|_{\infty} \\
\leq \|S\|_{\infty}\alpha\|d\|_{\infty} = \|S\|_{\infty}\alpha\|S^{-1}b\|_{\infty} \\
\leq \|S\|_{\infty}\alpha\|S^{-1}\|_{\infty}\|b\|_{\infty},
$$

and so it suffices to take $\gamma = \|S\|_{\infty}\alpha\|S^{-1}\|_{\infty}$.

**Definition 3.7.** We call the Bohr-Neugebauer constant of $A$ the least constant $\gamma$ which satisfies the last assertion of Lemma 3.6.

**4. An existence result**

In this section we obtain an existence result by using the Z. Lin theorem and the Picard-Banach fixed point theorem.

**Theorem 4.1.** Let $A \in \mathcal{P}_{\omega}^{0}(M_{n}(\mathbb{R}))$ and $f \in \mathcal{P}_{\omega}^{0}(\mathbb{R} \times \mathbb{R}^{n}, \mathbb{R}^{n})$. We assume that (3.2), (3.3) and (3.9) are fulfilled. Let $\gamma$ denote the Bohr-Neugebauer constant of $A$. We also assume that the following condition is fulfilled:

$$
\begin{cases}
\text{there exists } c \in (0,(\|A\|\gamma + 1 + \gamma)^{-1}) \text{ such that } \\
\|f(t,x) - f(t,y)\| \leq c\|x - y\| \\
\text{for all } t \in \mathbb{R} \text{ and for all } x,y \in \mathbb{R}^{n}.
\end{cases}
$$

Then equation (1.1) possesses a unique solution in $\mathcal{P}_{\omega}^{1}(\mathbb{R}^{n})$. 
Proof. We consider the following linear operator \( L : QP^1_\omega(\mathbb{R}^n) \rightarrow QP^0_\omega(\mathbb{R}^n) \) defined by \( Lx = [t \mapsto x(t) - A(t)x(t)] \). By using Lemma 3.6 we know that \( L \) is invertible, and for all \( b \in QP^0_\omega(\mathbb{R}^n) \), \( L^{-1}(b) = x_b \) the unique solution of \( \dot{x}(t) = A(t)x(t) + b(t) \) in \( QP^1_\omega(\mathbb{R}^n) \).

By using the Bohr-Neugebauer constant we know that \( \|x_b\|_\infty \leq \|b\|_\infty \), and moreover we have \( \|\dot{x}_b\| \leq \|A\|\|x_b\| + \|b\| \leq (\|A\|\|\gamma + 1\|)\|b\|_\infty \). And so we obtain

\[
\|L^{-1}(b)\| \leq (\|A\|\|\gamma + 1\|)\|b\|_\infty,
\]

that implies the following inequality for the norm of the inverse operator:

\[
\|L^{-1}\| \leq \|A\|\|\gamma + 1\|. \tag{4.2}
\]

We note that when \( x \in QP^0_\omega(\mathbb{R}^n) \) there exists \( \varphi \in C^0(\mathbb{T}^N, \mathbb{R}^n) \) such that \( x(t) = \varphi(t\omega) \) for all \( t \in \mathbb{R} \); [3, Theorem 2, p.97]. Since \( f \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \), by using Remark p.101 in [3], we know that there exists \( F \in C^0(\mathbb{T}^N \times \mathbb{R}^n, \mathbb{R}^n) \) such that \( f(t,x) = F(t\omega, x) \) for all \( t \in \mathbb{R} \) and for all \( x \in \mathbb{R}^n \). It is clear that the function \( \psi \), defined by \( \psi(\theta) = F(\theta, \varphi(\theta)) \) for all \( \theta \in \mathbb{T}^N \), belongs to \( C^0(\mathbb{T}^N, \mathbb{R}^n) \) as a composition of continuous periodic functions. Consequently, we have \( [t \mapsto f(t,x(t))] = \psi(t\omega) \in QP^0_\omega(\mathbb{R}^n) \). And so the superposition operator build on \( f \), \( N_f : QP^0_\omega(\mathbb{R}^n) \rightarrow QP^0_\omega(\mathbb{R}^n) \), \( N_f(x) = [t \mapsto f(t,x(t))] \), is well defined. From the assumption (4.1) it is easy to obtain the following inequality:

\[
\|N_f(x) - N_f(y)\|_\infty \leq c\|x - y\|_\infty \tag{4.3}
\]

for all \( x, y \in QP^0_\omega(\mathbb{R}^n) \).

Consequently by setting \( c_1 = c(\|A\|\|\gamma + 1\|)^{-1} \) we have \( c_1 \in (0, 1) \) and by using (4.2) and (4.3), the following inequality holds:

\[
\|L^{-1} \circ N_f(x) - L^{-1} \circ N_f(y)\|_\infty \leq c_1\|x - y\|_\infty
\]

for all \( x, y \in QP^0_\omega(\mathbb{R}^n) \). And so the operator \( L^{-1} \circ N_f : QP^0_\omega(\mathbb{R}^n) \rightarrow QP^0_\omega(\mathbb{R}^n) \) is a contraction. Then by using the Picard-Banach Fixed Point Theorem, we obtain that there exists a unique \( x \in QP^0_\omega(\mathbb{R}^n) \) such that \( L^{-1} \circ N_f(x) = x \).

We note that, for \( x \in QP^0_\omega(\mathbb{R}^n) \), \( L^{-1} \circ N_f(x) = x \) is equivalent to say that \( x \) is a solution of (1.1) in \( QP^1_\omega(\mathbb{R}^n) \), and so the theorem is proven.

5. A continuous dependence result

In this section, we establish the existence of quasi-periodic solutions of equation (1.2) and a continuous dependence result with respect to the parameters functions \( u \).

First we recall a theorem on fixed points which is proven in [14, p.103].

**Theorem 5.1.** (Parametrized fixed point) Let \( E \) be a complete metric space, let \( \Lambda \) be a topological space and let \( \phi : E \times \Lambda \rightarrow E \) be a mapping which satisfies the two following properties:

\[
\text{for all } x \in E, \lambda \mapsto \phi(x, \lambda) \text{ is continuous from } \Lambda \text{ into } E, \tag{5.1}
\]
and

\[
\left\{ \begin{array}{l}
\text{there exists } k \in (0, 1) \text{ such that,} \\
\text{for all } \lambda \in \Lambda \text{ and for all } x, y \in E, \\
\text{the following inequality holds:}
\end{array} \right.
\]
\[
d(f(x, \lambda), f(y, \lambda)) \leq k \cdot d(x, y).
\]

(5.2)

Then, for all \( \lambda \in \Lambda \), denoting by \( a[\lambda] \) the unique fixed point of the partial mapping \( f(., \lambda) \), the mapping \( \lambda \mapsto a[\lambda] \) is continuous from \( \Lambda \) into \( E \).

**Theorem 5.2.** Let \( A \in \text{QP}_0(M_{n}(\mathbb{R})) \) and \( g \in \text{QP}_U(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \). We assume that (3.2), (3.3) and (3.9) are fulfilled, and \( \gamma \) denotes the Bohr-Neugebauer constant. We also assume that the following condition is fulfilled.

\[
\left\{ \begin{array}{l}
\text{There exists } d \in (0, (\|A\|_{\infty} \gamma + 1 + \gamma)^{-1}) \\
\text{such that } ||g(t, x, u) - g(t, y, u)|| \leq d \cdot ||x - y||
\end{array} \right.
\]
\[
\text{for all } u \in \mathbb{R}^p.
\]

(5.3)

Then, for all \( u \in \text{QP}_0(\mathbb{R}^p) \) there exists a unique solution \( x[u] \in \text{QP}_1(\mathbb{R}^n) \) of (1.2), and moreover the mapping \( u \mapsto x[u] \) is continuous from \( \text{QP}_0(\mathbb{R}^p) \) into \( \text{QP}_1(\mathbb{R}^n) \).

**Proof.** We consider the operator \( L \) defined in the proof of Theorem 4.1. By using on \( g \) arguments similar to these ones used on \( f \) in the proof of Theorem 4.1, we obtain that the superposition operator \( N_g : \text{QP}_0(\mathbb{R}^n) \times \text{QP}_0(\mathbb{R}^p) \rightarrow \text{QP}_0(\mathbb{R}^n) \), \( N_g(x, u) = [t \mapsto g(t, x(t), u(t))] \), is well defined. By using (5.3) we easily verify that the following property holds:

\[
||N_g(x, u) - N_g(y, u)||_{\infty} \leq d \cdot ||x - y||_{\infty}
\]

(5.4)

for all \( x, y \in \text{QP}_0(\mathbb{R}^n) \) and for all \( u \in \text{QP}_0(\mathbb{R}^p) \).

We define the nonlinear operator \( \phi : \text{QP}_0(\mathbb{R}^n) \times \text{QP}_0(\mathbb{R}^p) \rightarrow \text{QP}_0(\mathbb{R}^n) \) by setting:

\[
\phi(x, u) = L^{-1} \circ N_g(x, u) \text{ for all } (x, u) \in \text{QP}_0(\mathbb{R}^n) \times \text{QP}_0(\mathbb{R}^p).
\]

With

\[
E = \text{QP}_0(\mathbb{R}^n) \text{ and } \Lambda = \text{QP}_0(\mathbb{R}^p),
\]

by using (4.2) and (5.4) by setting \( k = d \cdot (\|A\|_{\infty} \gamma + 1 + \gamma) \in (0, 1) \), we see that \( \phi \) satisfies (5.2). By using [2, Theorem 3.5, p.47], we know that

\[
N_g^1 : AP_0(\mathbb{R}^n) \times AP_0(\mathbb{R}^p) \rightarrow AP_0(\mathbb{R}^n), \quad N_g^1(x, u) = [t \mapsto g(t, x(t), u(t))],
\]

is continuous, and since \( N_g \) is a restriction of \( N_g^1 \), \( N_g \) is also continuous. Since \( L^{-1} \) is linear continuous, \( \phi \) is continuous as a composition of continuous operators, and consequently the partial operator \( u \mapsto \phi(x, u) \) is continuous for all \( x \in \text{QP}_0(\mathbb{R}^n) \), and so \( \phi \) satisfies (5.1).

Now we can use the theorem of parametrized fixed point, and we can assert that, for all \( u \in \text{QP}_0(\mathbb{R}^p) \) there exists a unique \( x[u] = L^{-1} \circ N_g(x[u], u) \), and moreover the mapping \( u \mapsto x[u] \) is continuous from \( \text{QP}_0(\mathbb{R}^p) \) into \( \text{QP}_0(\mathbb{R}^n) \).
To say that \( \mathcal{X}[u] \) satisfies the equation \( \mathcal{X}[u] = L^{-1} \circ N_g(\mathcal{X}[u], u) \) is equivalent to say that \( \mathcal{X}[u] \in QP^1_\omega(\mathbb{R}^n) \) and \( \mathcal{X}[u] \) is a solution of (1.2).

We note that

\[
\dot{\mathcal{X}}[u](t) = A(t)\mathcal{X}[u](t) + g(t, \mathcal{X}[u](t), u(t)).
\]

Since \( u \mapsto \mathcal{X}[u] \) is continuous from \( QP^0_\omega(\mathbb{R}^p) \) into \( QP^0_\omega(\mathbb{R}^n) \) and since \( v \mapsto Av = [t \mapsto A(t)v(t)] \) is linear continuous from \( QP^0_\omega(\mathbb{R}^n) \) into \( QP^0_\omega(\mathbb{R}^n) \), we obtain that \( u \mapsto Ax[u] \) is continuous from \( QP^0_\omega(\mathbb{R}^p) \) into \( QP^0_\omega(\mathbb{R}^n) \). We have yet seen that the superposition operator \( N_g \) is continuous from \( QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p) \) into \( QP^0_\omega(\mathbb{R}^n) \), and it is clear that the operator \( u \mapsto (\mathcal{X}[u], u) \) is continuous from \( QP^0_\omega(\mathbb{R}^p) \) into \( QP^0_\omega(\mathbb{R}^n) \times QP^0_\omega(\mathbb{R}^p) \), and so \( u \mapsto N_g(\mathcal{X}[u], u) \) is continuous from \( QP^0_\omega(\mathbb{R}^p) \) into \( QP^0_\omega(\mathbb{R}^n) \) as a composition of continuous operators. Finally \( u \mapsto \dot{\mathcal{X}}[u] = Ax[u] + N_g(\mathcal{X}[u], u) \) is continuous from \( QP^0_\omega(\mathbb{R}^p) \) into \( QP^1_\omega(\mathbb{R}^n) \) as a sum of continuous operators. Therefore \( u \mapsto \mathcal{X}[u] \) is continuous from \( QP^0_\omega(\mathbb{R}^p) \) into \( QP^1_\omega(\mathbb{R}^n) \).

6. A differentiable perturbation result

We fix \( \omega = (\omega_1, \ldots, \omega_n) \) a list of \( \mathbb{Z} \)-linearly independent real numbers. We consider, about the vector-field of the equation (1.3), the following condition:

\[
\begin{aligned}
g &\in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \cap C^{\tau - 1}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n), \\
D_xg &\in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_n(\mathbb{R})) \cap C^\tau(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_n(\mathbb{R}))
\end{aligned}
\]

with \( \tau = 2(N + 1)\left(\frac{n+n+1}{2} + 1\right) \), and

\[
D_ug \in QPU_\omega(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_{n,p}(\mathbb{R})).
\]

In this condition, \( D_xg \) denotes the partial differential of \( g \) with respect to the second vector variable and \( D_ug \) denotes the partial differential of \( g \) with respect to the third vector variable.

**Theorem 6.1.** Let \( g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n \) be a function which satisfies (6.1) where \( \omega \) satisfies (3.3). Let \( u_* \in QP^1_\omega(\mathbb{R}^p) \) and let \( x_* \in QP^1_\omega(\mathbb{R}^n) \) be a solution of (1.3) where \( u = u_* \). We set \( J(t) = D_xg(t, x_*(t), u_*(t)) \) for all \( t \in \mathbb{R} \) and we denote by \( \beta_1, \ldots, \beta_n \) the FL-CER of \( J \). Moreover we assume that the following condition is fulfilled.

\[
\text{For all } j = 1, \ldots, n, \quad \beta_j \text{ is non zero.} \quad (6.2)
\]

Then there exists \( r \in (0, \infty) \) such that, for all \( u \in QP^0_\omega(\mathbb{R}^p) \) satisfying

\[
||u - u_*||_\infty < r,
\]

there exists \( \mathcal{X}[u] \in QP^1_\omega(\mathbb{R}^n) \) which is a solution of (1.3).

Moreover the nonlinear operator \( u \mapsto \mathcal{X}[u] \) is of class \( C^1 \) from

\[
\{u \in QP^0_\omega(\mathbb{R}^p) : ||u - u_*||_\infty < r\} \rightarrow QP^1_\omega(\mathbb{R}^n),
\]

and there exists a neighborhood \( \mathcal{N} \) of \( x_* \) in \( QP^0_\omega(\mathbb{R}^n) \) such that \( \mathcal{X}[u] \) is the unique solution of (1.3) in \( QP^1_\omega(\mathbb{R}^n) \) which belongs to \( \mathcal{N} \).

Before to do the proof of this theorem we need a lemma of Differential Calculus.
LEMMA 6.2. When \( g \in QPU_{\omega}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) is such that its partial differentials with respect to the second and the third vector variables exist and satisfy

\[ D_xg \in QPU_{\omega}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_n(\mathbb{R})) \quad \text{and} \quad D_u g \in QPU_{\omega}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, M_n, p(\mathbb{R})), \]

then the operator

\[ \Gamma : QP^1_{\omega}(\mathbb{R}^n) \times QP^0_{\omega}(\mathbb{R}^p) \longrightarrow QP^0_{\omega}(\mathbb{R}^n), \quad \Gamma(x, u) = [t \mapsto \dot{x}(t) - g(t, x(t), u(t))], \]

is well-defined and it is of class \( C^1 \).

The formula of its partial differential with respect to its first variable is the following one:

\[ D_1 \Gamma(x, x, u_*)y = [t \mapsto \dot{y}(t) - D_xg(t, x_*(t), u_*(t))y(t)] \]

for all \( y \in QP^1_{\omega}(\mathbb{R}^n) \).

Proof. When \( x \in QP^0_{\omega}(\mathbb{R}^n) \) and \( u \in QP^0_{\omega}(\mathbb{R}^p) \) there exist \( \varphi \in C^0(\mathbb{T}^N, \mathbb{R}^n) \) and \( \psi \in C^0(\mathbb{T}^N, \mathbb{R}^p) \) such that \( x(t) = \varphi(t\omega) \) and \( u(t) = \psi(t\omega) \) for all \( t \in \mathbb{R} \), [3, Theorem 2, p.97]. By using Remark p.101 in [3], since \( g \in QPU_{\omega}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \), there exists \( G \in C^0(\mathbb{T}^N \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n) \) such that \( g(t, x, u) = G(t\omega, x, u) \) for all \( (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \). We set \( \chi(\theta) = G(\theta, \varphi(\theta), \psi(\theta)) \), then \( \chi \in C^0(\mathbb{T}^N, \mathbb{R}^n) \) as a composition of continuous periodic functions. Consequently we have:

\[ [t \mapsto g(t, x(t), u(t))] \in QP^0_{\omega}(\mathbb{R}^n). \]

And so the operator \( \Gamma \) is well-defined.

By using Theorem 5.1, p.54 in [2], we know that the superposition operator:

\[ N^1_g : AP^0_{\omega}(\mathbb{R}^n) \times AP^0_{\omega}(\mathbb{R}^p) \longrightarrow AP^0_{\omega}(\mathbb{R}^n), \quad N^1_g(x, u) = [t \mapsto g(t, x(t), u(t))], \]

is of class \( C^1 \). And so the superposition operator

\[ N_g : QP^0_{\omega}(\mathbb{R}^n) \times QP^0_{\omega}(\mathbb{R}^p) \longrightarrow QP^0_{\omega}(\mathbb{R}^n), \quad N_g(x, u) = [t \mapsto g(t, x(t), u(t))] \]

is of class \( C^1 \) as a restriction of \( N^1_g \). And so the following assertion holds:

\[ N_g \in C^1 \left( QP^0_{\omega}(\mathbb{R}^n) \times QP^0_{\omega}(\mathbb{R}^p), QP^0_{\omega}(\mathbb{R}^n) \right). \quad (6.3) \]

The operator \( \Pi_1 : QP^1_{\omega}(\mathbb{R}^n) \times QP^0_{\omega}(\mathbb{R}^p) \longrightarrow QP^1_{\omega}(\mathbb{R}^n) \), defined by \( \Pi_1(x, u) = x \), is linear continuous, therefore the following assertion holds:

\[ \Pi_1 \in C^1 \left( QP^1_{\omega}(\mathbb{R}^n) \times QP^0_{\omega}(\mathbb{R}^p), QP^1_{\omega}(\mathbb{R}^n) \right). \quad (6.4) \]

The operator \( \frac{d}{dt} QP^1_{\omega}(\mathbb{R}^n) \longrightarrow QP^0_{\omega}(\mathbb{R}^n), \quad \frac{d}{dt} x = \dot{x} \), is linear continuous, therefore the following assertion holds:

\[ \frac{d}{dt} \in C^1 \left( QP^1_{\omega}(\mathbb{R}^n), QP^0_{\omega}(\mathbb{R}^n) \right). \quad (6.5) \]
The operator \( in : QP^1_{p^0}(\mathbb{R}^n) \times QP^0_{p^0}(\mathbb{R}^p) \rightarrow QP^0_{p^0}(\mathbb{R}^n) \times QP^0_{p^0}(\mathbb{R}^p) \), \( in(x,u) = (x,u) \), is linear continuous, and so the following assertion holds:

\[
in \in C^1(QP^1_{p^0}(\mathbb{R}^n) \times QP^0_{p^0}(\mathbb{R}^p), QP^0_{p^0}(\mathbb{R}^n) \times QP^0_{p^0}(\mathbb{R}^p)).
\]  \tag{6.6}

We note that \( \Gamma = \frac{d}{dt} \circ \Pi_1 - N_g \circ in \), and so by using (6.3)-(6.6), \( \Gamma \) is of class \( C^1 \) as the difference of compositions of operators of class \( C^1 \).

Now, by using Theorem 5.1, p.54, in [2] and the chain rule of the differential calculus in Banach spaces, we obtain the following calculations:

\[
D_1 \Gamma(x_*,u_*) \cdot y = D \Gamma(x_*,u_*) \cdot (y,0)
\]

\[
= D \left( \frac{d}{dt} \circ \Pi_1 \right) (x_*,u_*) \cdot (y,0) - D(N_g \circ in)(x_*,u_*) \cdot (y,0)
\]

\[
= \frac{d}{dt} \circ \Pi_1(y,0) - DN_g(x_*,u_*) \cdot (y,0)
\]

\[
= \{ t \mapsto \dot{y}(t) - Dg(t,x_*(t),u_*(t)).y(t) \}.
\]

**Proof of Theorem 6.1.** Since \( g \) is of class \( C^{r-1} \), by using a bootstrapping argument we see that \( x_* \) is also of class \( C^r \). And so the matrix \( J(t) \) satisfies the condition (3.2). The assumption (6.2) ensures that (3.9) is fulfilled for \( A = J \). And so can use Lemma 3.6 to assert that for all \( b \in QP^0_{p^0}(\mathbb{R}^n) \), there exists a unique \( y \in QP^1_{p^0}(\mathbb{R}^n) \) such that \( \dot{y}(t) = J(t)y(t) + b(t) \) for all \( t \in \mathbb{R} \). And so, by using Lemma 6.2, we can translate this result in the following form:

\[
D_1 \Gamma(x_*,u_*) \text{ is a bijection from } QP^1_{p^0}(\mathbb{R}^n) \text{ onto } QP^0_{p^0}(\mathbb{R}^n). \]  \tag{6.7}

Since \( \dot{x}_*(t) = g(t,x_*(t),u_*(t)) \) for all \( t \in \mathbb{R} \), the following assertion holds:

\[
\Gamma(x_*,u_*) = 0. \]  \tag{6.8}

Since \( \Gamma \) is of class \( C^1 \), (6.7) and (6.8) permit to use the implicit function theorem of the differential calculus in Banach spaces, see [4, Theorem 4.7.1, p.61]. And so we can assert that there exist \( \mathcal{V} = \{ x \in QP^1_{p^0}(\mathbb{R}^n) : \| x - x_* \|_{C^1} < r \} \) with \( r \in (0,\infty) \), a neighborhood \( \mathcal{N} \) of \( u_* \) in \( QP^0_{p^0}(\mathbb{R}^p) \), and a \( C^1 \)-mapping \( \mathcal{X} : \mathcal{V} \rightarrow \mathcal{N} \) such that, for all \( (x,u) \in \mathcal{V} \times \mathcal{N} \), we have \( \Gamma(x,u) = 0 \) if and only if \( x = \mathcal{X}[u] \).

Notice that \( \Gamma(x,u) = 0 \) is equivalent to say that \( x \) is solution of (1.3) in \( QP^1_{p^0}(\mathbb{R}^n) \). And so Theorem 6.1 is proven.

**References**


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