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POSITIVE SOLUTIONS OF SECOND ORDER MULTI–POINT BOUNDARY VALUE PROBLEMS WITH NON–HOMOGENEOUS BOUNDARY CONDITIONS

JAMES S. W. WONG

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Abstract. We are interested in the existence of positive solutions to multi-point boundary value problems for second order nonlinear differential equations with non-homogeneous boundary conditions. We show that results for the multi-point problems can be proved much in a similar way by methods available for the three point problem.

1. Introduction

We are interested in the existence of positive solutions of second order differential equation

$$y'' + a(t)f(y) = 0, \qquad 0 \leqslant t \leqslant 1, \tag{1.1}$$

where $a(t), f(y)$ are continuous and non-negative functions of $t \in (0,1)$ and $y \in [0,\infty)$ and $a(t) \not\equiv 0$ in $(0,1)$, subject to a variety of boundary conditions. When such boundary conditions involve one or more interior points in $(0,1)$, equation (1.1) and the associated boundary conditions together are commonly referred to as multi-point boundary value problems.

We shall be interested in multi-point boundary conditions at m interior points, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, of the following types:

$$(BC1) \qquad y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i), \;\; y'(1) = \sum_{i=1}^{m} \beta_i y'(\xi_i) + b, \tag{1.2}$$

$$(BC2) \qquad y'(0) = \sum_{i=1}^{m} \alpha_i y'(\xi_i), \;\; y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) + b, \tag{1.3}$$

$$(BC3) \qquad y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i), \;\; y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) + b, \tag{1.4}$$

where $0 \leqslant \alpha_i < 1, 0 \leqslant \beta_i < 1, i = 1,2,\cdots,m$ and b real constant. We refer to (1.1), (1.2); (1.1), (1.3) and (1.1), (1.4) as (BVP1), (BVP2) and (BVP3) respectively. When

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$b = 0$, (BVP1), (BVP2) and (BVP3) are referred to as homogeneous boundary value problems and they are called inhomogeneous boundary value problems if $b \neq 0$.

Multi-point boundary value problems are also known as non-local boundary value problems and were initiated in the study by Ilin and Moiseev [20],[21]. These problems arise from a variety of problems in applied physics notably in heat conduction, Cannon [2], [3], Ionkin [22], Kamyuin [23], the vibration of cables with non-uniform weights, Moshinsky [42] and other problems in nonlinear elasticity, Timoshenko [47].

In the simplest case that of a three-point boundary value problem, i.e. $m = 1$ in (1.2), (1.3), (1.4) with one interior point $\xi_1 \in (0,1)$, Gupta [14], [15], [16] first applied functional analytical methods to prove existence of solutions followed by Eloe and Henderson [5], Ma [36],[37], Liu [31],[32], Webb [49] and many others. For existence of positive solutions, the fixed point theorem on cones by Krasnoselskü and Guo is commonly used, see [4],[11],[28]. The origin of applying this theorem can be found in Erbe and Wang [8] with application to semilinear elliptic equations on annuli, see Wang [48], Bandle, Coffman and Marcus [1], Lee and Lin [30], and Hai [18].

Denote the following limits of $f(y)/y$ which are assumed to exist

$$f_0 = \lim_{y\to 0} \frac{f(y)}{y} \quad \text{and} \quad f_\infty = \lim_{y\to\infty} \frac{f(y)}{y}. \tag{1.5}$$

When $f(y)$ satisfies $f_0 < f_\infty$, the boundary value problem (1.1) subject to various boundary conditions, such as (1.2), (1.3), (1.4), is said to be superlinear. Likewise if $f_\infty < f_0$, then it is said to be sublinear. When $f(y) = y^p$, it is superlinear if $p > 1$ and sublinear if $0 < p < 1$.

We are interested in a result of Ma for the inhomogeneous three point boundary value problem in the superlinear case:

$$(E_b) \qquad \begin{cases} y'' + a(t)f(y) = 0, \quad 0 < t < 1, \\ y(0) = 0, \quad y(1) = \beta y(\xi) + b, \end{cases} \tag{1.6}$$

where $0 < \xi < 1$, $0 < \beta < 1/\xi, b \geqslant 0$. Boundary value problem (E_b) is a special case of (BVP3).

THEOREM A. (Ma [37]) *Suppose that $f_0 = 0$ and $f_\infty = \infty$. Then there exists $b^* > 0$ such that the boundary value problem (E_b) has a positive solution for b satisfying $0 < b < b^*$ and no positive solution for $b > b^*$.*

Theorem A has been extended by Guo, Shan, and Ge [12] to a special case of (BVP3) with $\alpha_i = 0$ $i = 1, 2, \cdots, m$. A similar result was given by Sun, Chen, Zhang and Wang [45] for (BVP2).

In the homogeneous case when $b = 0$, we are interested in a results of Zhang and Sun [54] concerning boundary value problem (E_0), i.e. with $b = 0$ in (1.6), which relates f_0, f_∞ to the smallest positive eigenvalue λ_1 of the linear boundary value problem

$$(D) \qquad \begin{cases} u'' + \lambda a(t)u = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \beta u(\xi), \end{cases}$$

with $0 < \xi < 1$, $0 < \beta < 1/\xi$ as in (1.6).

THEOREM B. *Suppose that* $f_\infty < \lambda_1 < f_0$. *Then the* $(BVP3)$ *with* $\alpha_i = 0$, $i = 1, 2, \cdots, m$, *has a positive solution.*

The first paper which relates f_0, f_∞ to the eigenvalue of a linear problem seems to be Gupta and Trofimchuk [17] and more recently by Webb and Lan [50], Han [19], Sun [46], Kwong and Wong [29].

More recently, Zhang and Sun [55] studied (BVP1) (note these authors are not the same as that of [54]) and improved the results of Liu [33] for the homogeneous case, i.e. (1.2) with $b = 0$. They proved

THEOREM C. *Suppose that* $0 < \sum_{i=1}^{m} \alpha_i < 1$ *and* $\sum_{i=1}^{m} \beta_i < 1$. *If* $f_0 = 0$ *and* $f_\infty = \infty$ *and* $f(y)$ *is non-decreasing in* y, *then there exists* $b^* > 0$ *such that* $(BVP1)$ *has a positive solution for all* $b, 0 \leqslant b \leqslant b^*$ *and no positive solution for* $b > b^*$.

THEOREM D. *Suppose that* $0 < \sum_{i=1}^{m} \alpha_i < 1$ *and* $\sum_{i=1}^{m} \beta_i < 1$. *If* $f_0 = \infty$ *and* $f_0 = 0$, *then* $(BVP1)$ *has a positive solution for every* $b > 0$.

The purpose of this paper is to show that Theorems A and B for the three point boundary value problem (E_b) remain valid in their entirety for the more general boundary value problems (BVP1), (BVP2), and (BVP3). For Theorem C concerning (BVP1), we show that the assumption that $f(y)$ is non-decreasing is superfluous. In the sublinear case of Theorem D, it becomes a corollary to the "optimal existence theorems" in the form of Theorem B for $b \geqslant 0$ and for all three type of boundary conditions (1.2), (1.3), (1.4)

This paper is organized as follows. In section 2, we introduce a short hand notation for the summation given in (1.2), (1.3), (1.4) which allows us to apply techniques used for the three point problem for the more general boundary value problems (BVP1), (BVP2), (BVP3). Here we used the equivalent integral operator formulation originated from the earlier works of Gupta [14] and Ma [36] for the three point case where the fixed points of the Hammerstein operator give rise to the positive solutions of multi-point boundary value problems (BVP1), (BVP2), (BVP3). In section 3, we employ the standard Krasnoselskii- Guo fixed point theorem on cones and obtain extensions of Theorems A and C. In section 4, we use topological degree theory together with Krein-Rutman theorem to prove "optimal existence theorems" for (BVP1), (BVP2), (BVP3), thereby extending Theorems B and D in the sublinear case. In section 5, we discuss examples, give remarks concerning the limitation of our methods and suggest related problem for further research.

2. Integral operators via scalar product formulation

In proving existence theorems for boundary value problems (BVP1), (BVP2), (BVP3), we convert (1.1) and its associated boundary conditions $(1.2), (1.3), (1.4)$

to an equivalent integral equation in the form of a Hammerstein operator. For multipoint boundary value problems, one often finds it cumbersome in the repetitive use of summation notations. We now introduce a simpler method by introducing a "scalar product" for two m-dimensional vectors. Consider the collection of interior points $\{\xi_i : i = 1,2,\cdots,m\}$ as a m-vector in $\mathbb{R}^m$. For any function in $C^1[0,1]$, e.g. $y(t), y'(t)$ we consider the set $\{y(\xi_i) : i = 1,2,\cdots,m\}$ as a m-vector function of $\xi = (\xi_1,\xi_2,\cdots,\xi_m)$. Likewise we denote $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_m)$ and $\beta = (\beta_1,\beta_2,\cdots,\beta_m)$ as m-vectors in $\mathbb{R}^m$. The standard scalar product of two m-vectors are then given by examples below

$$\langle\beta,\xi\rangle = \sum_{i=1}^{m} \beta_i\xi_i, \;\; \langle\beta,y'(\xi)\rangle = \sum_{i=1}^{m} \beta_i, y'(\xi_i).$$

Using this notation, we can restate the boundary conditions (BVP1), (BVP2), (BVP3) as follows:

$$(BVP1) \qquad \begin{cases} y''(t)+a(t)f\big(y(t)\big)=0, \quad 0<t<1, \\ y(0)=\langle\alpha,y(\xi)\rangle, \;\; y'(1)=\langle\beta,y'(\xi)\rangle+b, \end{cases} \tag{2.1}$$

$$(BVP2) \qquad \begin{cases} y''(t)+a(t)f\big(y(t)\big)=0, \quad 0<t<1, \\ y'(0)=\langle\alpha,y'(\xi)\rangle, \;\; y(1)=\langle\beta,y(\xi)\rangle+b, \end{cases} \tag{2.2}$$

$$(BVP3) \qquad \begin{cases} y''(t)+a(t)f\big(y(t)\big)=0, \quad 0<t<1, \\ y(0)=\langle\alpha,y(\xi)\rangle, \;\; y(1)=\langle\beta,y(\xi)\rangle+b. \end{cases} \tag{2.3}$$

We denote $\overline{\alpha},\overline{\beta}$ by

$$\overline{\alpha} = \langle\alpha,1\rangle = \sum_{i=1}^{m} \alpha_i, \quad \overline{\beta} = \langle\beta,1\rangle = \sum_{i=1}^{m} \beta_i, \tag{2.4}$$

where $\langle\alpha,1\rangle,\langle\beta,1\rangle$ are scalar products of α,β with the identity vector $(1,1,\cdots,1) \in \mathbb{R}^m$.

Using the notation introduced above, we introduce three Hammerstein integral operators A_1,A_2,A_3 in terms of kernels $K_1(t,s),K_2(t,s),K_3(t,s)$ by

$$A_jy(t) = \int_0^1 K_j(t,s)a(s)f\big(y(s)\big)ds + l_j(t), \; j=1,2,3, \tag{2.5}$$

where $K_j(t,s)$ and $l_j(t), j=1,2,3$ are to be determined from the boundary conditions $(2.1),(2.2),(2.3)$. Now write $A_jy(t)$ as

$$A_jy(t) = G_j(t) + C_jt + D_j, \;\; j=1,2,3, \tag{2.6}$$

where

$$G_j(t) = \int_0^1 g_j(t,s)a(s)f\big(y(s)\big)ds, \;\; j=1,2,3, \tag{2.7}$$

with

$$g_1(t,s) = \begin{cases} s, & 0\leqslant s\leqslant t\leqslant 1, \\ t, & 0\leqslant t\leqslant s\leqslant 1, \end{cases} \tag{2.8}$$

$$g_2(t,s) = \begin{cases} 1-s, & 0 \leqslant s \leqslant t \leqslant 1, \\ 1-t, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.9}$$

$$g_3(t,s) = \begin{cases} s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.10}$$

From boundary conditions (2.1), (2.2), (2.3), we obtain three sets of two linear equations in unknowns C_j and $D_j, j = 1,2,3$, which are given below

$$C_1 = (1-\overline{\beta})^{-1}\left[\langle \beta, G_1'(\xi)\rangle + b\right], \tag{2.11}$$

$$D_1 = (1-\overline{\alpha})^{-1}\left\{\langle \alpha, G_1(\xi)\rangle + \frac{\langle \alpha,\xi\rangle}{1-\overline{\beta}}, \left[\langle \beta_2 G_1'(\xi)\rangle + b\right]\right\}, \tag{2.12}$$

where

$$G_1'(t) = \int_0^1 \frac{\partial g_1}{\partial t}(t,s)a(s)f\big(y(s)\big)ds = \int_t^1 a(s)f\big(y(s)\big)ds. \tag{2.13}$$

Similarly, we have

$$C_2 = (1-\overline{\alpha})^{-1}\langle \alpha, G_2'(\xi)\rangle, \tag{2.14}$$

$$D_2 = (1-\overline{\beta})^{-1}\left\{\langle \beta, G_2(\xi)\rangle - \frac{(1-\langle \beta,\xi\rangle)}{1-\overline{\alpha}}\langle \alpha, G_2'(\xi)\rangle + b\right\}, \tag{2.15}$$

where

$$G_2'(t) = \int_0^1 \frac{\partial g_2}{\partial t}(t,s)a(s)f\big(y(s)\big)ds = -\int_0^t a(s)f\big(y(s)\big)ds. \tag{2.16}$$

Also, we have

$$C_3 = \frac{1}{\Lambda}\left\{(1-\overline{\alpha})\left[\langle \beta, G_3(\xi)\rangle + b\right] - (1-\overline{\beta})\langle \alpha, G_3(\xi)\rangle\right\}, \tag{2.17}$$

$$D_3 = \frac{1}{\Lambda}\left\{(1-\langle \beta,\xi\rangle)\langle \alpha, G_3(\xi)\rangle + \langle \alpha,\xi\rangle\left[\langle \beta, G_3(\xi)\rangle + b\right]\right\}, \tag{2.18}$$

where $\Lambda = (1-\overline{\alpha})(1-\langle \beta,\xi\rangle) + (1-\overline{\beta})\langle \alpha,\xi\rangle$, and

$$G_3(t) = t\int_t^1 (1-s)a(s)f\big(y(s)\big)ds + (1-t)\int_0^t s\,a(s)f\big(y(s)\big)ds. \tag{2.19}$$

Denote $g_j'(t,s) = \frac{\partial}{\partial t} g_j(t,s), j = 1,2,3$. We can also express $K_j(t,s)$, being kernel of the Hammerstine operator A_j defined by (2.5), as follows:

$$K_1(t,s) = g_1(t,s) + \frac{t}{(1-\overline{\beta})}\left\{\langle \beta, g_1'(\xi,s)\rangle\right\} + \frac{1}{(1-\overline{\alpha})}\left\{\langle \alpha, g_1(\xi,s)\rangle + \frac{\langle \alpha,\xi\rangle}{(1-\overline{\beta})}\left[\langle \beta, g_1'(\xi,s)\rangle\right]\right\}, \tag{2.20}$$

$$K_2(t,s) = g_2(t,s) + \frac{t}{(1-\overline{\alpha})}\langle \alpha, g_2'(\xi,s)\rangle + \frac{1}{(1-\overline{\beta})}\left\{\langle \beta, g_2(\xi,s)\rangle + \frac{\langle \beta,\xi\rangle - 1}{(1-\overline{\alpha})}\langle \alpha, g_2'(\xi,s)\rangle\right\}, \tag{2.21}$$

$$K_3(t,s) = g_3(t,s) + \frac{t}{\Lambda}\left\{(1-\overline{\alpha})\left[\langle \beta, g_3(\xi,s)\rangle\right] - (1-\overline{\beta})\langle \alpha, g_3(\xi,s)\rangle\right\} + \frac{1}{\Lambda}\left\{(1-\langle \beta,\xi\rangle)\langle \alpha, g_3(\xi,s)\rangle + \langle \alpha,\xi\rangle\left[\langle \beta, g_3(\xi,s)\rangle\right]\right\}, \tag{2.22}$$

where $\Lambda = (1-\overline{\alpha})(1-\langle \beta,\xi\rangle) + (1-\overline{\beta})\langle \alpha,\xi\rangle$, and

$$l_1(t) = \frac{b}{D}\left\{(1-\overline{\alpha})t + \langle \alpha,\xi\rangle\right\}, \quad D = (1-\overline{\alpha})(1-\overline{\beta}),$$
$$l_2(t) = \frac{b}{1-\overline{\beta}}, \quad l_3(t) = \frac{b}{\Lambda}\left\{(1-\overline{\alpha})t + \langle \alpha,\xi\rangle\right\}.$$

We find it more convenient to discuss our proofs by using the simpler formula (2.6) and the constants $C_j's, D_j's$ as given by (2.11), (2.12); (2.14), (2.15); (2.17), (2.18). For these formulas to make sense we require $\triangle = (1-\overline{\alpha})(1-\overline{\beta}) \neq 0$ and $\Lambda \neq 0$.

We first prove that $A_j, j = 1,2,3$ are positive operators in the sense that $A_j(P) \subseteq P$, where P is the cone of non-negative functions in $C[0,1]$, i.e.

$$P = \{y(t) \in C[0,1] : y(t) \geqslant 0,\ 0 \leqslant t \leqslant 1\},$$

when in addition $\triangle > 0$ and $\Lambda > 0$.

Since $G_1'(t) \geqslant 0$ by (2.13), we note from (2.11), (2.12) that $C_1, D_1 \geqslant 0$ so $A_1(P) \subseteq P$. Note that $G_2'(t) \leqslant 0$ by (2.16), therefore $C_0 \leqslant 0$ by (2.14) and $D_2 \geqslant 0$ by (2.15). Now the linear function $C_2t + D_2 \geqslant \min\ (D_2, C_2 + D_2)$. We note by adding up (2.14) and (2.15) that

$$D_2 \geqslant C_2 + D_2 = (1-\overline{\beta})^{-1}\left\{(\overline{\beta} - \langle \beta, G_2(\xi)\rangle + \frac{\langle \beta,\xi\rangle - \overline{\beta}}{1-\overline{\alpha}}\left[\langle \alpha, G_2'(\xi)\rangle + b\right]\right\} \geqslant 0,$$

which proves $A_2(P) \subseteq P$. Finally since $1 - \langle \beta,\xi\rangle \geqslant 0$, so $D_3 \geqslant 0$. Adding up (2.17), (2.18), we find

$$C_3 + D_3 = \frac{1}{\triangle}\left\{(\overline{\beta} - \langle \beta,\xi\rangle)\langle \alpha, G_3(\xi)\rangle \ + \ (1 - \overline{\alpha} + \langle \alpha,\xi\rangle)\langle \beta, G_3(\xi)\rangle\right\} \geqslant 0.$$

Thus $C_3t+D_3 \geqslant \min\ (D_3, C_3+D_3) \geqslant 0$ and $A_3(P) \subseteq P$.

We need the following lemmas concerning solutions of (1.1) satisfying boundary conditions (1.2), (1.3), (1.4) in the proofs of theorems in the next section.

LEMMA 2.1. *Let $u(t)$ be a solution of $u''+h(t)=0, h(t) \geqslant 0, t \in [0,1]$. If $u(t)$ satisfies (2.1) with $0<\overline{\alpha}<1$ and $0 \leqslant \overline{\beta} < 1$, then $u(t)$ is monotone nondecreasing and satisfies*

$$u(t) \geqslant \gamma_1 \|u\| \quad \textit{for } t \in [0,1],$$

where

$$\gamma_1 = \frac{\langle \alpha, \xi \rangle}{1-\overline{\alpha}+\langle \alpha, \xi \rangle}. \tag{2.23}$$

REMARK 2.1. This result is given by Zhang and Sun [55; Lemma 2.2] which was attributed to Liu [33] (for the case when $b=0$ in (2.1)). The proof is similar to Lemma 2.2 which we shall give in its entity below.

LEMMA 2.2. *Let $u''+h(t)=0, h(t) \geqslant 0, t \in [0,1]$. If $u(t)$ satisfies (2.2), where $0 \leqslant \overline{\alpha} < 1$ and $0 < \overline{\beta} < 1$, then $u(t)$ is monotone non-increasing and satisfies*

$$u(t) \geqslant \gamma_2 \|u\| \quad \textit{for } t \in [0,1],$$

where

$$\gamma_2 = \frac{\overline{\beta}-\langle \beta, \xi \rangle}{1-\langle \beta, \xi \rangle}. \tag{2.24}$$

Proof. Since $u''(t) \leqslant 0$, so by (2.2), $u'(0) = \langle \alpha, u'(\xi) \rangle$ and $(1-\overline{\alpha})u'(0) \leqslant 0$. Now $0 \leqslant \overline{\alpha} < 1$ implies $u'(0) \leqslant 0$ hence $u'(t) \leqslant u'(0) \leqslant 0$ and $u(t)$ is monotone non-increasing. Furthermore, we have $u(0) \geqslant u(t) \geqslant u(1)$. By (2.2), $u(1) = \langle \beta, u(\xi) \rangle + b$ which implies $u(1) \geqslant \overline{\beta} u(1) + b$. Now $0 < \overline{\beta} < 1$ and $b \geqslant 0$ imply $u(1) > 0$ thus $u(t) > 0$ for all $t \in [0,1]$.

For each $i=1,2,\cdots,m$, we have from concavity of $u(t)$ that

$$u(0) \leqslant u(1) + \frac{u(1)-u(\xi_i)}{1-\xi_i}(0-1),$$

that is,

$$u(0)(1-\xi_1) \leqslant u(1)(1-\xi_1) + u(\xi_1) - u(1). \tag{2.25}$$

Multiplying (2.25) by β_i and summing up, we find

$$u(0)(\overline{\beta}-\langle \beta, \xi \rangle) \leqslant \langle \beta, u(\xi) \rangle + u(1)\langle \beta, \xi \rangle,$$

which by (2.2) implies

$$u(0)(\overline{\beta}-\langle \beta, \xi \rangle) \leqslant u(1)(1-\langle \beta, \xi \rangle) - b. \tag{2.26}$$

Since $b \geqslant 0$, (2.26) proves $u(t) \geqslant \gamma_2 \|u\|$, where γ_2 is given by (2.24).

LEMMA 2.3. *Let $u(t)$ be a solution of $u''+h(t)=0, h(t)\geqslant 0$, $t\in[0,1]$ and $u(t)$ satisfies* (2.3). *If $0\leqslant\overline{\alpha}<1$ and $0<\langle\beta,\xi\rangle<1$, then $u(t)$ satisfies*

$$\inf_{\xi_1\leqslant t\leqslant 1} u(t)\geqslant\gamma_3\|u\|, \tag{2.27}$$

where

$$\gamma_3=\min_{0\leqslant s\leqslant m}\left\{\xi_1,\langle\beta,\xi\rangle,(1-\langle\beta,\xi\rangle)^{-1}\left(\overline{\beta}-\langle\beta,\xi\rangle\right),\theta_s\right\},$$

$$\theta_s=(1-\langle\beta,\xi\rangle)^{-1}\left\{\sum_{i=1}^{s-1}\beta_i\xi_i+\sum_{i=s}^{m}\beta_i(1-\xi_i)\right\}.$$

Proof. Let $u(t)$ attains its maximum at $t=\sigma$, i.e. $\|u\|=\max_{0\leqslant t\leqslant 1}u(t)=u(\sigma)$.
Case (i) $(0<\sigma<\xi_1)$ By concavity, we have

$$\inf_{\xi_i\leqslant t\leqslant 1}u(t)=u(1),$$

and for each $i=1,2,\cdots,m$,

$$u(\xi_1)\geqslant(1-\xi_1)u(\sigma)+x_iu(1). \tag{2.28}$$

Multiplying (2.28) through by β_i and summing from $i=1$ to $i=m$, we obtain by (2.3),

$$u(1)=\langle\beta,u(\xi)\rangle=(\overline{\beta}-\langle\beta,\xi\rangle)u(\sigma)+\langle\beta,\xi\rangle u(1)$$

from which we obtain

$$u(1)\geqslant(1-\langle\beta,\xi\rangle)^{-1}(\overline{\beta}-\langle\beta,\xi\rangle)\|u\|. \tag{2.29}$$

Case (ii) $\left(0<\xi_1<\sigma<1 \text{ and } \inf_{\xi_1\leqslant t\leqslant 1}u(t)=u(\xi_1)\right)$ Consider

$$(t^{-1}u(t))'=t^{-2}g(t),$$

where $g'(t)=tu''(t)\leqslant 0$ and $g(t)=tu'(t)-u(t)\leqslant g(0)=-u(0)$. Since $u(t)$ satisfies the integral equation (2.6), (2.7), $j=3$, where

$$G_3(t)=\int_0^1 g_3(t,s)h(s)ds\geqslant 0$$

and the linear function $C_3t+D_3\geqslant 0$, we conclude that $u(0)\geqslant 0$ so $g(t)\leqslant 0$ and $t^{-1}u(t)$ is non-increasing. Thus

$$\frac{u(\xi_1)}{\xi_1}\geqslant\frac{u(\sigma)}{\sigma}\geqslant u(\sigma)=\|u\|$$

and

$$\inf_{\xi_1\leqslant t\leqslant 1}u(t)=u(\xi_1)\geqslant\xi_1\|u\|. \tag{2.30}$$

Case (iii) $\big(\inf_{\xi_1\leqslant t\leqslant 1} u(t)=u(1)$ and $0\leqslant \xi_{s-1}<\sigma<\xi_s<1$, for $s=2,3,\cdots,m\big)$ Multiply (2.28) through by β_i and summing from $i=s$ to m, we obtain

$$\sum_{i=s}^{m}\beta_i u(\xi_1)\geqslant\left(\sum_{i=s}^{m}\beta_i\xi_1\right)u(1)+\sum_{i=s}^{m}\beta_i(1-\xi_1)u(\sigma). \tag{2.31}$$

On the other hand, for $1\leqslant i\leqslant s-1, u(\xi_1)\geqslant \xi_1 u(\sigma)/\sigma\geqslant \xi_1 u(\sigma)$, so

$$\sum_{i=1}^{s-1}\beta_i u(\xi_1)\geqslant u(\sigma)\left(\sum_{i=1}^{s-1}\xi_1\beta_i\right) \tag{2.32}$$

Adding (2.31) and (2.32), we obtain

$$\begin{aligned}\left(1-\sum_{i=s}^{m}\beta_i\xi_1\right)u(1)-b&\geqslant\left[\langle\beta,u(\xi)\rangle-\sum_{i=s}^{m}\beta_i\xi_1\right]u(1)\\ &\geqslant\left(\sum_{i=1}^{s-1}\beta_i\xi_1+\sum_{i=s}^{m}\beta_i(1-\xi_1)\right)u(\sigma),\end{aligned}$$

which gives

$$\inf_{\xi_1\leqslant t\leqslant 1}u(t)=u(1)\geqslant(1-\langle\beta,\xi\rangle)^{-1}\left\{\sum_{i=1}^{s-1}\beta_i\xi_1+\sum_{i=s}^{m}\beta_i(1-\xi_1)\right\}u(\sigma). \tag{2.33}$$

Case (iv) $\big(\inf_{\xi\leqslant t\leqslant 1}u(t)=u(1)$ and $\sigma>\xi_m\big)$ Note that

$$u(1)\geqslant\langle\beta,u(\xi)\rangle \text{ and } u(\xi_1)\geqslant\xi_1 u(\sigma)/\sigma\geqslant\xi_1 u(\sigma)$$

implies

$$\langle\beta,u(\xi)\rangle\geqslant\langle\beta,\xi\rangle u(\sigma),$$

so

$$\inf_{\xi\leqslant t\leqslant 1}u(t)\geqslant\langle\beta,u(\xi)\rangle\geqslant\langle\beta,\xi\rangle u(\sigma). \tag{2.34}$$

Combining (2.30), (2.31), (2.32), (2.33) and (2.34), we obtain (2.27) proving this lemma. □

REMARK 2.2. The constant γ_3 given in (2.27) differs from γ_1, γ_2 in (2.23), (2.24), which are valid for all $t\in[0,1]$, and gives a lower bound of $u(t)$ only for $t\in[\xi_1,1]$. This constant γ_3 reduces to that given by Guo, Shan and Ge [12 ; p.418] when $\alpha_i=0$ for all $i=1,2,\cdots,m$ in the boundary condition (2.3)

3. Superlinear boundary value problems

In this section, we prove the extensions of Theorem A to (BVP1), (BVP2), (BVP3) when $f(y)$ is superlinear. Instead of the original assumption in [37] that $f_0 = 0$ and $f_\infty = \infty$, we give upper bounds on f_0 and lower bounds on f_∞ in terms of boundary conditions $(1.2), (1.3), (1.4)$ and $a(t)$. Here the lower bounds of f_∞ are always larger than the upper bounds on f_0 and there remains a gap so that these existence theorems are not optimal in the sense that there exists only one positive number separating f_0 and f_∞. On the other hand, this is the case when $f(y)$ is sublinear which we shall show in section 4.

The main tool in proving the results in this section is the Krasnoselskii-Guo fixed point theorem, see Guo Laksmikantham [11], Deimling [4], Krasnoselskii [28].

THEOREM KG. *Let X be a Banack space with a cone $P \subseteq X$, and Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Let A be a completely continuous operator which maps $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ into P and satisfies either*

(a) (expanded form) $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$*; or*

(b) (compressed form) $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$.

Then A has a fixed point $\hat{x}$, i.e. $A\hat{x} = \hat{x}$, where $\hat{x} \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We now describe briefly the method of our proofs which are similar for all three boundary value problems. In particular, (BVP1) and (BVP2) having Neumann boundary conditions at $t = 1$ and $t = 0$ are simpler than (BVP3) because by Lemma 2.1 and 2.2 we know that solutions to $u'' + h(t) = 0$, $h(t) \geqslant 0$, $t \in [0,1]$, subject to (BC1) (BC2), i.e. (1.2), (1.3), have a lower bound in terms of its maximum, i.e. $u(t) \geqslant \gamma\|u\|, t \in [0,1]$ for some $\gamma > 0$. The conditions for non-resonance are imposed so that such solutions $u(t)$ are unique and that the constants C_i, D_i in (2.6) are solvable from the boundary conditions imposed at $t = 0$ and $t = 1$ (cf. Remark 5.2).

THEOREM 3.1. *Let $0 \leqslant \overline{\alpha} < 1, 0 < \overline{\beta} < 1$. Then $(BVP1)$ has the following properties with regard to b:*

(a) $(BVP1)$ *has a positive non-decreasing solution if $f_0 < \frac{1}{2}\Lambda_1$ and $f_\infty > \gamma_1^{-1}\Lambda_1$, for $b \geqslant 0$ in (1.2) sufficiently small, where*

$$\Lambda_1^{-1} = \int_0^1 sa(s)ds + (1-\overline{\alpha})^{-1}\langle \alpha, \int_0^1 g_1(\xi,s)a(s)ds\rangle + \frac{1-\overline{\alpha}+\langle\alpha,\xi\rangle}{(1-\overline{\alpha})(1-\overline{\beta})}\langle\beta, \int_\xi^1 a(s)ds\rangle \quad (3.1)$$

and

$$\gamma_1 = \langle\alpha,\xi\rangle/(1-\overline{\alpha}+\langle\alpha,\xi\rangle); \quad (3.2)$$

(b) $(BVP1)$ *has no positive solutions if* b *is sufficiently large;*

(c) *There exists* $b^* > 0$ *such that for all* $b,\ 0 < b < b^*$, $(BVP1)$ *has a positive non-decreasing solution and has no positive solution if* $b > b^*$.

Clearly statement (c) in Theorem 3.1 comprises both (a) and (b). Indeed part (a) and part (b) were proved in [55] whilst part (c) was proved under the additional assumption that $f(y)$ is monotone non-decreasing in y. Note that the constants Λ_1, γ_1 given in (3.1), (3.2) are the same as given in [55]. So we shall only prove part (c).

PROOF OF THEOREM 3.1 (c). From Part (b), there exists $b^* = \sup\{b : \text{(BVP1)}$ has a positive solution$\}$. We first show that for every $b,\ 0 \leqslant b < b^*$, (BVP1) has a positive solution. Let b be any such real constant. From the definition of b^*, there exists c, $b < c < b^*$, such that (BVP1) has a positive solution which will be denoted as $u_c(t)$. Using $u_c(t)$ we define the following function

$$F_1\big(u(t)\big) = \begin{cases} 0, & u'(t) < 0, \\ f\big(u_c(t)\big), & u'(t) > u_c'(t) > 0, \\ f\big(\overline{u}(t)\big), & u_c'(t) \geqslant u'(t) \geqslant 0, \end{cases} \tag{3.3}$$

where $\overline{u}(t) = \max\left\{0,\ \min\big(u(t), u_c(t)\big)\right\}$. Now consider the boundary value problem

$$u''(t) + a(t)F_1\big(u(t)\big) = 0 \tag{3.4}$$

subject to (BC1), i.e. (1.2). Note that $F_1\big(u(t)\big)$ is uniformly bounded by a constant M_1 which depends only on $u_c(t)$, i.e.

$$M_1 = \max\left\{f\big(u(t)\big) : 0 \leqslant u(t) \leqslant u_c(t)\right\}$$

since $0 \leqslant \overline{u}(t) \leqslant u_c(t)$. Therefore by Schauder's fixed point theorem applied to a bounded subset of P, we know that the boundary value problem $(3.4), (1.2)$ has a positive solution which we denote as $u_1(t)$. We shall now show that $u_1(t)$ satisfies (i) $0 \leqslant u_1'(t) \leqslant u_c'(t)$, and (ii) $0 \leqslant u_1(t) \leqslant u_c(t)$. Suppose that $v_1(t) = u_c'(t) - u_1'(t) < 0$ on an open interval $(\tau_1, \tau_2) \subseteq [0,1], 0 < \tau_1 \leqslant \tau_2 < 1$. By definition (3.3), we know that $v_1'(t) \equiv 0$ on (τ_1, τ_2) and $v_1'(t) = 0$ whenever $v_1(t) < 0$. Hence $v_1(t)$ equals to a negative constant for all $t \in [0,1]$. Let $v_1(t) \equiv c_1 < 0$ for some negative constant c_1. Using (1.2), we note $c_1 = v_1(1) = \langle \beta, v_1(\xi) \rangle + c - b$, or $(1 - \overline{\beta})c_1 = c - b > 0$. Since $0 < \overline{\beta} < 1$, so $c_1 > 0$. This contradiction proves that $v_1(t) \geqslant 0, t \in [0,1]$. On the other hand, $u_1(t)$ is a solution to (3.4), so $u_1''(t) \leqslant 0$ which implies $u_1'(0) \geqslant u_1'(t) \geqslant u_1'(1)$. Using (1.2) once again, we have

$$u_1'(1) = \langle \beta, u_1'(\xi) \rangle + b \geqslant \overline{\beta} u_1'(1) + b$$

which gives $(1 - \overline{\beta})u_1'(1) \geqslant b$, so $u_1'(t) \geqslant u_1'(1) \geqslant 0$ since $b \geqslant 0$. This proves (i) and so $F_1\big(u_1(t)\big) = f\big(\overline{u}_1(t)\big)$ for all $t \in [0,1]$.

To prove (ii), we let $w_1(t) = u_1(t) - u_c(t)$. Note that $w_1'(t) = -v_1(t) \leqslant 0$ for all $t \in [0,1]$ so we have $w_1(0) \geqslant w_1(t)$ so by (1.2) again we have

$$w(0) = \langle \alpha, w(\xi) \rangle \leqslant \overline{\alpha} w(0),$$

which implies by $0<\overline{\alpha}<1$ that $w(0)=0$. Hence $w(t)\leqslant 0$ for all $t\in[0,1]$, so $u_c(t)\geqslant u_1(t)$ and $\overline{u}_1(t)=u_1(t)$. This gives $F_1(u_1(t))=f(\overline{u}_1(t))=f(u_1(t))$, proving that $u_1(t)$ is a positive solution to (BVP1) since it is now the same as (3.4), (1.2).

To complete the proof, we also need to prove that (BVP1) has a positive solution when $b=b^*$. From the definition of b^*, we can choose a strictly increasing sequence $\{b_m : \lim_{m\to\infty} b_m = b^*\}$ such that (BVP1) has a positive solution for every b_m. We denote such solutions by $u_m(t)$ which satisfy $0\leqslant r\leqslant \|u_m\|\leqslant R$. Since $u_m=A_3u_m$ and A_3 is completely continuous, so $\{u_m\}$ has a convergent subsequence $\{u_{m_i}\}$ such that $\lim_{i\to\infty} u_{m_i}=u^*$. Thus $A_3u^*=u^*$ is a positive solution of (BVP1) satisfying boundary condition (1.2) with $b=b^*$. This completes the proof.

We now turn to (BVP2), i.e. (1.1) and (1.3). Here the integral operator A_2 is given by (2.5), (2.21) which by using (2.7) can be restated as

$$A_2y(t)=G_2(t)+(1-\overline{\alpha})^{-1}\langle\alpha,G_2'(\xi)\rangle t \\ +(1-\overline{\beta})^{-1}\left\{\langle\beta,G_2(\xi)\rangle-\frac{(1-\langle\beta,\xi\rangle)}{1-\overline{\alpha}}\langle\alpha,G_2'(\xi)\rangle+b\right\}, \quad (3.5)$$

where

$$G_2(t)=\int_0^1 g_2(t,s)a(s)f\big(y(s)\big)ds \ \text{ and } \ G_2'(t)=-\int_0^1 a(s)f\big(y(s)\big)ds,$$

as given in (2.16).

THEOREM 3.2. *Let* $0\leqslant\overline{\alpha}<1$, $0<\overline{\beta}<1$. *Suppose that* $f(y)$ *satisfies*

$$f_0<\frac{1}{2}\Lambda_2 \quad \text{and} \quad f_\infty>\gamma_2^{-1}\Lambda_2, \quad (3.6)$$

where

$$\Lambda_2^{-1}=\int_0^1(1-s)a(s)ds+(1-\overline{\beta})^{-1}\langle\beta,G_2[a](\xi)\rangle \\ -(1-\overline{\alpha})^{-1}(1-\overline{\beta})^{-1}\left\{(1-\langle\beta,\xi\rangle)\langle\alpha,G_2'[a](\xi)\rangle\right\}, \quad (3.7)$$

and

$$G_2[a](\xi)=\int_0^1 g_2(\xi,s)a(s)ds, \ \ G_2'[a](\xi)=-\int_0^\xi a(s)ds$$

are vector-valued function of $\xi=(\xi_1,\xi_2,\cdots,\xi_m)$. *Then there exists* $b^*>0$ *such that for all* b, $0\leqslant b<b^*$, $(BVP2)$ *has a positive non-increasing solution and no positive solution if* $b>b^*$.

Proof. We apply Theorem KG to the positive operator A_2 defined by (2.5) or (2.21) with $j=2$, or alternatively (2.6), (2.7), (2.14), (2.15) (2.16). We first show that A_2 maps the subcone $P_2\subseteq P$ defined by

$$P_2=\{y(t)\in P : y(t)\geqslant\gamma_2\|y\|, t\in[0,1]\}$$

into itself. For any $y \in P$, $A_2y(t)$ is a solution to $u'' + h(t) = 0$ with $h(t) = a(t)f\big(y(t)\big)$, so by Lemma 2.2, $A_2y(t) \geqslant \gamma_2\|A_2y(t)\|$. This shows $A_2(P_2) \subseteq P_2$.

Next we prove that for b sufficiently small, (BVP2) has a positive solution. Since $f_0 < \frac{1}{2}\Lambda_2$, we can choose $r > 0$ r sufficiently small such that $f(y) \leqslant \frac{1}{2}\Lambda_2 y$ for all $y \in [0,r]$. Denote

$$\Omega_1 = \{y \in P : \|y\| < r\} \text{ and } \partial\Omega_1 = \{y \in \overline{\Omega}_1 : \|y\| = r\}.$$

For $y \in \partial\Omega_1$, we estimate (3.5) from above using (2.7), (2.16) and obtain

$$\begin{aligned} A_2y(t) \leqslant \frac{1}{2}\Lambda_2\|y\|\big\{G_2[a](0) + (1-\overline{\beta})^{-1}\langle\beta, G_2[a](\xi)\rangle\big\} \\ - (1-\overline{\alpha})^{-1}(1-\overline{\beta})^{-1}\big\{(\overline{\beta} - \langle\beta,\xi\rangle)\langle\alpha, G_2'[a](\xi)\rangle\big\} + \frac{b}{1-\overline{\beta}} \end{aligned} \tag{3.8}$$

by setting $t = 1$ in (3.5). By definition of Λ_2 in (3.7), we obtain from (3.8) that

$$A_2y(t) \leqslant \frac{1}{2}\|y\| + \frac{b}{1-\overline{\beta}}.$$

We now choose $b < \frac{1}{2}(1-\overline{\beta})r$, and obtain $\|A_2y\| \leqslant \|y\|$ for all $y \in P_2 \cap \partial\Omega_1$, verifying the first part of Theorem KG in its expanded form. Here we use the fact that $G_2'[a](t) \leqslant 0$ so

$$G_2[a](t) \leqslant G_2[a](0) = \int_0^1 (1-s)a(s)ds.$$

To verify the second part of Theorem KG in its expanded form, we have from (3.6) $f_\infty > \gamma_2^{-1}\Lambda_2$ so choose $R > r > 0$ such that $f(y) \geqslant \gamma_2^{-1}\Lambda_2 y$ for all $y \geqslant R$. Let

$$\Omega_2 = \{y \in P : \|y\| < R\} \text{ and } \partial\Omega_2 = \{y \in \overline{\Omega}_2 : \|y\| = R\}.$$

For $y \in P_2 \cap \partial\Omega_2$ evaluate $A_2y(0)$ by (3.5) and obtain

$$\begin{aligned} \|Ay\| \geqslant A_2y(0) \geqslant \Lambda_2\|y\|\Big\{\int_0^1 (1-s)a(s)ds + (1-\overline{\beta})^{-1}\langle\beta, G_2[a](\xi)\rangle\Big\} \\ - (1-\overline{\alpha})^{-1}(1-\overline{\beta})^{-1}\big\{(1 - \langle\beta,\xi\rangle)\langle\alpha, G_2'[a](\xi)\rangle\big\} + \frac{b}{1-\beta} \end{aligned} \tag{3.9}$$

which implies $\|A_2y\| \geqslant \|y\|$ by definition of Λ_2 in (3.7). Now both assumptions of Theorem KG in its expanded form are satisfied, so A_2 has a fixed point $\hat{y} \in \overline{\Omega}_2 \setminus \Omega_1$, satisfying $0 < r \leqslant \|\hat{y}\| \geqslant R$. by Lemma 2.2, $\underline{\hat{y}(t)}$ is a positive solution of (BVP2) when b is sufficiently small, namely, $0 \leqslant b < \frac{1}{2}(1-\overline{\beta})r$.

We now show that under superlinearity condition (3.6), (BVP2) has no positive solution for sufficiently large b. Suppose the contrary that there exists a sequence $\{b_n\}$ with $\lim\limits_{n\to\infty} b_n = \infty$ such that (BVP2) has a positive solution $y_n(t)$ for every n satisfying (2.1) with $b = b_n$ By Lemma 2.2, we know that all such solutions are monotone non-increasing in $[0,1]$ and $A_2y_n = y_n$ for all n. Observe from (3.9),

$$\|y_n\| = y_n(0) = Ay_n(0) \geqq G_2(0) \geqslant b_n/(1-\overline{\beta}) \to \infty, \ \ n \to \infty. \tag{3.10}$$

Use (3.6) to choose $R > 0$ such that $f(y) \geqslant \gamma_2^{-1} y$ for all $y \in [\gamma_2 R, \infty)$. We evaluate $A_2 y_n(0)$ for $\|y_n\| \geqslant R$ which is possible for large n by (3.10), so

$$\|y_n\| \geqslant \Lambda_2 \{G_2(0) + D_2[a]\} \|y_n\| + b_n/(1-\overline{\beta}), \tag{3.11}$$

where $D_2[a]$ is given by (2.15) with $G_2(\xi)$, $G_2'(\xi)$ replaced by $G_2[a](\xi)$, $G_2'[a](\xi)$ as defined in (3.7). Using (3.7) in (3.11), we conclude $\|y_n\| \geqslant \|y_n\| + b_n/1 - \overline{\beta}$ which leads to a desired contradiction as $n \to \infty$. This proves that there exists $b^* > 0$ such that for $b > b^*$, (BVP2) has no positive solution.

To complete the proof, we need to show that (BVP2) has a positive solution for every b, $0 \leqslant b \leqslant b^*$. We follow the same approach as in the proof of Theorem 3.1 (c) by first setting $b^* = \sup\{b :$ (BVP2) has a positive solution$\}$ which exists because of the preceding arguments. For any $b, 0 \leqslant b < b^*$, there exists by the definition of b^* a constant $c, b < c < b^*$ such that (BVP2) has a positive solution which $b = c$ in its boundary condition (2.2). We denote such a solution by $u_c(t)$. Using $u_c(t)$ we define the $F_2\big(u(t)\big)$, a function similar to (3.3), by

$$F_2\big(u(t)\big) = \begin{cases} 0, & u'(t) > 0, \\ f\big(u_c(t)\big), & u'(t) < u_c'(t) < 0, \\ f\big(\overline{u}(t)\big), & u_0'(t) \leqslant u'(t) \leqslant 0, \end{cases} \tag{3.12}$$

where $0 \leqslant \overline{u}(t) = \max\big\{0, \min\big(u(t), u_c(t)\big)\big\} \leqslant u_c(t)$. This function $F_2(u(t))$ like $F_1\big(u(t)\big)$ in (3.3) is also uniformly bounded by a constant

$$M_2 = \max\ \big\{f\big(u(t)\big) : 0 \leqslant u(t) \leqslant u_c(t)\big\},$$

where $u_c(t)$ is the solution of (BVP2) given above.

Now consider the boundary value problem:

$$u''(t) + a(t)F_2\big(u(t)\big) = 0, \quad 0 < t < 1, \tag{3.13}$$

subject to boundary condition (1.3) or in the form of (2.2). Since $F_2(u(t))$ is uniformly bounded by a constant M_2, we know by an application of Schauder's fixed point theorem to the operator equation associated with (3.13), (2.2) that it has a positive solution which we denote it by $u_2(t)$.

To show that $u_2(t)$ is a positive solution of (1.1) (2.2) we shall prove firstly (i) $F_2\big(u_2(t)\big) = f\big(\overline{u}_2(t)\big)$ and secondly (ii) $\overline{u}_2(t) = u_2(t)$. Let $v_2(t) = u_c'(t) - u_2'(t)$. We claim $v_2(t) \leqslant 0$ for all $t \in [0,1]$ which proves $F_2\big(u_2(t)\big) = f\big(\overline{u}_2(t)\big)$. Suppose that $v_2(t) > 0$ for $t \in (\tau_1, \tau_2) \subset [0,1]$, $0 < \tau_1 < \tau_2 < 1$. Then by definition of $F_2\big(u(t)\big)$ in (3.12), $v_2'(t) \equiv 0$ in (τ_1, τ_2) so $v_2(t) \equiv c_2 > 0$ for $t \in (\tau_1, \tau_2)$. Again we can extend the sub-interval (τ_1, τ_2) by continuity to the entire interval $[0,1]$ so that $v_2(t) \equiv c_2$ for all $t \in [0,1]$. Now boundary condition (2.2) show that $c_2 = v_2(0) = \langle \alpha, v(\xi) \rangle = \overline{\alpha} c_2$, so $c_2 = 0$ since $0 \leqslant \overline{\alpha} < 1$. This contradiction proves (i).

Next we let $w_2(t) = u_2(t) - u_c(t)$ and claim $w_2(t) \leqslant 0$. Suppose otherwise that $w_2(t) > 0$ in $(\tau_1, \tau_2) \subseteq [0,1], 0 < \tau_1 < \tau_2 < 1$. Then $w_2'(t) = -v_2(t) \geqslant 0$ for all $t \in [0,1]$. Using boundary condition (2.2), we find

$$w_2(1) = \langle \alpha, w_2(\xi) \rangle + b - c < \overline{\alpha} w_2(1),$$

which implies $w_2(1) < 0$ since $0 \leqslant \overline{\alpha} < 1$, so $w_2(t) \leqslant w_2(1) < 0$. This contradiction proves $w_2(t) \leqslant 0$. Hence $u_2(t) \leqslant u_c(t)$. Therefore, $\overline{u}_2(t) = u_2(t)$ and $u_2(t)$ is the desired solution to (BVP2).

The proof that (BVP2) has a positive solution for $b = b^*$ in (2.2) is similar to that given in the proof of Theorem 3.1 (c). This completes the proof of Theorem 3.2.

REMARK 3.1. We note that Theorem 3.1 improves Theorem C in two ways. It shows that the condition that $f(y)$ is non-decreasing in y is superfluous and it also shows that the result is valid when $\overline{\alpha} = 0$ and including both $b = 0$ and $b = b^*$.

REMARK 3.2. (BVP2) when $b = 0$ was studied in Ma and Castaneda [40] where it was proved under the stronger assumption that $f_0 = 0$ and $f_\infty = \infty$. The finite bounds on f_0, f_∞ used in [40] are more stringent than (3.6) which bears a close resemblance with the bounds used in Theorem 3.1.

We now study (BVP3) where the boundary condition (2.3) includes that of the three-point problem (E_b) namely (1.6). Here the operator A_3 defined by (2.5), (2.22), or alternatively (2.6), (2.17), (2.18), takes the following form:

$$A_3y(t) = G_3(t) + \frac{t}{\Lambda}\{(1-\overline{\alpha})[\langle\beta, G_3(\xi)\rangle + b] - (1-\overline{\beta})\langle\alpha, G_3(\xi)\rangle + b(1-\overline{\alpha})\}$$
$$+\frac{1}{\Lambda}\{(1-\langle\beta,\xi\rangle)\langle\alpha, G_3(\xi)\rangle + \langle\alpha,\xi\rangle\langle\beta, G_3(\xi)\rangle + b\langle\alpha,\xi\rangle\},$$

where $\Lambda = (1-\overline{\alpha})(1-\langle\beta,\xi\rangle) + (1-\overline{\beta})\langle\alpha,\xi\rangle$. Like the case with (BVP2) the constant C_3 associated with t in (2.17) may be negative so A_3 is a positive operator only if $C_3\, t + D_3 \geqslant 0$ for $t \in [0,1]$. The linear function satisfies

$$0 < m = \min(D_3, C_3 + D_3) \leqslant C_3\, t + D_3 \leqslant \max(D_3, C_3 + D) = M_0.$$

Since $G_3(t) \geqslant 0, D_3 > 0$ by (2.18), and

$$C_3 + D_3 = \frac{1}{\Lambda}\Big\{(\overline{\beta} - \langle\beta,\xi\rangle)\langle\alpha, G_3(\xi)\rangle + (1-\overline{\alpha} + \langle\alpha,\xi\rangle)[\langle\beta, G_3(\xi)\rangle + b]\Big\} \geqslant 0,$$

this shows that $A_3 : P \to P$. We now state and prove:

THEOREM 3.3. *Let* $0 \leqslant \overline{\alpha} < 1, 0 < \langle\beta,\xi\rangle < 1$. *Suppose that* $f(y)$ *satisfies*

$$f_0 < \frac{1}{2}\Lambda_3 \ \textit{and}\ f_\infty > \Lambda_4,$$

where

$$\Lambda_3^{-1} = \frac{1}{\Lambda}\left\{(|1-\overline{\alpha}| + \langle\alpha,\xi\rangle)\int_0^1 (1-s)a(s)ds\right\}, \tag{3.14}$$

$$\Lambda_4^{-1} = \frac{\gamma_3}{\Lambda}\{(1-\overline{\alpha} + \alpha\xi)\langle\beta, J[a](\xi)\rangle\}, \tag{3.15}$$

γ_3 is given by (2.27) and

$$J[a](\xi) = (J(\xi_1), \cdots, J(\xi_m)),\ J(\xi_i) = \xi_i \int_{\xi_i}^1 (1-s)a(s)ds.$$

Then there exists $b^* > 0$ such that (BVP3) has a positive solution for all $b, 0 \leqslant b \leqslant b^*$, and has no positive solution for $b > b^*$.

Proof. Define

$$P_3 = \{y \in P : \inf_{\xi_1 \leqslant t \leqslant 1} y(t) \geqslant \gamma_3 \|y\|\}$$

which is a subcone of P. Note that $\big(A_3 y(t)\big)'' + a(t) f\big(y(t)\big) = 0$ and $A_3 y(t)$ satisfies the boundary condition (2.3) so by Lemma 2.3 we conclude $A_3(P_3) \subset P_3$.

By assumption that $f_0 < \frac{1}{2}\Lambda_3$, there exists $r > 0$ so that $f(y) \leqslant \frac{1}{2}\Lambda_3 y$ for all $y \in [0, r]$. Denote $\Omega_1 = \{y \in P_0 : \|y\| < r\}$ and $\partial\Omega_1 = \{y \in \overline{\Omega}_1 : \|y\| = r\}$.

To find an upper bound for f_0, it is more convenient to use that alternative representation of A_3 by replacing $G_3(t)$ in (2.6) and (2.19) by the integral operator $I[y](t)$ as defined by

$$I(t) = I[y](t) = \int_0^t (t-s)a(s) f(y(s))ds. \tag{3.16}$$

Now we write

$$A_3 y(t) = G_3(t) + C_3 t + D_3 = -I(t) + B_1 t + B_2. \tag{3.17}$$

From (3.17) we find, upon setting $t = 0$ and $t = 1$, that

$$D_3 = B_2,\ \ C_3 + D_3 = -I(1) + B_1 + B_2.$$

Here B_1, B_2 are given by

$$\begin{aligned} B_1 &= \frac{1}{\Lambda}\big\{(1-\overline{\alpha})\,[I(1) - \langle \beta, I(\xi)\rangle + b] + (1-\overline{\beta})\langle \alpha, I(\xi)\rangle\big\}, \\ B_2 &= \frac{1}{\Lambda}\big\{\langle \alpha, \xi\rangle\,[I(1) - \langle \beta, I(\xi)\rangle + b] - (1 - \langle \beta, \xi\rangle)\langle \alpha, (\xi)\rangle\big\}, \end{aligned} \tag{3.18}$$

where $\Lambda = (1-\overline{\alpha})(1 - \langle \beta, \xi\rangle) + (1-\overline{\beta})\langle \alpha, \xi\rangle$. Note that $I(t) \geqslant 0$ implies by (3.17) that

$$\begin{aligned} A_3 y(t) &\leqslant \max(B_2, B_1 + B_2) \\ &\leqslant \frac{1}{\Lambda}\{(|1-\overline{\alpha}| + \langle \alpha, \xi\rangle)\,[I(1) + b]\}. \end{aligned} \tag{3.19}$$

For $y \in \partial\Omega_1$, we obtain from (3.19) that

$$A_3 y(t) \leqslant \frac{1}{\Lambda}\left\{(|1-\overline{\alpha}| + \langle \alpha, \xi\rangle)\frac{1}{2}\Lambda_3 \|y\| \int_0^1 (1-s)a(s)ds + b\right\}. \tag{3.20}$$

Choose $0 \leqslant b \leqslant \frac{r}{2}\Lambda(|1-\overline{\alpha}| + \langle \alpha, \xi\rangle)^{-1}$, then using the definition of Λ_3 in (3.14), we obtain from (3.20) that $\|A_3 y\| \leqslant \|y\| = r$, for all $y \in P_3 \cap \partial\Omega_1$.

Next we apply $f_\infty > \Lambda_4$ to choose $R > r > 0$ so that $f(y) \geqslant \Lambda_4 y$ for all $y \in [R,\infty)$. Define

$$\Omega_2 = \{y \in C[0,1] : \|y\| < R\} \text{ and } \partial\Omega_2 = \{y \in \overline{\Omega}_2 : \|y\| = R\}.$$

For $y \in P_3 \cap \partial\Omega_2$, we evaluate $A_3y(t)$ at $t = 1$ by (2.6) and use Lemma 2.3 and (2.17), (2.18) to find

$$\begin{aligned}\|A_3y\| &\geqslant A_3y(1) \geqslant C_3 + D_3\\ &= \frac{1}{\Lambda}\{(1-\overline{\alpha}+\langle\alpha,\xi\rangle)\,[\langle\beta, G_3(\xi)\rangle + b]\}\\ &\geqslant \frac{1}{\Lambda}\gamma_3\Lambda_4\|y\|\,\{(1-\overline{\alpha}+\langle\alpha,\xi\rangle)\langle\beta, J[a](\xi)\rangle\}.\end{aligned} \tag{3.21}$$

Using (3.15) in (3.21), we obtain $\|A_3y\| \geqslant \|y\|$. We can apply Theorem KG to obtain a fixed point $\hat{y} \in P_3 \cap (\Omega_2 \setminus \Omega_1)$ satisfying $0 < r < \|y\| < R$. When $\xi_1 \leqslant t \leqslant 1$, Lemma 2.3 shows that $\hat{y}(t) \geqslant \gamma_3\|\hat{y}\| \geqslant \gamma_3 r > 0$. For $0 \leqslant t < \xi_1$, we note that boundary condition (2.3) implies $\hat{y}(0) = \langle\alpha, \hat{y}(\xi)\rangle \geqslant 0$. Observe that

$$\left(t^{-1}\hat{y}(t)\right)' = t^{-2}\left(t\hat{y}'(t) - \hat{y}(t)\right) = t^{-2}g(t)$$

and that $g'(t) = t\hat{y}''(t) \leqslant 0$, $g(0) = -\hat{y}(0) \leqslant 0$, so $\left(t^{-1}\hat{y}(t)\right)' \leqslant 0$ and $t^{-1}\hat{y}(t) \geqslant \xi_1^{-1}\hat{y}(\xi_1) > 0$ for $0 \leqslant t < \xi_1$. This proves that $\widehat{y}(t) > 0$ and is a positive solution of (BVP3) provided that b is sufficiently small.

We now prove that (BVP3) has no positive solution when b is sufficiently large. Instead of repeating similar argument given in the proof of Theorem 3.1 (c), we adopt a different argument suggested by Sun, Chen, Zhang, Wang [45]. Suppose that $(\overline{y})(t)$ is a positive solution of (BVP3), so $A_3\overline{y} = \overline{y}$ for all $b \geqslant 0$. Use $f_\infty > \Lambda_4$ to choose $R > 0$ such that $f(y) \geqslant \Lambda_4 y$ for all $y \in [R,\infty)$. Form (3.21), we have $\overline{y}(1) = A_3y(1) \geqslant \frac{1}{\Lambda}(1-\overline{\alpha}+\langle\alpha,\xi\rangle)b$. Choose $b_0 \geqslant \Lambda(1-\overline{\alpha}+\langle\alpha,\xi\rangle)^{-1}R$. Thus $\|\overline{y}_0\| \geqslant \overline{y}_0(1) \geqslant R$, where $\overline{y}_0(t)$ is a solution of (BVP3) with $b = b_0$. Apply (3.21) once again, we find

$$\begin{aligned}\|\overline{y}_0\| \geqslant \overline{y}_0(1) = C_3 + D_3 &\geqslant \frac{1}{\Lambda}\left\{(1-\overline{\alpha}+\langle\beta.\xi\rangle)[\langle\beta, G_3(\xi) + b]\right\}\\ &\geqslant \frac{1}{\Lambda}\Lambda_4\gamma_3\|\overline{y}_0\|\left\{(1-\overline{\alpha}+\langle\alpha,\xi\rangle)\langle\beta, J[a](\xi)\rangle\right\} + \frac{1-\overline{\alpha}+\langle\alpha,\xi\rangle}{\Lambda}b_0\\ &\geqslant \|\overline{y}_0\| + R\end{aligned}$$

because of definition of Λ_4 given in (3.15). This is the desired contradiction.

Turning to the last portion of Theorem 3.3. From the proofs given in previous paragraphs, we can define $b^* = \sup\{b : \text{(BVP3) has a positive solution}\}$. We now prove that for any $b, 0 \leqslant b < b^*$, (BVP3) has a positive solution. By the definition of b^*, there must exist c, $b < c < b^*$ such that (BVP3) has a positive solution satisfying (2.3) with $b = c$ and we note it by $u_c(t)$. In terms of this solution $u_c(t)$, we define

$$F_3\big(u(t)\big) = \begin{cases} 0, & u(t) < 0,\\ f\big(u_c(t)\big), & u(t) > u_0(t),\\ f\big(u(t)\big), & 0 \leqslant u(t) \leqslant u_c(t), \end{cases} \tag{3.22}$$

and consider the boundary value problem

$$f(x) = \begin{cases} u'' + a(t)F_3\big(u(t)\big) = 0, \quad 0 < t < 1, \\ u(0) = \langle \alpha, u(\xi)\rangle, u(1) = \langle \beta, u(\xi)\rangle + b. \end{cases} \tag{3.23}$$

From (3.22), we know the $F_3\big(u(t)\big)$ is uniformly bounded by

$$M_3 = \max\left\{f\big(u(t)\big) : 0 \leqslant u(t) \leqslant u_c(t)\right\},$$

a constant dependent only on the known function $u_c(t)$. Now we can apply Schauder's fixed point theorem to (3.23) and obtain a positive solution $u_3(t)$ satisfying the boundary condition (2.3). We shall prove that $u_3(t)$ is a positive solution of (BVP3) by showing $F_3\big(u_3(t)\big) = f\big(u_3(t)\big)$, hence (3.23) becomes (BVP3).

Let $w(t) = u_3(t) - u_c(t)$ and we need to show that $w(t) \leqslant 0$ for all $t \in [0,1]$. At first we suppose that $w(t) > 0$ on a sub-interval (τ_1, τ_2), $0 < \tau_1 < \tau_2 < 1$, with $w(\tau_1) = w(\tau_2) = 0$. Note that $w(t) > 0$ implies by (3.22), (3.23) that $w''(t) = 0$ which in turn yields $w(t) \equiv 0$ in (τ_1, τ_2) contradicting $w(t) > 0$ in (τ_1, τ_2). Secondly, we note that between any two ξ_i, ξ_{i+j}, $1 \leqslant i, j \leqslant m-1$ at which $w(\xi_j) < 0$ and $w(\xi_{i+j}) < 0$, there can be no intermitent $\xi_k' s$ such that $w(\xi_j) > 0$, $i < k < i+j$ for otherwise this will lead to the case where exists a subinterval $(\tau_1, \tau_2) \subseteq [0,1], 0 < \tau_1, < \tau_2 < 1, w(t) > 0$ in (τ_1, τ_2) and $w(\tau) = w(\tau_2) = 0$ which has just been ruled out.

There remain three separate cases : (1) $w(t) > 0$ for all $t \in [0,1]$, (2) $w(t) > 0$ for $t \in [0, \tau_3), 0 < \tau_3 < 1$, and (3) $w(t) > 0$ for $t \in (\tau_4, 1], 0 < \tau_4 < 1$. In case (1), since $w''(t) \equiv 0$ in $[0,1]$, so $w(t) = [w(1) - w(0)]t + w(0)$ with $w(0), w(1) > 0$. Using the boundary condition (2.3) at $t = 0$ with $w(\xi) = [w(1) - w(0)]\xi + w(0)$, we have

$$w(0)(1 - \overline{\alpha} + \langle \alpha, \xi\rangle) = w(1)\langle \alpha, \xi\rangle. \tag{3.24}$$

Likewise with $t = 1$, in (2.3), we obtain

$$w(1) = \langle \beta, [w(1) - w(0)]\xi + w(0)\rangle + b - c$$

and since $b < c$, this leads to

$$w(1) = (1 - \langle \beta, \xi\rangle) < w(0)(\overline{\beta} - \langle \beta, \xi\rangle). \tag{3.25}$$

Combining (3.24), (3.25), we obtain

$$(1 - \overline{\alpha} + \langle \alpha, \xi\rangle)(1 - \langle \beta, \xi\rangle) < \langle \beta, \xi\rangle(\overline{\beta} - \langle \beta, \xi\rangle)$$

which contradicts the non-resonance condition $\Lambda > 0$ which is implied by $0 \leqslant \overline{\alpha} < 1$, $0 < \langle \beta, \xi\rangle < 1$.

In this case (2), $w''(t) \equiv 0$ on $[0, \tau_3]$ implies $w(t) = w'(0)(t - \tau_3)$ with $w(0) > 0$ and $w'(0) < 1$. Let $\xi_s = \max\{\xi_i : w(\xi_i) > 0\}$ for some $s = 1, \cdots, m-1$. If no such s exists, then this case reduces to Case (3). On the other hand, if $s = m$, then $w(t) > 0$ for all $t \in [0,1]$ and it becomes case (1). Using boundary condition (2.3) at $t = 0$, we observe

$$-w'(0)\tau_3 = w(0) =< \alpha, w(\xi) >\leqslant \sum_{i=1}^{s} \alpha_i w(\xi_i) = w'(0)\sum_{i=1}^{s} \alpha_i(\xi_i - \tau_3),$$

which reduces to

$$-w'(0)\tau_3\left(1-\sum_{i=1}^{s}\alpha_i\right)\leqslant w'(0)\sum_{i=1}^{s}\alpha_i\xi_i. \tag{3.26}$$

Since $-w'(0)>0$, and $\overline{\alpha}<1$, so we obtain from (3.26), $\tau_3<0$ which is impossible.

Case 3. Suppose that $w(t)>0$ on $(\tau_4,1]$ some τ_4 with $w(\tau_4)=1, 0<\tau_4<1$. We have by (3.22), (3.23) that $w''(t)\equiv 0$ in $[\tau_4,1]$, so $w(t)=w(1)(1-\tau_4)^{-1}(t-\tau_4)$. Evaluating boundary condition (2.3) at $t=1$, we find

$$w(1)=\langle\beta,w(\xi)\rangle+b-c<\langle\beta,w(1)(1-\tau_4)^{-1}(\xi-\tau_4)\rangle. \tag{3.27}$$

Since $\xi_i-\tau_4\leqslant\xi_i(1-\tau_4)$, (3.27) reduces to $w(1)<w(1)\langle\beta,\xi\rangle$ which contradicts the assumption $0<\langle\beta,\xi\rangle<1$.

Now that these four cases exhaust all possibilities for $w(t)>0$, we conclude that $w(t)\leqslant 0$ or $u_3(t)\leqslant u_c(t)$. This proves that (BVP3) has a positive solution for every $b=b^*$. Finally, the proof that (BVP3) also has a positive solution for $b=b^*$ is similar to that given in the proof of Theorem 3.1 (c). The completes the proof of Theorem 3.3.

REMARK 3.3. We remark that the conditions on $\alpha_i's,\beta_i's$ required by Theorems 3.1, 3.2, 3.3 are stronger than the usual non-resonance conditions for positive solutions. In case of (BVP1), (BVP2), to ensure the boundary conditions (2.1), (2.2) with $b=0$ do not give rise to nontrivial solutions for the base equation $y''=0$, the non-resonance condition is $\triangle=(1-\overline{\alpha})(1-\overline{\beta})>0$ which is implied by $0\leqslant\overline{\alpha}<1,\ 0<\overline{\beta}<1$ in Theorems 3.1 and 3.2. Likewise the condition that $0\leqslant\overline{\alpha}<1,\ 0<\langle\beta,\xi\rangle<1$ for (BVP3) implies the non-resonance condition $\Lambda=(1-\overline{\alpha})(1-\beta,\xi\rangle)+(1-\overline{\beta})\langle\alpha,\xi\rangle>0$. However, in all three theorems, we require $\overline{\beta}>0$, a condition required in the proofs to ensure the solutions established are positive throughout the entire interval $[0,1]$.

REMARK 3.4. In all three Theorems, we require $f_0<\frac{1}{2}\Lambda_j, j=1,2,3$ to accommodate $b\neq 0$. When $b\geqslant 0$ is sufficiently small in (2.1), (2.2), (2.3), these assumptions can be relaxed to $f_0<\Lambda_j, j=1,2,3$.

4. Optimal existence theorems. The Sublinear Case

Theorems in the previous section are intended as extensions of Theorem A when the nonlinear function $f(y)$ satisfies a superlinear condition given in Theorems 3.1, 3.2 and 3.3 but we say nothing about the situation when $f(y)$ satisfies a sublinear condition. In fact in a special case of (BVP2) when all $\alpha_i's$ are zero, Sun Chen, Zhang and Wang [45 ; Theorem 1.2] stated without proof an analogous result when $f_0=\infty$ and $f_\infty=0$. This is incorrect as pointed out in our earlier work [29 ; p.3-4] by the simple counterexample:

$$y''(t)+1=0,\ y'(0)=0,\ y(1)=\frac{1}{2}y(1/2)+b,$$

which has the unique solution $y(t)=-t^2/2+2b+7/8$ which is positive for all $b\geqslant 0$.

We observe that the superlinear conditions concerning (BVP1), (BVP2), (BVP3) provide upper and lower bounds for f_0 and f_∞. In the case of (BVP1), (BVP2) for $b=0$, we require

$$f_0 < \Lambda_i < \gamma_i^{-1}\Lambda_i < f_\infty,\ i=1,2 \tag{4.1}$$

and for (BVP3) with $b=0$,

$$f_0 < \Lambda_3 < \Lambda_4 < \gamma_3^{-1}\Lambda_4 < f_\infty. \tag{4.2}$$

Both (4.1) and (4.2) leave an interval between f_0 and f_∞. For the simpler problem (E_b) with $b=0$, this is also the case amongst studies reported by Ma [36], Liu [31], [32] Zhang and Wang [56]. In [29], we compare f_0, f_∞ with the smallest eigenvalue λ_1 of the linear boundary value problem:

$$(R) \quad \begin{cases} u'' + \lambda a(t)u = 0, \\ \sin\theta u(0) = \cos\theta u'(0), \\ u(1) = \sum\limits_{i=1}^{m} \beta_i u(\xi_i) + b, \end{cases} \tag{4.3}$$

where $\beta_i > 0, i=1,2,\cdots,m$ and $b \geqslant 0$ and proved

THEOREM E. (*a*) *Suppose that $f_0 < \lambda_1 < f_\infty$. Then there exists a constant $b^* > 0$ such that the boundary value problem $BVP(R)$ has a positive solution for all b, $0 \leqslant b \leqslant b^*$, and no positive solution for $b > b^*$.*

(*b*) *Suppose that $f_\infty < \lambda_1 < f_0$. Then the boundary value problem $BVP(R)$ has a positive solution for all $b \geqslant 0$.*

Theorem E was proved by using classical "shooting method" and Sturm's Comparism Theorem. We note that this approach is unable to deal with boundary value problems (BVP1), (BVP2), (BVP3) because the boundary conditions at $t=0$ involves interior boundary points. On the other hand, application of Theorem KG does not involve the linear problem (R), so we must find alternative methods.

Theorem B by Zhang and Sun [54] was proved by applying the Krein-Rutman theorem together Krasnoselskii fixed point theorem via topological degree theory and established optimal existence theorem for the three point homogeneous boundary value problem (E_b) with $b=0$. In this section, we intend to do the same for the more general boundary value problems (BVP1), (BVP2), (BVP3).

We require the following variant of Krasnoselskii's fixed point theorem stated in the form using fixed point indices from topological degree theory, see e.g. Erbe [8], Han[19], Webb [49], Webb and Lan [50], Zhang and Sun [54].

THEOREM K. *Let X be a Banach space and $P \subseteq X$ be an ordered cone. Suppose that Ω is an open subset of X with non-empty interior. Let $A : P\cap\Omega \to P$ be a completely continuous operator.*

(a) *If there exists $p \in P, p \neq 0$ such that $u - Au \neq \mu p$ for all $u \in P\cap\partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω, and all $\mu \geqslant 0$, then the fixed point index of A with regard to $P\cap\partial\Omega$ satisfies $i(A; P\cap\Omega, P) = 0$;*

(b) *If* $u \neq \mu Au$ *for all* $u \in P \cap \partial\Omega$ *and all* $\mu, 0 \leqslant \mu \leqslant 1$, *then* $i(A, P \cap \Omega, P) = 1$.

We apply Theorem K to operators A_1, A_2, A_3 as defined by (2.5) with kernels $K_j(t,s), j = 1,2,3$ as given by (2.20), (2.21), (2.22) and prove existence of positive solutions of (BVP1), (BVP2), (BVP3) in the sublinear case, i.e. $f_0 > f_\infty$. We note that it is a standard argument to prove that A_1, A_2, A_3 are completely continuous operators. A word of caution is required for operators A_1, A_2 since unlike the kernel function $K_3(t,s)$ as given in (2.22) the kernel functions $K_1(t,s), K_2(t,s)$ given by (2.20), (2.21) are not continuous in $(t,s) \in [0,1] \times [0,1]$. This is because $g_1'(t,s),\ g_2'(t,s)$ is discontinuous along the lines $s = \xi_i, i = 1,2,\cdots,m$. However, we note that for $t \neq \tau,\ j = 1,2$

$$\left| \int_0^1 (K_j(t,s) - K_j(\tau,s))\, a(s) f\big(y(s)\big) ds \right| \leqslant B_0 \int_0^1 |K_j(t,s) - K_j(\tau,s)| ds$$

and

$$|K_j(t,s) - K_j(\tau,s)| \leqslant |g_j(t,s) - g_j(\tau,s)| + B_1 |t - \tau|, \tag{4.4}$$

where B_0 is a constant depending on continuous functions $a(t), f(y)$ and

$$B_1 = \operatorname*{Max}_{0 \leqslant s \leqslant 1} \left\{ \frac{1}{1 - \overline{\beta}} \left[\langle \beta, g_1'(\xi, s) \rangle + b \right], \frac{1}{1 - \overline{\alpha}} \langle \alpha, g_2'(\xi, s) \rangle \right\}.$$

Since $|g_j(t,s) - g_j(\tau,s)| \leqslant |t - \tau|$, and (4.4) holds uniformly for all $s \in [0,1]$ hence $K_j(t,s)$ is equicontinuous, $j = 1,2$. This shows that A_1, A_2 are completely continuous operators.

We need the following result of Krein-Rutman. See Zeildler [52, p.290, Theorem 7.C], Krein and Rutman [27]:

THEOREM K-R. (Krein and Rutman [25]) *Let* X *be a Banach space with an ordered cone* P *and its interior int* P *is non-empty. Suppose that* L *is a linear operator which maps* P *into itself. If* L *is completely continuous and strongly positive, i.e.* $Ly \in$ *int* P *for every* $Ly \in$ *int* P, *then* L *has a positive eigenvector* φ *corresponding to the positive eigenvalue* $r(L)$, *where* $r(L)$ *denotes the spectral radius of* L.

We define the linear operators L_j associated with $A_j,\ j = 1,2,3$ by

$$L_j y(t) = \int_0^1 K_j^0(t,s) a(s) y(s) ds,\ j = 1,2,3, \tag{4.5}$$

where $K_j^0(t,s)$ equal to $K_j(t,s)$ given by (2.20), (2.21), (2.22) with $b = 0$. Clearly since $A_j{}'s$ are completely continuous, so are the $L_j{}'s$. To apply Theorem K-R to L_j, we need to show that they are strongly positive.

Note that $C_1, D_1 \geqslant 0$, so $K_1^0(t,s) \geqslant g_1(t,s) > 0$ all $t,s \in (0,1)$. Since $a(t) \not\equiv 0$ in $[0,1]$, there exists $t_0 \in (0,1)$ such that $a(t) > 0$ in $(\tau_1, \tau_2), 0 < \tau_1 < \tau_2 < 1$. Now for $y(t) > 0, t \in [0,1]$, we have

$$L_1 y(t) \geqslant \int_{\tau_1}^{\tau_2} K_1^0(t,s) a(s) u(s) ds > 0 \tag{4.6}$$

showing that L_1 is strongly positive.

For L_2, we note that $C_2 \leqslant 0$ since $G_2'(\xi_i) \leqslant 0, i = 1,2,\cdots,m$ in (2.14), so $D_2 \geqslant C_2 + D_2$. Furthermore,

$$C_2 + D_2 = \frac{1}{1-\overline{\beta}} \left\{ \langle \beta, G_2(\xi) \rangle - \frac{(\overline{\beta} - \langle \beta, \xi \rangle)}{1-\overline{\alpha}} \langle \alpha, G_2'(\xi) \rangle + b \right\} \geqslant 0,$$

because $\overline{\beta} \geqslant 0, \overline{\alpha} \geqslant 0, b \geqslant 0$. Thus $K_2^0(t,s) \geqslant g_2(t,s) > 0$ all $t,s \in (0,1)$, and a similar argument applies proving that L_2 is strongly positive.

Turning to L_3, we know from the proof of $A_3(P_3) \subseteq P_3$ that both $C_3, D_3 \geqslant 0$, so $K_3^0(t,s) \geqslant g_3(t,s)$. Clearly $g_3(t,s) > 0$ for all $t,s, \in (0,1)$, again by a similar argument as (4.6), we can show that $L_3 y(t) > 0$ whenever $y(t) > 0$ for all $t \in [0,1]$. Thus, L_3 is also strongly positive.

Now we are in position to apply Theorem K-R to the three linear operators L_j, $j = 1,2,3$ given in (4.5). Denote μ_j the corresponding positive eigenvalue of the linear boundary value problem

$$u'' + \mu a(t) u = 0 \tag{4.7}$$

subject to homogeneous boundary conditions (2.1), (2.2), (2.3) with $b = 0$ for $j = 1,2,3$ respectively.

THEOREM 4.1. *Let* $0 < \overline{\alpha} < 1, 0 \leqslant \overline{\beta} < 1$. *Suppose that* $f(y)$ *satisfies* $f_\infty < \mu_1 < f_0$, *where* μ_1 *is the smallest eigenvalue of the boundary value problem* (4.7) *for* $j = 1$, *then the* $(BVP1)$ *has a positive solution for every* $b \geqslant 0$.

REMARK 4.1. We remark that Theorem 4.1 improves upon Theorem D substantively in two directions. To see this, let φ_1, be the positive eigenvector μ_1, i.e. $\varphi_1 = \mu_1 L_1 \varphi_1$. Using $K_1^0(t,s)$ given by (2.20) with $b = 0$, we find

$$\varphi_1(t) = \mu_1 \int_0^1 K_1^0(t,s) a(s) ds \leqslant \mu_1 \Lambda_1^{-1} \|\varphi_1\|$$

for all $t \in [0,1]$. Taking $\bar{t} \in [0,1]$ such that $\varphi_1(\bar{t}) = \|\varphi_1\|$ we obtain $\Lambda_1 \leqslant \mu_1$. On the other hand, since φ_1 is a solution of (4.7), satisfying boundary condition (2.1) with $b = 0$ then $\varphi_1(t) \geqslant \gamma_1 \|\varphi_1\|$ by Lemma 2.1. Observe that

$$\|\varphi_1\| \geqslant \varphi_1(1) = \mu_1 \int_0^1 K_1^0(l,s) a(s) \varphi_1(s) ds \geqslant \mu_1 \Lambda_1^{-1} \gamma_1 \|\varphi_1\|,$$

so $\mu_1 \leqslant \gamma_1^{-1} \Lambda_1$. Thus condition (3.1), and $f_0 = \infty, f_\infty = 0$ in Theorem D, both imply $f_\infty < \mu_1 < f_0$. Furthermore, Theorem D assumes $\overline{\alpha} > 0$ and $b > 0$, but the proof of Theorem 4.1 shows that either $\overline{\alpha} > 0$ or $b > 0$ will suffice but not both.

PROOF OF THEOREM 4.1 Let $P_1 = \{y(t) \in P : y(t) \geqslant \gamma_1 \|y\|\}$, where γ_1 is given by (2.23). When we note from the fact that $A_1 y(t)$ is a solution to $u'' + h(t) = 0$ with $h(t) = a(t) f\big(y(t)\big)$ and satisfying boundary condition (2.1) so by Lemma 2.1, $A_1(P_1) \subseteq$

P_1. Now since $C_1, D_1 \geqslant 0$ so we have $K_1^0(t.s) \geqslant g_1(t,s) > 0$ for all $t \in (0,1)$. Referring to the argument relating to (4.6), we conclude that L_1 is strongly positive. Let μ_1 be the smallest eigenvalue of the boundary value problem

$$\begin{cases} u'' + \mu a(t)u = 0, \quad 0 < t < 1, \\ u(0) = \langle \alpha, u(\xi) \rangle, \ u'(1) = \langle \beta, u'(\xi) \rangle. \end{cases} \tag{4.8}$$

By assumption that $f_0 > \mu_1$, we can find $r > 0$ such that $f(y) \geqslant \mu_1 y$ for all $y \in [0,r]$. Define $\Omega_1 = \{y \in C[0,1] : \|y\| < r\}$ and $\partial\Omega_1 = \{y \in \overline{\Omega_1} : \|y\| = r\}$. To verify condition (a) of Theorem K, we suppose the contrary that there exists $y_0 \in P_1 \cap \partial\Omega_1$ such that $y_0 = A_1 y_0 + \sigma_0 \varphi_1$, for some $\sigma_0 > 0$. (If $\sigma_0 = 0$, then $y_0 \in P_1$ is a fixed point of A_1 which is a positive solution of (BVP1).) Define $\sigma^* = \sup\{\sigma : y_0 \geqslant \sigma\varphi_1\}$. Since A_1 is positive so $\sigma_0 \leqslant \sigma^*$ and σ^* exists. Observe that $y_0 \geqslant \sigma^* \varphi_1$ implies

$$\begin{aligned} y_0 &= A_1 y_0 + \sigma_0 \varphi_1 \geqslant \mu_1 L_1 y_0 + \sigma_0 \varphi_1 \\ &\geqslant \mu_1 \sigma^* L_1 \varphi_1 + \sigma_0 \varphi_1 = (\sigma^* + \sigma_0)\varphi_1, \end{aligned}$$

which contradicts the definition of σ^*. Now by Theorem K (a), $i(A_1; P_1 \cap \Omega_1, P_1) = 0$.

Turning to the second part of Theorem K, we use the assumption that $f_\infty < \mu_1$ to choose $R > r > 0$ so that $f(y) \leqslant \mu_1 y$ if $y \in [R, \infty)$. Suppose that condition (b) of Theorem K does not hold, then the set $W = \{v \in P_1 : v = \sigma A_1 v, 0 \leqslant \sigma \leqslant 1\}$ is non-empty. We shall show that the set W is bounded. Let $v \in W$. and $v = \sigma A_1 v$ for some σ, $0 \leqslant \sigma < 1$. (Note that if $\sigma = 1$, then v is a fixed point of A_1, hence it is a positive solution of (BVP1) because $v \in P_1$). We now use the Hammerstein operator A_1 to estimate $v(t)$,

$$v(t) = \sigma A_1 v(t) = \sigma \left\{ \int_E + \int_{E^c} \right\} K_1(t,s)a(s)f\big(v(s)\big)ds + \sigma \frac{b}{D}, \tag{4.9}$$

where $E = \{s \in [0,1] : v(s) \leqslant R\}$ and $E^c = [0,1] \setminus E$. Define $\overline{v}(t) = \min\{v(t), R\}$. We obtain form (4.9),

$$\begin{aligned} v(t) &\leqslant \sigma \int_{E^c} K_1(t,s)a(s)f\big(v(s)\big)ds + \sigma \int_0^1 K_1(t,s)a(s)f\big(\overline{v}(s)\big)ds + \frac{b}{D} \\ &\leqslant \sigma\mu_2 \int_0^1 K_1(t,s)a(s)v(s)ds + M + \frac{b}{D}, \end{aligned} \tag{4.10}$$

where

$$M = \sup_{0 \leqslant t \leqslant 1} \int_0^1 K_1(t,s)a(s)f\big(\overline{v}(s)\big)ds + \frac{b}{D} < \infty$$

is a finite constant depending on $\max\limits_{0 \leqslant u \leqslant R} f(u)$ but independent of $v(t)$. From (4.10) we have

$$v \leqslant \sigma\mu_1 L_1 v + M_1, \quad v \in W, \tag{4.11}$$

where $M_1 = M + \frac{b}{D}$. Denote the linear operator $L_\sigma = \sigma\mu_2 L_1$ and by Gelfand's formula for spectral radius $r(L_\sigma)$ we have

$$r(L_\sigma) = \lim_{n\to\infty} \|(\sigma\mu_1 L_1)^n\|^{1/n} = \sigma\mu_1 \lim_{n\to\infty} \|L_1^n\|^{1/n}$$
$$= \sigma\mu_1 r(L_1) = \sigma < 1. \tag{4.12}$$

Write (4.11) as $(I - L_\sigma)v \leqslant M_1$. By (4.12), we know that $(I - L_\sigma)^{-1}$ exists so we conclude $v \leqslant (I - L_\sigma)^{-1} M_1 = M_\sigma$ where M_σ is a constant independent of $v \in W$. Thus $\sup W \leqslant M_\sigma < \infty$. We define $\Omega_2 = \{v : \|v\| \leqslant R_1\}$, where $R_1 > \max\,(R, \sup W)$. For $y \in P_2 \cap \partial\Omega_2$, we have $\|y\| = R > \sup W$ so $y \neq \sigma A_1 y$ for all $0 \leqslant \sigma \leqslant 1$. This shows that condition (b) of Theorem K holds, so $i(A_1; P_1 \cap \Omega_2, P_1) = 1$. Finally, by additivity of the index function

$$i\big(A_1; (P_1 \cap \Omega_2) \setminus (P_1 \cap \overline{\Omega_1})\big) = i(A_1; P_1 \cap \Omega_2, P_1) - i(A_1; P_1 \cap \Omega_1, P_1) = 1.$$

It follows that A_1 has a fixed point $\widehat{y_1} \in P_1 \cap (\Omega_2 \setminus \overline{\Omega_1})$ satisfying $0 < r \leqslant \|\widehat{y_1}\| \leqslant R_1$. By (2.26) in Lemma 2.2, $\widehat{y_1}(t) > 0$ for $t \in [0,1]$ and is a positive solution of (BVP1). This completes the proof.

REMARK 4.2. The use of spectral radius of the linear operators L_j given by (4.5) in the study of multi-point boundary value problems was initiated by Gupta and Trofimchuk [17] followed by many others, e.g. [56], [46], [19].

Similarly, we state and can prove

THEOREM 4.2. *Let* $0 \leqslant \overline{\alpha} < 1, 0 < \overline{\beta} < 1$. *Suppose that* $f_\infty < \mu_2 < f_0$, *where* μ_2 *is the smallest positive eigenvalue of* (4.7) *for* $j = 2$, *then the* $(BVP2)$ *has a positive solution for every* $b \geqslant 0$.

Since the proof of Theorem 4.2 is similar to that of Theorem 4.1, we leave the details to the readers.

THEOREM 4.3. *Let* $0 \leqslant \overline{\alpha} < 1, 0 \leqslant \langle \beta, \xi \rangle < 1$. *Suppose that* $f(y)$ *satisfies* $f_\infty < \mu_3 < f_0$, *where* μ_3 *is the smallest positive eigenvalue of* (4.7) *for* $j = 3$, *then the* $(BVP3)$ *has a positive solution for every* $b \geqslant 0$.

Proof. Let

$$P_3 = \{y(t) \in P : \inf_{\xi_1 \leqslant t \leqslant 1} y(t) \geqslant \gamma_3 \|y\|\},$$

where γ_3 is given by (2.27). Note that for any given $y(t) \in P_3$, $A_3 y(t)$ is a solution to $u'' + h(t) = 0$ with $h(t) = a(t) f\big(y(t)\big)$ and in addition $A_3 y(t)$ satisfies the boundary condition (3). So by Lemma 2.3 we conclude $A_3(P_3) \subseteq P_3$.

Like a similar argument in (4.6), we know that L_3 is strongly positive. We can now apply Krein-Rutman theorem to L_3 and P_3 and conclude that $r(L_3) > 0$ and $\mu_3 = [r(L_3)]^{-1}$ is the smallest eigenvalue of the linear boundary value problem (4.7) for $j = 3$.

We again apply Krasnoselskii's Theorem K to A_3 with regard to P_3 and use the sublinearity condition $f_\infty < \mu_3 < f_0$ to verify conditions (a) and (b) in much the same way as Theorem 4.1 to conclude that A_3 has a fixed point $\widehat{y_3} \in P_3 \cap (\Omega_2 \setminus \overline{\Omega_1})$ where Ω_2, Ω_1 are defined similar to that given in the proof of Theorem 4.1, so $0 < r \leqslant \|\widehat{y_3}\| \leqslant R_1$. Since $\widehat{y_3}(t) \geqslant \gamma_3 \|y\|$ for $t \in [\xi_1, 1]$, together with $\widehat{y_3}(0) = \langle \alpha, \widehat{y_3}(\xi) \rangle \geqslant 0$ and concavity of $\widehat{y_3}(t)$, we conclude that $\widehat{y_3}(t) > 0$ on $(0,1]$ and is a solution of (BVP3). ($\widehat{y_3}(t) > 0$ for all $t \in [0,1]$ if in addition $\overline{\alpha} > 0$). This completes the proof.

REMARK 4.3. Theorem 4.3 extends Theorem B which deals with a special case of (BVP3) with $\overline{\alpha} = 0$ and $b = 0$. In addition, Theorem B requires $0 \leqslant \overline{\beta} < 1$ which is stronger than $0 \leqslant \langle \beta, \xi \rangle < 1$.

REMARK 4.4. Similar to Remark 4.1 we can also show for (BVP2), $\Lambda_2 \leqslant \mu_2 \leqslant \gamma_2^{-1}\Lambda_2$; and for (BVP3), $\Lambda_3 \leqslant \mu_3 \leqslant \gamma_3^{-1}\Lambda_4$. It is perhaps useful to state these conclusions separately below as corollaries to Theorems 4.1, 4.2 and 4.3.

COROLLARY 4.1. *Let* $0 \leqslant \overline{\alpha} < 1, 0 \leqslant \overline{\beta} < 1$. *If* $f(y)$ *satisfies*

$$f_\infty \leqslant \Lambda_1 < \mu_1 < \gamma_1^{-1}\Lambda_1 \leqslant f_0, \tag{4.13}$$

then $(BVP1)$ *has a positive solution for every* $b \geqslant 0$.

COROLLARY 4.2. *Let* $0 < \overline{\alpha} < 1, 0 \leqslant \overline{\beta} < 1$. *If* $f(y)$ *satisfies*

$$f_\infty \leqslant \Lambda_2 < \mu_2 < \gamma_2^{-1}\Lambda_2 \leqslant f_0, \tag{4.14}$$

then $(BVP2)$ *has a positive solution for every* $b \geqslant 0$.

COROLLARY 4.3. *Let* $0 \leqslant \overline{\alpha} < 1, 0 \leqslant \langle \beta, \xi \rangle < 1$. *If* $f(y)$ *satisfies*

$$f_\infty \leqslant \Lambda_3 < \mu_3 < \gamma_3^{-1}\Lambda_4 \leqslant f_0, \tag{4.15}$$

then $(BVP3)$ *has a positive solution for every* $b \geqslant 0$.

5. Examples and discussion

Because of the laborious calculations involved in determining the upper and lower bounds of f_0, f_∞ as defined by (1.5), we found few examples in literature. However, we first select two examples from papers by Liu [31], [32] and compare the estimates on f_0, f_∞ using results reported in this paper with that of Liu's.

EXAMPLE 5.1. Consider the three point problem

$$y'' + 3ty\left(1 + \frac{c}{1+y^2}\right) = 0, \quad 0 < t < 1, \tag{5.1}$$

$$y'(0) = 0, \quad y(1) = \frac{1}{4}y(1/3) + b, \tag{5.2}$$

where c is a positive constant and $b \geqslant 0$. This example was given by Liu [31; p.27, Example 5.5] where $b=0$. Here $f_0 = 1+c$ and $f_\infty = 1$. In [31], it was reported that if $c > 9593/208 = 46.12$ then the (BVP) (5.1),(5.2) has a positive solution with $b=0$. This problem is a special case of (BVP2) in the sublinear case, and we can apply Corollary 4.2 with γ_2, Λ_2 given by (2.24),(3.7). Here $\gamma_2 = \frac{2}{11}, \Lambda_2 = 1656/1458 > f_\infty = 1$, and $f_0 = 1+c > \gamma_2^{-1}\Lambda_2 = 18238/2916$, so $c > 5.2469$ which is an improvement up on $c > 46.12$. In addition, we proved that BVP(5.1),(5.2) has a positive solution for all $b \geqslant 0$.

EXAMPLE 5.2. Consider the boundary value problem

$$y'' + \frac{1}{3}\frac{aye^{2y}}{c+e^y+e^{2y}} = 0, \quad 0<t<1, \tag{5.3}$$

$$y(0) = 0, \quad y(1) = 2y(1/3)+b, \tag{5.4}$$

which was studied in Liu [32; p210-211, Ex. 4.3] with $b=0$. Here $f_0 = a/(c+2), f_\infty = a$ and the problem is a special case of (BVP3), with $\alpha_1 = 0, \beta_1 = 2, \xi_1 = 1/3$. Suppose that $a>0, c>-1$, then it is in the superlinear case and Theorem 3.3 is applicable. According to Liu [32; p.208, Corollary 3.1], existence of positive solution requires $f_0 < \triangle_1, f_\infty > \gamma_3^{-1}\triangle_2$ where $\gamma_3 = \min\{\xi_1, \beta_1\xi_1, \frac{\beta_1(1-\xi_1)}{1-\beta_1\xi_1}\}$ given by (2.27) and

$$\triangle_1 = (1-\beta_1\xi_1)\left\{\int_0^1 (1-s)a(s)ds\right\}^{-1},$$

$$\triangle_2 = (1-\beta_1\xi_1)\left\{\xi_1\gamma_3\int_{\xi_1}^1 (1-s)a(s)ds\right\}^{-1}.$$

Here $\triangle_1 = 2, \triangle_2 = 81/2$, and $\gamma_3 = \frac{1}{3}$, so $f_\infty = a > 243/2$ and $f_0 = a/c+2 < 2$. Liu [32] concluded that for $c > 235/4 = 58.75$ and $a > 243/2$, the boundary value problem (5.3), (5.4) has a positive solution for $b=0$.

To apply Theorem 3.3, we compute Λ_3, Λ_4 according to (3.14), (3.15). Here $\Lambda_3 = \triangle_1 = 2$ and $\Lambda = 1/3$. By (3.15) and (2.27), we require $f_\infty = a > \gamma_3^{-1}\Lambda_4 = 27/4$ and $c > 65/8$ which improves upon that reported in [32], $c > 58.75$.

We now give two other examples to illustrate the versatility of the results given in previous two sections.

EXAMPLE 5.3. Consider the boundary value problem

$$y'' + \frac{a+cy+\sin y}{1+y}y = 0, \quad 0<t<1, \tag{5.5}$$

$$y(0) = \frac{1}{2}y(1/2), \quad y'(1) = b \geqslant 0. \tag{5.6}$$

Here $f_0 = a, f_\infty = c$ and the (BVP) (5.5),(5.6) is superlinear if $0<a<c$ and sublinear if $a>c>0$. It is a special case of (BVP1).

When $a < c$, we apply Theorem 3.1 with γ_1, Λ_1 given by (2.23), (3.1). Here $\alpha_1 = \frac{1}{2}, \xi_1 = \frac{1}{2}$, so $\gamma_1 = \frac{1}{3}$ and $\Lambda_1 = 4/5$. So if $a < 4/5, c > 12/5$ then there exists $b^* > 0$ such that the (BVP) (5.5), (5.6) has a positive solution for any $b, 0 \leqslant b \leqslant b^*$ and no solution for $b > b^*$. However, since $f(y)$ is not monotone for any set of values a, c because

$$f'\left[(2n+1)\pi\right] = \left[1+(2n+1)\pi\right]^{-2}\left[c-a-\left(1+(2n+1)\pi\right)\right] < 0$$

for large values of n and $f'(0) = c-a+1$, $f'(\infty) = c$. Hence Theorem C is not applicable to this example.

In the sublinear case $a > c$. We can use Corollary 4.1 and determine the smallest positive eigenvalue of the linear boundary value problem $y'' + \lambda y = 0$, (5.6). Here the eigen functions satisfying (5.6) take the form $\cos\sqrt{\lambda}(1-t)$ and the eigenvalues are zeros of the $4x^2 - x - 2 = 0$, $x = \cos\sqrt{\lambda}/2$, yielding $\lambda_1 = 1.2897$. This shows from (4.14) of Corollary 4.1

$$c = f_\infty < \frac{4}{5} < 1.2897 < \frac{12}{5} < f_0 = a.$$

Indeed, Theorem 4.1 proves that whenever $c < 1.2897 < a$, then the (BVP) (5.5), (5.6) has a positive solution for every $b \geqslant 0$.

EXAMPLE 5.4. Consider the boundary value problem

$$y'' + \frac{c\log(y+1)+a}{\log(y+1)+1}y = 0, \quad 0 < t < 1, \tag{5.7}$$

$$y(0) = \frac{1}{2}y(1/3), \quad y(1) = \frac{1}{3}y(1/2) + b, \tag{5.8}$$

with $a, b, c \geqslant 0$, which is a special 4-point case of (BVP3). Here $\alpha_1 = \frac{1}{2}, \xi_1 = \frac{1}{3}, \beta_2 = \frac{1}{3}, \xi_2 = \frac{1}{2}$; and $\gamma_3 = \frac{1}{6}$ by (2.27), $f_0 = a, f_\infty = c$ and $\Lambda = 19/36$.

When $c > a > 0$ in the superlinear case, we find from (3.14),(3.15) that $\Lambda_3 = 19/12, \Lambda_4 = 12.19 = 228$. So if $0 < a < 19/12$, $c > 228$, then there exists $b > 0$ such that the (BVP) (5.7), (5.8) has a positive solution for any b, $0 \leqslant b \leqslant b^*$ and no positive solution if $b > b^*$.

In the sublinear case when $f_0 = a > f_\infty = c$. We can avail to Theorem 4.3 and obtain optimal condition in terms of the smallest eigenvalue λ_1 of the linear problem $y'' + \lambda y = 0$ subject to boundary condition (5.8). We find by numerical methods, $\lambda_1 = 5.3163775$ with the corresponding eigen function $\sin\{(2.3057271)t + 0.4971368\}$. Hence if $a > 5.316775 > c$, then the boundary value problem (5.7),(5.8) has a positive solution for all $b \geqslant 0$.

REMARK 5.1. We note that the bounds Λ_3, Λ_4 given in (3.14), (3.15) of Theorem 3.3 are not the best possible using Theorem KG when compared with Theorems 3.1 and 3.2. Instead of Λ_3, Λ_4, we can introduce the following "sharper bounds"

$$\overline{\Lambda}_3^{-1} = \max\{D_3[a], C_3[a] + D_3[a]\},$$
$$\overline{\Lambda}_4^{-1} = \min\{D_3[a], C_3[a] + D_3[a]\}.$$

In the case of Example 5.4, we have

$$D_3[a] = \frac{1}{\Lambda}\{(1-\beta_2\xi_2)\alpha_1 G_3(\xi_1) + \alpha_1\xi_1[\beta_2 G_3(\xi_2)+b]\}$$

$$C_3[a]+D_3[a] = \frac{1}{\Lambda}\{(\beta_2-\beta_2\xi_2)\alpha_1 G_3(\xi_1) + (1-\alpha_1+\alpha_1\xi_1)[\beta_2 G_3(\xi_2)+b]\}.$$

Note that

$$G_3(t) = t\int_t^1 (1-s)a(s)ds + (1-t)\int_0^t sa(s)ds,$$

so $G_3(\frac{1}{3}) = \frac{1}{9}$, $G_3(\frac{1}{2}) = \frac{1}{8}$ since $a(t) \equiv 1$. Thus, we obtain

$$D_3[a] = 69/342,\ \ C_3[a]+D_3[a] = 12/171,$$

and can conclude that if $0 < a = f_0 < \frac{1}{2}\Lambda_3 = \frac{1}{2}D_3[a]^{-1} = \frac{171}{69} = 2.4782$, and $c > \Lambda_4 = (C_3[a]+D_3[a])^{-1} = 171/12 = 14.25$ which improve upon the estimates on a, c given in Example 5.4.

We now close our discussion with additional remarks and problems for further research.

REMARK 5.2. (*uniqueness and non-resonance*) Consider the simple linear equation $u'' + h(t) = 0$ with $h(t) \in C[0,1]$, $h(t) \geqslant 0$ subject to boundary conditions (2.1), (2.2), (2.3) with $b = 0$. It is easy to show that these particular boundary value problems have a unique solution if and only if $u'' = 0$ has no non-trivial solutions satisfying (2.1), (2.2), (2.3). In this case, (BVP1), (BVP2), (BVP3) are commonly referred to as non-resonant cases. Alternatively, uniqueness of the zero solution is equivalent to the solvability of the 2×2 system of linear algebraic equations with unknowns $C_j, D_j, j = 1,2,3$. This in turn is equivalent to the condition $\triangle = (1-\overline{\alpha})(1-\overline{\beta}) \neq 0$ for (BVP1),(BVP2) and $\Lambda = (\Lambda = (1-\overline{\alpha})(1-\langle\beta,\xi\rangle) + (1-\overline{\beta})\langle\alpha,\xi\rangle \neq 0$ for (BVP3).

REMARK 5.3. Robin boundary condition at $t = 0$. In section 4, we discuss Theorem E which deals with the Robin boundary condition at $t = 0$, i.e. (4.4), which included both the Dirichlet $\left(\theta = \frac{\pi}{2}\right)$ and Neumann $(\theta = 0)$ conditions. When the boundary conditions involves both the solution $y(t)$ and its derivative $y'(t)$ at a boundary point, say $t = 0, 1$, or ξ_i, they are known as mixed boundary conditions. The method introduced in this paper is unable to handle problems involving mixed boundary conditions. In addition, Theorem E shows that for b, $0 < b < b^*$, there are at least two positive solutions, when $b = 0$, b^* there is one positive solution. There exists large literature on the subject of multiplicity. We refer the reader to [43], [44], [39], [29] on earlier results. As far as we know, there is no result on multiplicity of solutions for (BVP1), (BVP2), (BVP3) and problems with Robin boundary condition.

REMARK 5.4. (*sign-changing nonlinearities*) Our discussion is confined to proving existence of positive solutions with non-negative nonlinear function $f(y)$. There are also many papers on existence theorems with sign changing nonlinearities. We refer the reader to recent work of Kong and Kong [24], [25] and the references therein.

REMARK 5.5. (*higher order equations and others*) One can obviously consider more general equations than (1.1) such as $Ly + a(t)f(y) = 0$, where $Ly = y'' + p(t)y$ or $Ly = y^{(n)}$, the n^{th} derivative of y. There are also many papers involving $p-L$aplacians in the form $Ly = \left(\Phi_p(y')\right)'$, where $\Phi_p(u) = |u|^{p-2}u, p > 1$. We contend by listing a few recent publications on these subjects in References for the interested reader, [38], [19], [6], [9], [26], [10], [13], [55], [34], [35].

REMARK 5.6. We introduce the notion of scalar product formulation of (BVP1), (BVP2), (BVP3) which heuristically treats the interior boundary points $(\xi_1, \xi_2, \cdots, \xi_m)$ as one vector in $\mathbb{R}^m$. By this approach, (BVP1), (BVP2), (BVP3) can be viewed as corresponding three-point problems. On the other hand, the boundary conditions at interior points can also be viewed as special cases of certain integral boundary conditions involving Stieltjes integrals. We refer the reader to recent papers on second order boundary value problem with integral conditions, e.g. Webb and Infante [51], Ma and An [41].

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James S. W. Wong
Institute of Mathematical Research
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
China
e-mail: jsww@chinneyhonkwok.com