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EXISTENCE OF NON–RADIALLY SYMMETRIC VISCOSITY SOLUTIONS TO SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS WITH RADIALLY SYMMETRIC COEFFICIENTS IN THE PLANE, PART II

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Abstract. We study continuous viscosity solutions for a semilinear degenerate elliptic equation with radially symmetric coefficients in the plane. If the equation satisfies certain relations with respect to the behavior of coefficients at the infinity, then it is known that there exist many solutions. Our purpose is to construct many non radially symmetric solutions satisfying the similar behavior with radial symmetric solutions at the infinity. The solutions are obtained as a small perturbation from a radially symmetric solution. We construct super- and sub-solution by using the series expansion of $r^{\alpha-j\beta}\cos n\theta$ $(j,n=1,2,\dots)$, where (r,θ) is the polar coordinate and α and β are certain positive constants.

1. Introduction

We consider the following semilinear degenerate elliptic equation:

$$\mathscr{L}u=-g(|x|)\Delta u(x)+u(x)|u(x)|^{p-1}-f(|x|)=0 \quad \text{in} \quad \mathbb{R}^2, \tag{1}$$

where $g:[0,\infty)\to[0,\infty)$ is a differentiable and non-negative function and $p>1$ is a constant. We assume that $g(T_0)=0$ for some $T_0>0$ and $g(t)>0$ for any $t>T_0$. Moreover, we assume that $f,g\in C^\infty((T_0,\infty))\cap C^2(\mathbb{R})$ satisfy

$$\lim_{t\to\infty}\frac{g(t)}{t^\ell}=1, \quad \lim_{t\to\infty}\frac{f(t)}{t^{\alpha p}}=\kappa^p,$$

where $\kappa>0$, $\ell>2$ and $\alpha=\dfrac{\ell-2}{p-1}$. The case $0<\ell\leqq 2$ has already considered in [6]. In this case the solution of (1) exists uniquely and radially symmetric.

In the preceding paper [4], under certain assumptions of $f(t)$ and $g(t)$, we have shown the existence of non-radially symmetric viscosity solutions of (1) satisfying

$$\overline{\lim_{|x|\to\infty}}\frac{u(x)}{|x|^\alpha}-\underline{\lim_{|x|\to\infty}}\frac{u(x)}{|x|^\alpha}>0.$$

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Our purpose of this paper is to show the existence of non-radially symmetric solutions which are different from those in [4].

The solution obtained in [4] has the phase at infinity as the solution of Laplace-Beltrami equation associated with (1). On the contrary, we show the existence of a small perturbed solution from a radially symmetric solution. We are looking for a solution which has the asymptotic

$$\overline{\lim_{|x|\to\infty}} \frac{u(x)}{|x|^{\alpha}} = \underline{\lim_{|x|\to\infty}} \frac{u(x)}{|x|^{\alpha}}.$$

To prove our assertion we shall construct super- and sub-solutions that suit our purpose and apply the comparison theorem developed in [4] to (1). We are then able to obtain the desired results. We assume that f and g have power series expansion with respect to the radius $t = |x|$. First we construct a radially symmetric solution of the power series form by using these expansion. Then next we add a small perturbation of power series which has osscillation with respect to the angular variable θ. In these arguments, we have to make sure that the coefficients of this power series are determined recurcively from the coefficients of f and g. This is the main caluculation of this paper.

The outline of the present paper is as follows. In section 2, we state the assumptions and our theorem. In sections 3, 4 we show the existence of asymptotically radially symmetric and non-radially symmetric solutions of (1) respectively. Section 5 is devoted to the study of the existence of super- and sub-solution of (1) for large t. In Section 6 our theorem is proved.

2. Assumptions and main theorem

In this section we list the detailed assumptions of $f(t),\ g(t) \in C^{\infty}((T_0,\infty)) \cap C^2([0,\infty))$ and state the theorem. Recall that $\kappa > 0$, $\ell > 2$ and $\alpha = \dfrac{\ell-2}{p-1}$ and

$$\lim_{t\to\infty} \frac{g(t)}{t^{\ell}} = 1, \quad \lim_{t\to\infty} \frac{f(t)}{t^{\alpha p}} = \kappa^p > 0.$$

We consider the following algebraic equation:

$$X|X|^{p-1} - \kappa^p - \alpha^2 X = 0. \tag{2}$$

We assume that (2) has a positive single root ω_+ and negative single roots ω_0 and ω_-, where $\omega_- < \omega_0 < 0 < \omega_+$.

We assume the following two assumptions.

(H-1) Constants p, α and ω_0 satisfy $\alpha^2 - p|\omega_0|^{p-1} > 1$.

Let $N = [\alpha - \sqrt{p|\omega_0|^{p-1}}]$, where $[x]$ is the maximum integer which does not exceed x.

(H-2) Functions $f(t)$ and $g(t)$ behave near $t \to \infty$ as follows:

$$f(t) = t^{\alpha p}\left(\kappa^p + \tau_1 t^{-1} + \tau_2 t^{-2} + \cdots + \tau_N t^{-N} + O\left(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}}-\varepsilon}\right)\right),$$

$$g(t)^{-1} = t^{-\ell}\left(1+\sigma_1 t^{-1}+\sigma_2 t^{-2}+\cdots+\sigma_N t^{-N}+O\left(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}}-\varepsilon}\right)\right),$$

where $\varepsilon > 0$.

We will introduce some notation often used in this paper:

$$\begin{aligned}
&r_n = \alpha - \sqrt{p|\omega_0|^{p-1}+n^2}, \ \ r_\varepsilon = \alpha - \sqrt{p|\omega_0|^{p-1}+\varepsilon},\\
&r_0 = \alpha - \sqrt{p|\omega_0|^{p-1}}, \ \ X^p = X|X|^{p-1},\\
&N = \left[\alpha - \sqrt{p|\omega_0|^{p-1}}\right], \ \ N_0 = \left[\sqrt{\alpha^2 - p|\omega_0|^{p-1}}\right],\\
&M_n = \left[\frac{\alpha - \sqrt{p|\omega_0|^{p-1}}}{\alpha - \sqrt{p|\omega_0|^{p-1}+n^2}}\right],\\
&a \vee b = \max\{a,\, b\}, \ \ a \wedge b = \min\{a,\, b\}.
\end{aligned}$$

Note that $r_\varepsilon = \alpha - \sqrt{p|\omega_0|^{p-1}} + \varepsilon + o(\varepsilon)$ for sufficiently small $\varepsilon > 0$, because $\omega_0 \neq 0$ since $\kappa > 0$. We use this fact frequently.

It is known by [6], Theorem 4.1, that there exist radially symmetric solutions of (1) satisfying $\lim_{|x|\to\infty} y(|x|)/|x|^\alpha = \omega_0$. The next theorem states that there exist non-radially symmetric solutions near $y(|x|)$. More precisely, the main theorem of this paper is the following.

THEOREM 2.1. *Let $y(|x|)$ be any radially symmetric solution of (1) such that*

$$\lim_{|x|\to\infty} \frac{y(|x|)}{|x|^\alpha} = \omega_0.$$

Assume (H-1), (H-2) *and $r_{N_0} > 0$. Then for any $n = 1,\ldots,N_0$, there exist a constant $\delta > 0$ and N continuous viscosity solutions $\{u_n(x)\}$ of* (1) *such that*

$$\lim_{|t|\to\infty} \frac{u_n(t,\theta)-y(t)}{|t|^{\sqrt{p|\omega_0|^{p-1}+n^2}}} = C_n \cos n\theta \quad (0 \leqq \theta < 2\pi), \tag{3}$$

where $x = (t,\theta)$ is the polar coordinate, C_n are any constants satisfying $|C_n| < \delta$. The convergence is uniformly with respect to θ.

REMARK 2.1. In order to avoid complicated calculation and to close the argument in the framework of the power order with respect to t, we prove the theorem by assuming an additional condition (H-3).

(H-3) For any $n = 1,\ldots,N_0$, there are no pairs of non-negative integers (k, j) satisfying $kr_n + j = r_0$.

We will mention in Remark 6.1 below for an argument without (H-3).

REMARK 2.2. We mention the role of constants r_0, r_ε, r_{N_0}, M_n, N and N_0. The r_0 plays an important role when we construct an asymptotically non-radially symmetric solution of (1). The r_ε is used to control reminder terms. The assumption $r_{N_0} > 0$ in Theorem 2.1 is needed to construct a non-radially symmetric solution satisfying $\lim_{t\to\infty} u_n(t,\theta)/t^\alpha = 0$. Note that this condition is equivalent to $N_0 < \alpha < N_0+1$. If (H-3) is satisfied, then $M_n r_n + j \neq r_0$, $kr_n + j \neq r_0$ and $j \neq r_0$ $(n = 1,2,\ldots N_0)$ for any nonnegative integers k, j. Note also that M_n, N satisfy $M_n r_n < r_0 < (M_n+1) r_n$ and $N < r_0 < N+1$.

REMARK 2.3. Although the equation (1) is treated in $\mathbb{R}^2$ it is enough to consider the equation (1) in $\mathbb{R}^2 \backslash B_{T_0}$, where $B_{T_0} = \{x \in \mathbb{R}^2; |x| < T_0\}$. Indeed, if $g \in C^2([0,\infty))$, then there exists the continuous viscosity solution in $\mathbb{R}^2$ by combining continuous viscosity solutions on $\mathbb{R}^2 \backslash B_{T_0}$ and that in B_{T_0}. The boundary condition on ∂B_{T_0} is automatically determined by the degeneracy of the $g(t)$ such that $\lim_{|x|\to T_0} u(x) = \varphi(T_0)$, where $\varphi(T_0)|\varphi(T_0)|^{p-1} = f(T_0)$ ([5], Lemma 4.3). Also, it has been shown that a continuous viscosity solution in B_{T_0} exists uniquely and is radially symmetric (see [5], Theorem 2). Moreover, it is known that a continuous viscosity solution in $\mathbb{R}^2 \backslash B_{T_0}$ is $C^\infty(\mathbb{R}^2 \backslash B_{T_0})$ from the existence and regularity theorem of $C^\infty(B_T \backslash B_{T_0+\varepsilon})$ solution of (1) and the comparison theorem, where T is sufficiently large number.

In the following, we study the equation (1) in $\mathbb{R}^2 \backslash B_{T_0}$. Let n be a fixed integer such that $1 \leqq n \leqq N$ and we denote $r = r_n$ for the simplicity.

To prove the theorem we construct the super- and sub-solution in $\mathbb{R}^2 \backslash B_{T_0}$ of (1) satisfying (3) as an asymptotic solution of the variable t. Then the result follows by the comparison theorem. We need some asymptotic formulae of t which are derived from (H-2):

$$\frac{t^{\alpha p-\alpha}}{g(t)} = \frac{1}{t^2}\left(1 + \sum_{i=1}^{N} \sigma_i t^{-i} + O\left(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}}-\varepsilon}\right)\right),$$

$$\begin{aligned}\frac{f(t)t^{-\alpha}}{g(t)} &= \frac{t^{\alpha p-\alpha}}{t^\ell}\left(\left(\kappa^p + \sum_{i=1}^{N} \tau_i t^{-1}\right)\left(1 + \sum_{i=1}^{N} \sigma_i t^{-i}\right) + O\left(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}}-\varepsilon}\right)\right) \\ &= \frac{1}{t^2}\left(\kappa^p + \sum_{i=1}^{N} f_i t^{-i} + O\left(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}}-\varepsilon}\right)\right),\end{aligned}$$

where $\{f_i\}_{i=1}^N$ are constants depending only on $\{\kappa^p, \tau_j, \sigma_j; j = 1,2,\ldots,N\}$.

Let $w(t,\theta)$ be such that $u(x) = u(t,\theta) = t^\alpha(\omega_0 + w(t,\theta))$ and define

$$\begin{aligned}\widetilde{\mathscr{L}}w(t,\theta) = &-\left(\frac{\partial^2}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial}{\partial t} + \frac{\alpha^2 - p|\omega_0|^{p-1}}{t^2} + \frac{1}{t^2}\frac{\partial^2}{\partial\theta^2}\right) w(t,\theta) \\ &+ \frac{1}{t^2}\left(\sum_{i=0}^{N} \sigma_i t^{-i}\right)\left((\omega_0+w)^p - \omega_0^p - p|\omega_0|^{p-1} w\right) \\ &+ \frac{1}{t^2}\sum_{i=1}^{N} (\sigma_i p|\omega_0|^{p-1}) t^{-i} w + \frac{1}{t^2}\sum_{i=1}^{N} (\sigma_i \omega_0^p - f_i) t^{-i}.\end{aligned}$$

Here we set $\sigma_0 = 1$. Then it follows from (1) and the relation $\omega_0^p - \alpha^2\omega_0 - \kappa^p = 0$ that

$$\widetilde{\mathscr{L}}w = O\left(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}}-\varepsilon-2}\right) \tag{4}$$

for a sufficiently large t. In section 5 we construct the super- and sub solution by using this $\omega(t,\theta)$.

3. Asymptotically radially symmetric solution

In this section we construct an asymptotically radially symmetric solution $h_{-1}(t)$ of (4) for sufficiently large t. That is, $h_{-1}(t)$ satisfies the following equation:

$$\begin{aligned} &\widetilde{\mathscr{L}}h_{-1}(t) \\ &\quad = -\left(\frac{\partial^2}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial}{\partial t} + \frac{\alpha^2 - p|\omega_0|^{p-1}}{t^2}\right)h_{-1}(t) \\ &\qquad + \frac{1}{t^2}\left(\sum_{i=0}^{N}\sigma_i t^{-i}\right)\left((\omega_0 + h_{-1})^p - \omega_0^p - p|\omega_0|^{p-1}h_{-1}\right) \\ &\qquad + \frac{1}{t^2}\left(\sum_{i=1}^{N}\sigma_i p|\omega_0|^{p-1}t^{-i}\right)h_{-1} + \frac{1}{t^2}\sum_{i=1}^{N}(\sigma_i\omega_0^p - f_i)t^{-i} = O(t^{-r_\varepsilon-2}). \end{aligned} \tag{5}$$

We construct $h_{-1}(t)$ such as

$$h_{-1}(t) = \sum_{k=1}^{N} c_{-1,k}t^{-k}. \tag{6}$$

Our purpose in this section is to decide the coefficients of $h_{-1}(t)$ as to satisfy the equation (5). Remark that $N < r_\varepsilon < N+1$ for sufficiently small $\varepsilon > 0$. Note that since the equation (5) contains the order term $O(t^{-r_\varepsilon-2})$, it is sufficient to calculate h_{-1} in the modulus $O(t^{-r_\varepsilon})$.

LEMMA 3.1. *Let q be an integer such that $q \geqq 2$. Then it follows that*

$$h_{-1}(t)^q = \sum_{k=q}^{N} f_{q,k}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-(q-1)})t^{-k} + O(t^{-r_\varepsilon}), \tag{7}$$

where $f_{q,k}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-(q-1)})$ are homogeneous q-th degree polynomials in $\{c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-(q-1)}\}$.

Proof. We use the induction with respect q. Let $q = 2$. As $N < r_\varepsilon < N+1$ it holds that

$$h_{-1}(t)^2 = \left(\sum_{k_1=1}^{N} c_{-1,k_1}t^{-k_1}\right)\left(\sum_{k_2=1}^{N} c_{-1,k_2}t^{-k_2}\right) + O(t^{-r_\varepsilon})$$

$$= \sum_{k_1,k_2=1}^{N} c_{-1,k_1} c_{-1,k_2} t^{-(k_1+k_2)} + O(t^{-r\varepsilon})$$
$$= \sum_{k=2}^{N} \Big(\sum_{k_1=1}^{k-1} c_{-1,k_1} c_{-1,k-k_1} \Big) t^{-k} + O(t^{-r\varepsilon})$$
$$= \sum_{k=2}^{N} f_{2,k}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-(2-1)}) t^{-k} + O(t^{-r\varepsilon}),$$

where we changed the variables from (k_1,k_2) to (k,k_1) such that $k = k_1 + k_2$. Then (7) is proved for $q = 2$.

Suppose that (7) is true for q. Then the change of variables yields

$$h_{-1}(t)^{q+1}$$
$$= \Big(\sum_{k_2=1}^{N} c_{-1,k_2} t^{-k_2} \Big) \Big(\sum_{k_1=q}^{N} f_{q,k_1}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k_1-(q-1)}) t^{-k_1} \Big) + O(t^{-r\varepsilon})$$
$$= \sum_{k=q+1}^{N} \Big(\sum_{k_1=q}^{k-1} c_{-1,k-k_1} f_{q,k_1} \Big) t^{-k} + O(t^{-r\varepsilon}).$$

Since $q \leqq k_1 \leqq k-1$, it follows that

$$k_1 - (q-1) \leqq k - 1 - (q-1) = k - q \text{ and } k - k_1 \leqq k - q.$$

Then $c_{-1,k-k_1} f_{q,k}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-(q-1)})$ are homogeneous $q+1$-st degree polynomials in $\{c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-q}\}$. Thus, we obtain that

$$h_{-1}(t)^{q+1} = \sum_{k=q+1}^{N} f_{q+1,k}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,k-(q+1-1)}) t^{-k} + O(t^{-r\varepsilon}).$$

This completes the proof. □

LEMMA 3.2. *It holds*

$$\sum_{i=0}^{N} \sigma_i t^{-i} \left((\omega_0 + h_{-1})^p - \omega_0^p - p|\omega_0|^{p-1} h_{-1} \right)$$
$$= \sum_{j=2}^{N} F_{j,1}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}) t^{-j} + O(t^{-r\varepsilon}), \quad (8)$$

where $F_{j,1}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1})$ *are sum of homogeneous* q*-th* $(2 \leqq q \leqq N)$ *degree polynomials in variables* $\{c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}\}$.

Proof. Let

$$I(t) = \sum_{i=0}^{N} \sigma_i t^{-i} \left((\omega_0 + h_{-1})^p - \omega_0^p - p|\omega_0|^{p-1} h_{-1} \right).$$

Note that by Taylor's theorem, there exist constants $e_q (q = 2, 3, \dots, N)$ satisfying

$$(\omega_0 + h_{-1})^p - \omega_0^p - p|\omega_0|^{p-1} h_{-1} = \sum_{q=2}^{N} e_q h_{-1}(t)^q + O(t^{-r\varepsilon}).$$

Then it follows that

$$\begin{aligned} I(t) &= \sum_{i=0}^{N} \sigma_i t^{-i} \sum_{q=2}^{N} e_q h_{-1}(t)^q + O(t^{-r\varepsilon}) \\ &= \sum_{q=2}^{N} \Big(\sum_{i=0}^{N} \sigma_i e_q t^{-i} \Big) h_{-1}(t)^q + O(t^{-r\varepsilon}). \end{aligned}$$

Lemma 3.1 yields that

$$I(t) = \sum_{q=2}^{N} \Big(\sum_{i=0}^{N} \Big(\sum_{k=q}^{N} \sigma_i e_q f_{q,k}(c_{-1,1}, c_{-1,2}, \dots, c_{-1,k-(q-1)}) t^{-(k+i)} \Big) \Big) + O(t^{-r\varepsilon}).$$

By the change of variables from (i,k) to (j,k) such that $j = i + k$ and exchanging the order of sum with respect to q and j, it follows

$$\begin{aligned} I(t) &= \sum_{q=2}^{N} \Big(\sum_{j=q}^{N} \Big(\sum_{k=q}^{j} \sigma_{j-k} e_q f_{q,k} \Big) \Big) t^j + O(t^{-r\varepsilon}) \\ &= \sum_{j=2}^{N} \Big(\sum_{q=2}^{j} \Big(\sum_{k=q}^{j} \sigma_{j-k} e_q f_{q,k} \Big) \Big) t^j + O(t^{-r\varepsilon}). \end{aligned}$$

Let $F_{j,1} = \sum_{q=2}^{j} (\sum_{k=q}^{j} \sigma_{j-k} e_q f_{q,k})$. Note that

$$\bigcup_{2 \leqq q \leqq j,\ q \leqq k \leqq j} \{c_{-1,1}, c_{-1,2}, \dots, c_{-1,k-(q-1)}\} \subset \{c_{-1,1}, c_{-1,2}, \dots, c_{-1,j-1}\}$$

and $\sum_{k=q}^{j} \sigma_{j-k} e_q f_{q,k}$ are the sum of homogeneous q-th degree polynomials in

$$\{c_{-1,1}, c_{-1,2}, \dots, c_{-1,k-(q-1)}\}.$$

Then $F_{j,1}$ is the sum of homogeneous q-th degree polynomials in

$$\{c_{-1,1}, c_{-1,2}, \dots, c_{-1,j-1}\}$$

such that $2 \leqq q \leqq j$. The assertion is proved. □

LEMMA 3.3. *The following equality holds:*

$$\Big(\sum_{i=1}^{N} \sigma_i t^{-i} \Big) p|\omega_0|^{p-1} h_{-1} = \sum_{j=2}^{N} F_{j,2}(c_{-1,1}, c_{-1,2}, \dots, c_{-1,j-1}) t^{-j} + O(t^{-r\varepsilon}), \quad (9)$$

where $F_{j,2}$ *are homogeneous polynomial of degree* 1 *in* $\{c_{-1,1}, c_{-1,2}, \dots, c_{-1,j-1}\}$.

Proof. Put

$$II(t) = \Big(\sum_{i=1}^{N} \sigma_i t^{-i}\Big) p|\omega_0|^{p-1} h_{-1}.$$

The change of variables yields

$$\begin{aligned} II(t) &= \sum_{i=1}^{N}\sum_{k=1}^{N} (\sigma_i p|\omega_0|^{p-1}) c_k t^{-(k+i)} + O(t^{-r_\varepsilon}) \\ &= \sum_{j=2}^{N} \Big(\sum_{k=1}^{j-1} (\sigma_{j-k} p|\omega_0|^{p-1}) c_k\Big) t^{-j} + O(t^{-r_\varepsilon}). \end{aligned}$$

By putting $F_{j,2} = \sum_{k=1}^{j-1} (\sigma_{j-k} p|\omega_0|^{p-1}) c_k$ we have the assertion. □

By combining the above arguments, the following property holds.

LEMMA 3.4. *The coefficients $c_{-1,j}$ of $h_{-1} = \sum_{j=1}^{N} c_{-1,j} t^{-j}$ are represented by α, ω_0, σ_i and f_i $(i \leqq j \leqq N)$.*

Our goal of this section is the following Proposition.

PROPOSITION 3.1. *$h_{-1}(t) = \sum_{j=1}^{N} c_{-1,j} t^{-j}$ defined as above satisfies the equation* (5).

Proof. Let F_j be

$$\begin{aligned} &F_j(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}) \\ &\qquad = F_{j,1}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}) + F_{j,2}(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}), \end{aligned}$$

where $F_{j,1}$ and $F_{j,2}$ are those in Lemmas 3.2 and 3.3, respectively. Then,

$$I(t) + II(t) = \sum_{j=2}^{N} F_j(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}) t^{-j} + O(t^{-r_\varepsilon}).$$

By substituting this into $\mathscr{L} h_{-1}$ in (5), and moving the linear part to the left hand side, it holds:

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial}{\partial t} + \frac{\alpha^2 - p|\omega_0|^{p-1}}{t^2}\right) h_{-1}(t) \\ &\quad = \sum_{j=1}^{N} \left((\alpha-j)^2 - p|\omega_0|^{p-1}\right) c_{-1,j} t^{-j-2} \\ &\quad = \sum_{j=2}^{N} F_j(c_{-1,1}, c_{-1,2}, \ldots, c_{-1,j-1}) t^{-j-2} + \sum_{j=1}^{N} (\sigma_j \omega_0^p - f_j) t^{-j-2} + O(t^{-r_\varepsilon - 2}). \quad (10) \end{aligned}$$

By comparing the coefficients of t^{-j-2} in the both sides of (10), we have:

$$\begin{aligned} \left((1-\alpha)^2 - p|\omega_0|^{p-1}\right) c_{-1,1} &= -(\sigma_1 \omega_0^p - f_1) \\ \left((j-\alpha)^2 - p|\omega_0|^{p-1}\right) c_{-1,j} &= -(\sigma_j \omega_0^p - f_j) \\ &\quad + F_j(c_{-1,1}, c_{-1,2}, \dots, c_{-1,j-1}), \end{aligned} \tag{11}$$

where $2 \leqq j \leqq N$. Note that $(1-\alpha)^2 - p|\omega_0|^{p-1} \neq 0$ by (H-1). By applying (H-3), assume $\alpha - N > \sqrt{p|\omega_0|^{p-1}}$. Then $\alpha - j - \sqrt{p|\omega_0|^{p-1}} > \alpha - j - (\alpha - N) = N - j \geqq 0$ for $j = 1, 2, \dots, N$. Thus $\alpha - j - \sqrt{p|\omega_0|^{p-1}} > 0$. Therefore the equations (11) are the recursive system. Hence all $\{c_{-1,j} : j = 1, 2, \dots, N\}$ can be determined as we desire. $\square$

4. Asymptotically non-radially symmetric solutions

In this section we construct an asymptotically non-radially symmetric solution $w(t,\theta)$ of (4) in order to compose a super-solution of (1). Fix $n = 1, \dots, N_0$. Let $M = M_n$ for the simplicity. Recall that $r = r_n$.

We shall construct the solution in the form

$$w(t,\theta) = h_{-1}(t) + \sum_{s=0}^{M} h_s(t) \cos(sn\theta).$$

Here, h_{-1} be the asymptotically radially symmetric solution in the previous section.

The coefficients $h_0(t)$ and $h_s(t)$ are polynomials defined by

$$\begin{cases} h_0(t) = \displaystyle\sum_{k=2}^{M} \sum_{j=0}^{[r_0 - rk]} c_{0,k,j} t^{-kr-j}, \\ h_s(t) = \displaystyle\sum_{k=s}^{M} \sum_{j=0}^{[r_0 - rk]} c_{s,k,j} t^{-kr-j}, \quad s = 1, \dots, M, \end{cases} \tag{12}$$

where $c_{0,k,j}, c_{s,k,j}$ are certain constants.

If $h_1(t) \neq 0$ is constructed, then our assertion is proved. Our strategy in this section is as follows.

(1) We prove that there is an appropriate self contained relation in the set of $c_{s,k,j}$ (Proposition 4.1 and Lemmas 4.2, 4.3).

(2) We introduce a dictionary-like order relation in the set of (s,j,k) and represent the relation above to a system of equations.

(3) We calculate the Jacobian of the system (Lemmas 4.7, 4.8). The Jacobian has simple form by using the order relation.

(4) By the implicit function theorem and Lemma 4.9, we can establish the existence of the coefficients $c_{s,k,j}$ (Lemma 4.10). The non-triviality of $c_{s,k,j}$ is guaranteed by the fact $c_{1,1,0} \neq 0$ in Proposition 4.2.

REMARK 4.1. We regard $c_{s,k_1,j_1}t^{-k_1r-j_1}$ and $c_{s,k_2,j_2}t^{-k_2r-j_2}$ as different terms in case of $k_1r+j_1=k_2r+j_2$ and $k_1\neq k_2$, where $s=0,1,\dots,M$.

Let $\mathbf{S}$ be a set of coefficients in the polynomials $h_0(t)$ and $h_s(t)$ in (12). Denote $F_q(\mathbf{S})$ by

$$F_q(\mathbf{S})=\{f\equiv 0\}\cup\{f;\text{homogeneous } q\text{-th degree polynomials in } \{c_{s,k,j}\}\in\mathbf{S}\},$$

where $q=1,2,\dots,M$.

LEMMA 4.1. *Let q be $2\leqq q\leqq M$. Then there exist polynomials $E_{s,k,j,q}(\mathbf{S})\in F_q(\mathbf{S})$ such that*

$$\Big(\sum_{s=0}^{M}h_s(t)\cos sn\theta\Big)^q=\sum_{k=q}^{M}\sum_{j=0}^{[r_0-kr]}E_{0,k,j,q}(\mathbf{S})t^{-kr-j}$$
$$+\sum_{s=1}^{M}\Big(\sum_{k=s\vee q}^{M}\sum_{j=0}^{[r_0-kr]}E_{s,k,j,q}(\mathbf{S})t^{-kr-j}\Big)\cos sn\theta+O(t^{-r_\varepsilon}).$$

Proof. We use the induction with respect to q. Let $q=2$. Then it follows

$$\begin{aligned}&\Big(\sum_{s=0}^{M}h_s(t)\cos sn\theta\Big)^2\\&=h_0(t)^2+2h_0(t)\Big(\sum_{s=1}^{M}h_s(t)\cos sn\theta\Big)+\Big(\sum_{s=1}^{M}h_s(t)\cos sn\theta\Big)^2\\&=I_1+I_2+I_3.\end{aligned}$$

For I_1 the following representation holds:

$$I_1=\sum_{k_1=2}^{M}\sum_{j_1=0}^{[r_0-k_1r]}\sum_{k_2=2}^{M}\sum_{j_2=0}^{[r_0-k_2r]}c_{0,k_1,j_1}c_{0,k_2,j_2}t^{-r(k_1+k_2)-(j_1+j_2)}.\tag{13}$$

If $r(k_1+k_2)+j_1+j_2>r_0$, then $t^{-r(k_1+k_2)-(j_1+j_2)}=O(t^{-r_\varepsilon})$. Moreover, by using the change of variables from (k_1,k_2) to $(k=k_1+k_2,k_2)$ and (j_1,j_2) to $(j=j_1+j_2,j_2)$, respectively, it holds

$$\begin{aligned}I_1&=\sum_{k=4}^{M}\sum_{j=0}^{[r_0-kr]}\Big(\sum_{k_2=2}^{k-2}\sum_{j_2=0}^{j}c_{0,k-k_2,j-j_2}c_{0,k_2,j_2}\Big)t^{-rk-j}+O(t^{-r_\varepsilon})\qquad(14)\\&=\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}E^1_{0,k,j,2}(\mathbf{S})t^{-rk-j}+O(t^{-r_\varepsilon}).\end{aligned}$$

Here we defined $E^1_{0,k,j,2}$ by the terms inside the parentheses in the upper line, and $E^1_{0,k,j,2}(\mathbf{S})=0$ for $k=2,3$. Then it follows that $E^1_{0,k,j,2}(\mathbf{S})$ belong to $F_2(\mathbf{S})$.

Next consider I_2 by the similar method for the case I_1. I_2 is written as follows.

$$I_2 = \sum_{s=1}^{M} \Big(\sum_{k_1=2}^{M} \sum_{j_1=0}^{[r_0-k_1 r]} \sum_{k_2=s}^{M} \sum_{j_2=0}^{[r_0-k_2 r]} 2c_{0,k_1,j_1} c_{s,k_2,j_2} t^{-r(k_1+k_2)-(j_1+j_2)} \Big) \cos sn\theta \qquad (15)$$
$$= \sum_{s=1}^{M} (I_{2,1}) \cos sn\theta.$$

Here we defined $I_{2,1}$ by the terms inside the parentheses in the upper line. Then the change of variables of (14) and $s \vee 2 < s+2$ imply that

$$\begin{aligned} I_{2,1} &= \sum_{k=s+2}^{M} \sum_{k_2=s}^{k-2} \sum_{j=0}^{[r_0-kr]} \sum_{j_2=0}^{j} 2c_{0,k-k_2,j-j_2} c_{s,k_2,j_2} t^{-rk-j} + O(t^{-r\varepsilon}) \\ &= \sum_{k=s+2}^{M} \sum_{j=0}^{[r_0-kr]} \Big(\sum_{k_2=s}^{k-2} \sum_{j_2=0}^{j} 2c_{0,k-k_2,j-j_2} c_{s,k_2,j_2} \Big) t^{-rk-j} + O(t^{-r\varepsilon}) \\ &= \sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^2_{s,k,j,2}(\mathbf{S}) t^{-rk-j} + O(t^{-r\varepsilon}), \end{aligned}$$

where $E^2_{s,k,j,2}(\mathbf{S}) = 0$ if $k = s \vee 2, \ldots, s+1$. Then $E^2_{s,k,j,2}(\mathbf{S})$ belong to $F_2(\mathbf{S})$. Moreover, it follows

$$I_2 = \sum_{s=1}^{M} \sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^2_{s,k,j,2}(\mathbf{S}) t^{-rk-j} \cos sn\theta + O(t^{-r\varepsilon}).$$

Consider I_3. I_3 is also rewritten as:

$$I_3 = \sum_{s_1,s_2=1}^{M} \Big(\sum_{k_1=s_1}^{M} \sum_{j_1=0}^{[r_0-k_1 r]} \sum_{k_2=s_2}^{M} \sum_{j_2=0}^{[r_0-k_2 r]}$$
$$c_{s_1,k_1,j_1} c_{s_2,k_2,j_2} t^{-r(k_1+k_2)-(j_1+j_2)} \Big) \cos s_1 n\theta \cos s_2 n\theta \qquad (16)$$
$$= \sum_{s_1,s_2=1}^{M} (I_{3,1}) \cos s_1 n\theta \cos s_2 n\theta.$$

Here we defined $I_{3,1}$ by the terms inside the parentheses in the upper line. From $s_1, s_2 \geqq 1$, it follows that $(s_1+s_2) \vee 2 = s_1+s_2$. Then the change of variables of (14) yields

$$\begin{aligned} I_{3,1} &= \sum_{k=s_1+s_2}^{M} \sum_{j=0}^{[r_0-kr]} 2\Big(\sum_{k_2=s_2}^{k-2} \sum_{j_2=0}^{j} c_{s_1,k-k_2,j-j_2} c_{s_2,k_2,j_2}/2 \Big) t^{-rk-j} + O(t^{-r\varepsilon}) \\ &= \sum_{k=(s_1+s_2)\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} 2E^3_{s_1,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} + O(t^{-r\varepsilon}). \end{aligned}$$

Here we defined $E^3_{s_1,s_2,k,j,2}$ by the terms inside the parentheses in the upper line. Since $\cos s_1 n\theta \cos s_2 n\theta = (\cos(s_1+s_2)n\theta + \cos(s_1-s_2)n\theta)/2$, it follows that

$$\begin{aligned} I_3 &= \sum_{s_1,s_2=1}^{M} \Big(\sum_{k=(s_1+s_2)\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^3_{s_1,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} \Big) \cos(s_1+s_2)n\theta \\ &\quad + \sum_{s_1,s_2=1}^{M} \Big(\sum_{k=(s_1+s_2)\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^3_{s_1,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} \Big) \cos(s_1-s_2)n\theta + O(t^{-r_\varepsilon}) \\ &= I_{3,2} + I_{3,3} + O(t^{-r_\varepsilon}). \end{aligned}$$

Here we defined $I_{3,2}$, $I_{3,3}$ by the first and the second terms in the upper line, respectively. Then $E^3_{s_1,s_2,k,j,2}(\mathbf{S})$ belongs to $F_2(\mathbf{S})$. By using the change of variable from (s_1,s_2) to $(s=s_1+s_2,s_2)$ for $I_{3,2}$, it follows

$$\begin{aligned} I_{3,2} &= \sum_{s=2}^{M} \Big(\sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} \big(\sum_{s_2=1}^{s-1} E^3_{s-s_2,s_2,k,j,2}(\mathbf{S}) \big) t^{-rk-j} \Big) \cos sn\theta + O(t^{-r_\varepsilon}) \\ &= \sum_{s=1}^{M} \Big(\sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^{3,2}_{s,k,j,2}(\mathbf{S}) t^{-rk-j} \Big) \cos sn\theta + O(t^{-r_\varepsilon}), \end{aligned} \tag{17}$$

where $E^{3,2}_{s,k,j,2}(\mathbf{S}) = 0$ if $s=1$.

Next, divide $I_{3,3}$ into three terms as follows:

$$\begin{aligned} I_{3,3} &= \sum_{1\leqq s_2<s_1\leqq M}^{M} (I_{3,1}) \cos(s_1-s_2)n\theta + \sum_{1\leqq s_2=s_1\leqq M}^{M} I_{3,1} \\ &\quad + \sum_{1\leqq s_1<s_2\leqq M}^{M} (I_{3,1}) \cos(s_1-s_2)n\theta \\ &= I_{3,3,1} + I_{3,3,2} + I_{3,3,3}. \end{aligned}$$

By using the change of variables from (s_1,s_2) to $(s=s_1-s_2,s_2)$ for $I_{3,3,1}$ and noting $1 \leqq s_2 \leqq M_s$ if $s_1+s_2 \leqq k \leqq M$, it follows

$$I_{3,3,1} = \sum_{s=1}^{M-2} \sum_{s_2=1}^{M_s} \Big(\sum_{k=s+2s_2}^{M} \sum_{j=0}^{[r_0-kr]} E^3_{s+s_2,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} \Big) \cos sn\theta + O(t^{-r_\varepsilon}), \tag{18}$$

where $M_s = [(M-s)/2]$. Since $s+2s_2 \geqq s\vee 2$, redefine $E^3_{s+s_2,s_2,k,j,2}$ by $E^3_{s+s_2,s_2,k,j,2} = 0$ if $s\vee 2 \leqq k < s+2s_2$. Then it follows

$$\begin{aligned} I_{3,3,1} &= \sum_{s=1}^{M-2} \sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} \Big(\sum_{s_2=1}^{M_s} E^3_{s+s_2,s_2,k,j,2}(\mathbf{S}) \Big) t^{-rk-j} \cos sn\theta + O(t^{-r_\varepsilon}) \\ &= \sum_{s=1}^{M} \sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^{3,3,1}_{s+s_2,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} \cos sn\theta + O(t^{-r_\varepsilon}), \end{aligned}$$

where

$$\begin{cases} E^{3,3,1}_{s,k,j,2}(\mathbf{S}) = 0 & \text{if } s = M-1, M \\ E^{3,3,1}_{s,k,j,2}(\mathbf{S}) = \sum_{s_2=1}^{M_s} E^3_{s+s_2,s_2,k,j,2}(\mathbf{S}) & \text{otherwise.} \end{cases}$$

Hence it holds

$$I_{3,3,1} = \sum_{s=1}^{M} \sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^{3,3,1}_{s,k,j,2}(\mathbf{S}) t^{-rk-j} \cos sn\theta + O(t^{-r_\varepsilon}).$$

Redefine $E^3_{s_2,s_2,k,j,2}$ in $I_{3,3,2}$ by $E^3_{s_2,s_2,k,j,2} = 0$ if $2 \leqq k < 2s_2$. Then $E^{3,3,2}_{0,k,j,2}(\mathbf{S}) \in F_2(\mathbf{S})$ such that

$$\begin{aligned} I_{3,3,2} &= \sum_{s_2=1}^{M} \sum_{k=2s_2}^{M} \sum_{j=0}^{[r_0-kr]} 2E^3_{s_2,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} \\ &= \sum_{k=2}^{M} \sum_{j=0}^{[r_0-kr]} \Big(2 \sum_{s_2=1}^{M} E^3_{s_2,s_2,k,j,2}(\mathbf{S})\Big) t^{-rk-j} \\ &= \sum_{k=2}^{M} \sum_{j=0}^{[r_0-kr]} E^{3,3,2}_{0,k,j,2}(\mathbf{S}) t^{-rk-j}. \end{aligned}$$

The change of variables from (s_1, s_2) to (s_2, s_1) for $I_{3,3,3}$ yields

$$\begin{aligned} I_{3,3,3} &= \sum_{1\leqq s_1<s_2\leqq M} \Big(\sum_{k=s_1+s_2}^{M} \sum_{j=0}^{[r_0-kr]} E^3_{s_1,s_2,k,j,2}(\mathbf{S}) t^{-rk-j} \Big) \cos(s_1-s_2)n\theta \qquad (19) \\ &= \sum_{1\leqq s_2<s_1\leqq M} \Big(\sum_{k=s_1+s_2}^{M} \sum_{j=0}^{[r_0-kr]} E^3_{s_2,s_1,k,j,2}(\mathbf{S}) t^{-rk-j} \Big) \cos(s_1-s_2)n\theta. \end{aligned}$$

By the same change of variables as (18) it follows

$$I_{3,3,3} = \sum_{s=1}^{M-2} \sum_{s_2=1}^{M_s} \Big(\sum_{k=s+2s_2}^{M} \sum_{j=0}^{[r_0-kr]} (E^3_{s_2,s+s_2,k,j,2}(\mathbf{S})) t^{-rk-j} \Big) \cos sn\theta + O(t^{-r_\varepsilon}).$$

Let $E^3_{s_2,s+s_2,k,j,2}(\mathbf{S}) = 0$ if $s \vee 2 \leqq k < s + 2s_2$ and $s = M-1, M$. By putting

$$E^{3,3,3}_{s,k,j,2}(\mathbf{S}) = \sum_{s_2=1}^{M_s} E^3_{s_2,s+s_2,k,j,2}(\mathbf{S}),$$

it holds

$$I_{3,3,3} = \sum_{s=1}^{M} \sum_{k=s\vee 2}^{M} \sum_{j=0}^{[r_0-kr]} E^{3,3,3}_{s,k,j,2}(\mathbf{S}) t^{-rk-j} \cos sn\theta + O(t^{-r_\varepsilon}).$$

Denote $E_{0,k,j,2}$ and $E_{s,k,j,2}$ $(s = 1, \dots, M)$ by

$$\begin{cases} E_{0,k,j,2} = E^1_{0,k,j,2} + E^{3,3,2}_{0,k,j,2} \\ E_{s,k,j,2} = E^2_{s,k,j,2} + E^{3,2}_{s,k,j,2} + E^{3,3,1}_{s,k,j,2} + E^{3,3,3}_{s,k,j,2}. \end{cases}$$

Then it follows

$$I_1+I_2+I_3=\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}E_{0,k,j,2}(\mathbf{S})t^{-rk-j}$$
$$+\sum_{s=1}^{M}\Big(\sum_{k=s\vee 2}^{M}\sum_{j=0}^{[r_0-kr]}E_{s,k,j,2}(\mathbf{S})t^{-rk-j}\Big)\cos n\theta+O(t^{-r_\varepsilon}).$$

Moreover, it is trivial to $E_{s,k,j,2}\in F_2(\mathbf{S})$ for $s=0,1,\dots,M$. Therefore the assertion is true for $q=2$.

We skip the proof of the case $q>2$ until the appendix since the idea is similar and the calculation is rather long. □

In order to proceed the calculation smoothly, redefine $E_{s,k,j,q}(\mathbf{S})$ by

$$\begin{cases} E_{0,k,j,q}(\mathbf{S})=0 & \text{if } 2\leqq k<q,\\ E_{s,k,j,q}(\mathbf{S})=0 & \text{if } s\vee 2\leqq k<s\vee q \text{ and } s>0,\\ E_{s,k,j,q}(\mathbf{S})=\text{same as in Lemma 4.1} & \text{otherwise.}\end{cases}$$

The following lemma holds.

LEMMA 4.2. *Let q be $q\geqq 2$. Then it holds*

$$\Big(h_0(t)+\sum_{s=1}^{M}h_s(t)\cos sn\theta\Big)^q=\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}E_{0,k,j,q}(\mathbf{S})t^{-rk-j}$$
$$+\sum_{s=1}^{M}\sum_{k=s\vee 2}^{M}\sum_{j=0}^{[r_0-kr]}E_{s,k,j,q}(\mathbf{S})t^{-rk-j}\cos sn\theta+O(t^{-r_\varepsilon}),$$

where $E_{s,k,j,q}(\mathbf{S})\in F_q(\mathbf{S})$.

PROPOSITION 4.1. *Let $w(t,\theta)$ be $w=h_{-1}(t)+\sum_{s=0}^{M}h_s(t)\cos sn\theta$. Then,*

$$\Big(\sum_{j=0}^{N}\sigma_j t^{-j}\Big)\big((\omega_0+w)^p-\omega_0^p-p|\omega_0|^{p-1}w\big)$$
$$=\Big(\sum_{j=0}^{N}\sigma_j t^{-j}\Big)\big((\omega_0+h_{-1})^p-\omega_0^p-p|\omega_0|^{p-1}h_{-1}\big) \tag{20}$$
$$+\sum_{k=2}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_1=0}^{j-1}c^{**}_{j-j_1}c_{0,k,j_1}\Big)t^{-kr-j} \tag{21}$$
$$+\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_1=0}^{j-1}c^{**}_{j-j_1}c_{s,k,j_1}\Big)t^{-kr-j}\cos sn\theta \tag{22}$$
$$+\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}H_{0,k,j}(\mathbf{S})t^{-kr-j} \tag{23}$$

$$+\sum_{s=1}^{M}\Big(\sum_{k=s\vee 2}^{M}\sum_{j=0}^{[r_0-k]}H_{s,k,j}(\mathbf{S})t^{-kr-j}\Big)\cos sn\theta \qquad (24)$$
$$+O(t^{-r_\varepsilon}),$$

where $H_{0,k,j}(\mathbf{S})$ and $H_{s,k,j}(\mathbf{S})$ are the linear combination of terms $E_{0,k,j,q}(\mathbf{S})$ and $E_{s,k,j,q}(\mathbf{S})$ about $q \geqq 2$, respectively. Moreover, it follows that $H_{s,k,j}(\mathbf{S})$ belong to $\bigcup_{q=2}^{M} F_q(\mathbf{S})$ and $\{c_j^{**}\}$ are constants depending on $\{c_{-1,j}\}$.

Proof. Denote $\mathbf{F} = \bigcup_{q=2}^{M} F_q(\mathbf{S})$ for the simplicity. Remark that the coefficients $\{c_{-1,j}\}$ of $h_{-1}(t)$ are given in Proposition 3.1.

First consider the case $M > N$. By the Taylor expansion it follows that

$$(\omega_0+w)^p-\omega_0^p-p|\omega_0|^{p-1}w=\sum_{q=2}^{M}e_q w(t,\theta)^q+O(t^{-r_\varepsilon}),$$

where e_q are constants depending on q-th derivative at ω_0 of $(\omega_0+w)^p$. Denote $Y(t,\theta)$ by $Y(t,\theta)=\sum_{s=0}^{M}h_s(t)\cos sn\theta$. Then it follows that $w(t,\theta)=h_{-1}(t)+Y(t,\theta)$. The binomial expansion yields

$$\begin{aligned}(\omega_0+w)^p&-\omega_0^p-p|\omega_0|^{p-1}w\\ &=\sum_{q=2}^{M}e_q h_{-1}(t)^q+\sum_{q=2}^{M}qe_q h_{-1}(t)^{q-1}Y(t,\theta)\\ &\quad+\sum_{q=2}^{M}\sum_{q_1=2}^{q}\binom{q}{q_1}e_q h_{-1}(t)^{q-q_1}Y(t,\theta)^{q_1}+O(t^{-r_\varepsilon})\\ &=I_1+I_2+I_3+O(t^{-r_\varepsilon}).\end{aligned} \qquad (25)$$

Since $M \geqq N$ and $h_{-1}(t)^q = O(t^{-r_\varepsilon})$ if $q > N$, it follows

$$\sum_{q=2}^{M}e_q h_{-1}(t)^q=\sum_{q=2}^{N}e_q h_{-1}(t)^q+O(t^{-r_\varepsilon}).$$

From $\sum_{q=2}^{M}e_q h_{-1}(t)^q=(\omega_0+h_{-1})^p-\omega_0^p-p|\omega_0|^{p-1}h_{-1}+O(t^{-r_\varepsilon})$, it follows

$$I_1=(\omega_0+h_{-1})^p-\omega_0^p-p|\omega_0|^{p-1}h_{-1}+O(t^{-r_\varepsilon}).$$

Then (20) is equal to I_1.

Since $h_{-1}^{q-1}(t)=\sum_{j=q-1}^{N}c_{j,q-1}t^{-j}+O(t^{-r_\varepsilon})$ from Proposition 3.1, there exist constants $\{c_j^*\}$ such that

$$\sum_{q=2}^{M}qe_q h_{-1}(t)^{q-1}=\sum_{j=1}^{N}c_j^* t^{-j}+O(t^{-r_\varepsilon}).$$

Therefore it holds

$$
\begin{aligned}
I_2 &= \Big(\sum_{j_1=1}^{N} c^*_{j_1} t^{-j_1}\Big)\Big(\sum_{k=2}^{M}\sum_{j_2=0}^{[r_0-kr]} c_{0,k,j_2} t^{-kr-j_2}\Big)\\
&\quad + \Big(\sum_{j_1=1}^{N} c^*_{j_1} t^{-j_1}\Big)\Big(\sum_{s=1}^{M}\Big(\sum_{k=s}^{M}\sum_{j_2=0}^{[r_0-kr]} c_{s,k,j_2} t^{-kr-j_2}\Big)\cos sn\theta\Big) + O(t^{-r\varepsilon})\\
&= I_{2,1} + I_{2,2} + O(t^{-r\varepsilon}).
\end{aligned}
$$

On the other hand it follows that

$$
\Big(\sum_{j_1=0}^{N} \sigma_{j_1} t^{-j_1}\Big)\Big(\sum_{j_2=1}^{N} c^*_{j_2} t^{-j_2}\Big) = \sum_{j=1}^{N} c^{**}_j t^{-j} + O(t^{-r\varepsilon}).
$$

By the change of variable of (14) it follows

$$
\begin{aligned}
\Big(\sum_{j_1=0}^{N} \sigma_j t^{-j_1}\Big) I_{2,1} &= \sum_{k=2}^{M}\Big(\sum_{j_1=1}^{N}\sum_{j_2=0}^{[r_0-kr]} c^{**}_{j_1} c_{0,k,j_2} t^{-kr-(j_2+j_1)}\Big) + O(t^{-r\varepsilon})\\
&= \sum_{k=2}^{M}\Big(\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_2=0}^{j-1} c^{**}_{j-j_2} c_{0,k,j_2}\Big) t^{-kr-j}\Big) + O(t^{-r\varepsilon})\\
&= I_{2,1,1} + O(t^{-r\varepsilon}).
\end{aligned}
$$

Hence $I_{2,1,1}$ is equal to (21). By using the same method as $I_{2,1}$ for $I_{2,2}$ it follows

$$
\begin{aligned}
&\Big(\sum_{j=0}^{N} \sigma_j t^{-j}\Big) I_{2,2}\\
&\quad = \Big(\sum_{j_1=0}^{N} c^{**}_{j_1} t^{-j_1}\Big)\Big(\sum_{s=1}^{M}\Big(\sum_{k=s}^{M}\sum_{j_2=0}^{[r_0-kr]} c_{s,k,j_2} t^{-kr-j_2}\Big)\cos sn\theta\Big) + O(t^{-r\varepsilon})\\
&\quad = \sum_{s=1}^{M}\sum_{k=s}^{M}\Big(\sum_{j=0}^{N}\Big(\sum_{j_2=0}^{[r_0-kr]} c^{**}_{j-j_2} c_{s,k,j_2}\Big) t^{-kr-j}\Big)\cos sn\theta + O(t^{-r\varepsilon})\\
&\quad = I_{2,1,2} + O(t^{-r\varepsilon}).
\end{aligned}
$$

Then $I_{2,1,2}$ is equal to (22).

From $h^{q-q_1}_{-1}(t) = \sum_{j=q-q_1}^{N} c^{**}_{j,q-q_1} t^{-j}$, it follows

$$
\Big(\sum_{k=0}^{N} \sigma_k t^{-k}\Big) e_q e_{q,q_1} h^{q-q_1}_{-1}(t) = \sum_{j=q-q_1}^{N} c^*_{q,q_1,j} t^{-j} + O(t^{-r\varepsilon}),
$$

where $\{c^*_{q,q_1,j}\}$ are constants depending on the coefficients σ_k and of $h_{-1}(t)$. On the other hand, Lemma 4.2 implies that

$$
Y^{q_1}(t,\theta) = \sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]} E_{0,k,j,q_1} t^{-kr-j}
$$

$$+\sum_{s=1}^{M}\sum_{k=s\vee 2}^{M}\sum_{j=0}^{[r_0-kr]}E_{s,k,j,q_1}t^{-kr-j}\cos sn\theta+O(t^{-t_\varepsilon})$$
$$=I_{3,1,q_1}+I_{3,2,q_1}+O(t^{-r_\varepsilon}).$$

Then I_3 is represented as:

$$\Big(\sum_{j=0}^{N}\sigma_j t^{-j}\Big)I_3=\sum_{q=2}^{M}\sum_{q_1=2}^{q}\Big(\sum_{j=0}^{N}c^*_{q,q_1,j}t^{-j}\Big)(I_{3,1,q_1}+I_{3,2,q_1})+O(t^{-r_\varepsilon}). \tag{26}$$

Thus, it holds

$$\sum_{q=2}^{M}\sum_{q_1=0}^{q}\Big(\sum_{j=0}^{N}c^*_{q,q_1,j}t^{-j}\Big)I_{3,1,q_1}$$
$$=\sum_{q=2}^{M}\sum_{q_1=2}^{q}\sum_{j_1=0}^{N}\sum_{k=2}^{M}\sum_{j_2=0}^{[r_0-kr]}c^*_{q,q_1,j_1}E_{0,k,j_2,q_1}t^{-kr-j_1-j_2}+O(t^{-r_\varepsilon})$$
$$=I_{3,1}+O(t^{-r_\varepsilon}).$$

By using the change variables from (j_1,j_2) to $(j=j_1+j_2,j_2)$,

$$\Big(\sum_{j=0}^{N}\sigma_j t^{-j}\Big)I_{3,1}=\sum_{q=2}^{M}\sum_{q_1=2}^{q}\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}\sum_{j_2=0}^{j-1}c^*_{q,q_1,j-j_2}E_{0,k,j_2,q_1}t^{-kr-j}+O(t^{-r_\varepsilon})$$
$$=\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}\Big(\sum_{q=2}^{M}\sum_{q_1=2}^{q}\sum_{j_2=0}^{j-1}c^*_{q,q_1,j-j_2}E_{0,k,j_2,q_1}\Big)t^{-kr-j}+O(t^{-r_\varepsilon}).$$

Define $H_{0,k,j}$ by

$$H_{0,k,j}=\sum_{q=2}^{M}\sum_{q_1=1}^{q}\sum_{j_2=0}^{j-1}c^*_{q,q_1,j-j_2}E_{0,k,j_2,q_1}.$$

Since $\{E_{0,k,j_3,j_1}\}\subset\mathbf{F}$, we see $\{H_{0,k,j}\}\subset\mathbf{F}$. Moreover it holds

$$\Big(\sum_{j=0}^{N}\sigma_j t^{-j}\Big)I_{3,1}=\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}H_{0,k,j}t^{-kr-j}+O(t^{-r_\varepsilon}).$$

This implies (23). By using the similar method of $I_{3,1,q_1}$ for $I_{3,2,q_1}$ and putting

$$H_{s,k,j}=\sum_{q=2}^{M}\sum_{q_1=2}^{q}\sum_{j_2=0}^{j-1}c^*_{q,q_1,j-j_2}E_{s,k,j_2,q_1},$$

it follows that there exist $H_{s,k,j}\in\mathbf{F}$ satisfying

$$I_{3,2}=\sum_{s=1}^{M}\Big(\sum_{k=s\vee 2}^{M}\sum_{j=0}^{[r_0-kr]}H_{s,k,j}t^{-kr-j}\Big)\cos sn\theta+O(t^{-r_\varepsilon}).$$

This yields (24).

Let $M<N$. In this case (25) holds if we replace M by N. The term I_1 is trivial. We have $r>1$ since $M<N$. Then we can replace N in I_2 and I_3 by M since $kr+[r_0-kr]+r>r_\varepsilon$. The terms I_1 and I_2 can be treated by the same argument. For I_3, note that (26) holds by replacing q by $q_M=q\wedge M$. Then we can argue as the previous case. Hence we complete the proof. □

LEMMA 4.3. *Let* $w(t,\theta)=h_{-1}(t)+\sum_{s=0}^{M}h_s(t)\cos sn\theta$. *Then*

$$\begin{aligned}
&\sum_{j_1=1}^{N}\sigma_{j_1}p|\omega_0|^{p-1}t^{-j_1}w(t,\theta)\\
&\quad=\sum_{j_1=1}^{N}\sigma_{j_1}p|\omega_0|^{p-1}t^{-j_1}h_{-1}(t)+\sum_{k=2}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_2=0}^{j-1}\sigma_{j-j_2}p|\omega_0|^{p-1}c_{0,k,j_2}\Big)t^{-kr-j}\\
&\qquad+\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_2=0}^{j-1}\sigma_{j-j_2}p|\omega_0|^{p-1}c_{s,k,j_2}\Big)t^{-kr-j}\cos sn\theta+O(t^{-r_\varepsilon}).
\end{aligned}$$

Proof. Let I_i $(i=1,2,3)$ be defined by

$$\begin{aligned}
&\sum_{j_1=1}^{N}\sigma_{j_1}p|\omega_0|^{p-1}t^{-j_1}w(t,\theta)\\
&\quad=\sum_{j_1=1}^{N}\sigma_{j_1}p|\omega_0|^{p-1}t^{-j_1}h_{-1}(t)+\sum_{j_1=1}^{N}\sigma_{j_1}p|\omega_0|^{p-1}t^{-j_1}h_0(t)\\
&\qquad+\Big(\sum_{j_1=1}^{N}\sigma_{j_1}p|\omega_0|^{p-1}t^{-j_1}\Big)\Big(\sum_{s=1}^{M}h_s(t)\cos sn\theta\Big)+O(t^{-r_\varepsilon})\\
&\quad=I_1+I_2+I_3+O(t^{-r_\varepsilon}).
\end{aligned}$$

The change of variables of (14) and the definition of $h_0(t)$ imply that

$$I_2=\sum_{k=2}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_2=0}^{j-1}\sigma_{j-j_2}p|\omega_0|^{p-1}c_{0,k,j_2}\Big)t^{-kr-j}.$$

By the change of variables of (14) and the interchange of the order of summation in I_3, it follows

$$I_3=\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_2=0}^{j-1}\sigma_{j-j_2}p|\omega_0|^{p-1}c_{s,k,j_2}\Big)t^{-kr-j}\cos sn\theta.$$

Then the proof is complete. □

LEMMA 4.4. *Let $h_0(t)$, $h_s(t)$ be the polynomials in* (12). *Then it follows that*

$$\Big(\frac{\partial^2}{\partial t^2}+\frac{2\alpha+1}{t}\frac{\partial}{\partial t}+\frac{\alpha^2-p|\omega_0|^{p-1}}{t^2}+\frac{1}{t^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\sum_{s=0}^{M}h_s(t)\cos sn\theta\Big)$$
$$=\frac{1}{t^2}\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}\big((\alpha-kr-j)^2-p|\omega_0|^{p-1}\big)c_{0,k,j}t^{-kr-j}$$
$$+\frac{1}{t^2}\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=0}^{[r_0-kr]}\big((\alpha-kr-j)^2-p|\omega_0|^{p-1}-s^2n^2\big)c_{s,k,j}t^{-kr-j}\cos sn\theta.$$

Proof. Since the proof is a combination of above lemmas, we omit the detail. □

LEMMA 4.5. *Suppose $kr+j<r_0$ with $k\geqq 1$, $j\geqq 0$. Then it holds*

$$(\alpha-kr-j)^2-p|\omega_0|^{p-1}-s^2n^2\neq 0$$

except $(s,k,j)=(1,1,0)$. Moreover it follows $(\alpha-kr-j)^2-p|\omega_0|^{p-1}\neq 0$.

Proof. We argue by contradiction. Let $(\alpha-kr-j)^2=p|\omega_0|^{p-1}+s^2n^2$, where $k\geqq 1$, $j\geqq 0$. Then from $0<\alpha-r_0<\alpha-kr-j\leqq\alpha-r$, it follows

$$0\geqq(\alpha-kr-j)^2-(\alpha-r)^2=(s^2-1)n^2.$$

If $s\geqq 2$, then a contradiction occurs. In the case $s=1$ except $(1,1,0)$, we get a contradiction $0>(\alpha-kr-j)^2-(\alpha-r)^2=0$ by $\alpha-kr-j<\alpha-r$. The above inequalities are mutually contradicted except $(s,k,j)=(1,1,0)$. Then $(\alpha-kr-j)^2\neq p|\omega_0|^{p-1}+s^2n^2$ except $(s,k,j)=(1,1,0)$. Since $\alpha-kr-j>\alpha-\big(\alpha-\sqrt{p|\omega_0|^{p-1}}\big)=\sqrt{p|\omega_0|^{p-1}}$, it follows $(\alpha-kr-j)^2-p|\omega_0|^{p-1}>0$. The proof is complete. □

Let $w(t,\theta)=h_{-1}(t)+\sum_{s=0}^{M}h_s(t)\cos sn\theta$. Substitute w into (4). By applying Propositions 3.1, 4.1 and Lemma 4.3, it holds

$$\widetilde{\mathscr{L}}w(t,\theta)$$
$$=-\left(\frac{\partial^2}{\partial t^2}+\frac{2\alpha+1}{t}\frac{\partial}{\partial t}+\frac{\alpha^2-p|\omega_0|^{p-1}}{t^2}+\frac{1}{t^2}\frac{\partial^2}{\partial\theta^2}\right)\sum_{s=0}^{M}h_s(t)\cos sn\theta \tag{27}$$
$$+\frac{1}{t^2}\sum_{k=2}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_1=0}^{j-1}c^{**}_{j-j_1}c_{0,k,j_1}\Big)t^{-kr-j} \tag{28}$$
$$+\frac{1}{t^2}\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_1=0}^{j-1}c^{**}_{j-j_1}c_{s,k,j_1}\Big)t^{-kr-j}\cos sn\theta \tag{29}$$
$$+\frac{1}{t^2}\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}H_{0,k,j}(\mathbf{S})t^{-kr-j} \tag{30}$$

$$+\frac{1}{t^2}\sum_{s=1}^{M}\Big(\sum_{k=s\vee 2}^{M}\sum_{j=0}^{[r_0-kr]}H_{s,k,j}(\mathbf{S})t^{-kr-j}\Big)\cos sn\theta \tag{31}$$

$$+\frac{1}{t^2}\sum_{k=2}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_1=0}^{j-1}\sigma_{j-j_1}p|\omega_0|^{p-1}c_{0,k,j_1}\Big)t^{-kr-j} \tag{32}$$

$$+\frac{1}{t^2}\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=1}^{[r_0-kr]}\Big(\sum_{j_1=0}^{j-1}\sigma_{j-j_1}p|\omega_0|^{p-1}c_{s,k,j_1}\Big)t^{-kr-j}\cos sn\theta \tag{33}$$

$$+O(t^{-r_\varepsilon-2})$$

$$=I_{(27)}+I_{(28)}+I_{(29)}+I_{(30)}+I_{(31)}+I_{(32)}+I_{(33)}+O(t^{-r_\varepsilon-2}).$$

We treat each of $I_{(27)}$–$I_{(33)}$ separately. Lemma 4.4 yields that

$$I_{(27)}=\frac{1}{t^2}\sum_{k=2}^{M}\sum_{j=0}^{[r_0-kr]}\left(p|\omega_0|^{p-1}-(\alpha-kr-j)^2\right)c_{0,k,j}t^{-kr-j} \tag{34}$$

$$+\frac{1}{t^2}\sum_{s=1}^{M}\sum_{k=s}^{M}\sum_{j=0}^{[r_0-kr]}\left(p|\omega_0|^{p-1}+s^2n^2-(\alpha-kr-j)^2\right)c_{s,k,j}t^{-kr-j}\cos sn\theta. \tag{35}$$

Define $\sum_{j=0}^{-1}\{\cdots\}=0$ in this section for the convenience.

From (28), (30), (32), and (34), define the system of equations $\{D_{0,k,j}(\mathbf{S})\}$ of the coefficients of t^{-kr-j} such that

$$D_{0,k,j}(\mathbf{S})=-\left((\alpha-kr-j)^2-p|\omega_0|^{p-1}\right)c_{0,k,j}$$
$$+\sum_{j_1=0}^{j-1}(c^{**}_{j-j_1}+\sigma_{j-j_1}p|\omega_0|^{p-1})c_{0,k,j_1}+H_{0,k,j}(\mathbf{S}), \tag{36}$$

where $k=2,3,\ldots,M$ and $j=0,\ldots,[r_0-kr]$. Next, let $s,\ k,\ j$ be $s\geqq 1, k\geqq s,\ j\geqq 0$ except $(s,k,j)=(1,1,0)$. As the same method as (36) define the system of equations $\{D_{s,k,j}(\mathbf{S})\}$ of the coefficients of $t^{-kr-j}\cos sn\theta$ satisfying

$$D_{s,k,j}(\mathbf{S})=-\left((\alpha-kr-j)^2-p|\omega_0|^{p-1}-s^2n^2\right)c_{s,k,j}$$
$$+\sum_{j_1=0}^{j-1}\left(c^{**}_{j-j_1}+\sigma_{j-j_1}p|\omega_0|^{p-1}\right)c_{s,k,j_1}+H_{s,k,j}(\mathbf{S}) \tag{37}$$

from (29), (31), (33), and (35). Denote

$$D_{1,1,0}(\mathbf{S})=-\left((\alpha-r)^2-p|\omega_0|^{p-1}-n^2\right)c_{1,1,0}.$$

There exists the term $t^{-r}\cos n\theta$ in (27). However, this term does not appear in (28)-(33). Then it follows

$$D_{1,1,0}(\mathbf{S})=0. \tag{38}$$

Since $(\alpha-r)^2-p|\omega_0|^{p-1}-n^2=0$ by the definition of r, the equality (38) holds for any $c_{1,1,0}\in\mathbb{R}$.

Define a set $W(Int)$ with the order relation by

$$W(Int) = \left\{ (s,k,j) : \begin{array}{l} s,\ k,\ j \text{ are integers,} \\ 1 \leqq s \leqq M,\ s \leqq k \leqq M,\ 0 \leqq j \leqq [r_0 - kr], \\ \text{or} \\ s = 0,\ 2 \leqq k \leqq M,\ 0 \leqq j \leqq [r_0 - kr], \end{array} \right\}$$

where the order relation $\prec$ is defined by

$$\begin{cases} (s_1,k_1,j_1) \prec (0,k_2,j_2) & \text{if } 1 \leqq s_1, \\ (s_1,k_1,j_1) \prec (s_2,k_2,j_2) & \text{if } 1 \leqq s_1 < s_2, \\ (s_1,k_1,j_1) \prec (s_1,k_2,j_2) & \text{if } k_1 < k_2, \\ (s_1,k_1,j_1) \prec (s_1,k_1,j_2) & \text{if } j_1 < j_2. \end{cases}$$

Note that this order relation is the dictionary order except for the case $(s_1,k_1,j_1) \prec (0,k_2,j_2)$. Then $\mathbf{S} = \{c_{s,k,j};\ (s,k,j) \in W(Int)\}$. Consider the recursive system of equations such that

$$D_{s,k,j}(\mathbf{S}) = 0, \quad (s,k,j) \in W(Int), \tag{39}$$

where the order of equations follows the order in $W(Int)$. Note that by the order in $W(Int)$, the terms of degree one of $c_{s,j,k}$ in (36), (37) appear in the lower triangle part in the coefficient of (39). By finding all elements in $\mathbf{S}$ satisfying (39) and constructing h_s $(s = 0,\dots,M)$ by elements in $\mathbf{S}$, it follows that $w(t,\theta) = h_{-1}(t) + \sum_{s=0}^{M} h_s(t)\cos sn\theta$ satisfies (4).

Since (38) is the identity, we may consider the following recursive system of equations without (38);

$$D_{s,k,j}(\mathbf{S}) = 0 \quad \text{for any } (s,k,j) \in W(Int) \backslash \{(1,1,0)\}. \tag{40}$$

For the simplicity denote $W_1(Int) = W(Int) \backslash \{(1,1,0)\}$ and $\mathbf{S}_1 = \mathbf{S} \backslash \{c_{1,1,0}\}$.

LEMMA 4.6. *Let* $\mathbf{0} = (0,0,\dots,0)$. *Then* $\mathbf{0} \in \mathbf{S}$ *and*

$$D_{s,k,j}(\mathbf{0}) = 0 \quad \textit{for any } (s,k,j) \in W_1(Int).$$

Proof. $\mathbf{0} \in \mathbf{S}$ is obvious. From $H_{s,k,j} \subset \mathbf{F}$ it follows $H_{s,k,j}(\mathbf{0}) = 0$. Then (36) and (37) yield the assertion. □

LEMMA 4.7. *Let* $(s_1,k_1,j_1) \in W_1(Int)$ *and* $c_{s_2,k_2,j_2} \in \mathbf{S_1}$. *Then it follows*

$$\frac{\partial H_{s_1,k_1,j_1}}{\partial c_{s_2,k_2,j_2}}(\mathbf{0}) = 0.$$

Proof. Since $H_{s_1,k_1,j_1} \in \mathbf{F}$, $\dfrac{\partial H_{s_1,k_1,j_1}}{\partial c_{s_2,k_2,j_2}}(\mathbf{S})$ are polynomials which does not contain 0-th degree terms in variables $\mathbf{S}$. Then the proof is complete. □

LEMMA 4.8. *Let* (s_1,k_1,j_1), $(s_2,k_2,j_2)\in W_1(Int)$ *satisfy* $(s_1,k_1,j_1)\prec(s_2,k_2,j_2)$. *Then*

$$\frac{\partial D_{s_1,k_1,j_1}}{\partial c_{s_2,k_2,j_2}}(\mathbf{0})=0.$$

Moreover for any $(s_1,k_1,j_1)\in W_1(Int)$ *it holds*

$$\frac{\partial D_{s_1,k_1,j_1}}{\partial c_{s_1,k_1,j_1}}(\mathbf{0})=-\left((\alpha-k_1r-j_1)^2-p|\omega_0|^{p-1}-s_1^2n^2\right).$$

Proof. From Lemma 4.7, (36) and (37) imply

$$\begin{aligned}\frac{\partial D_{s_1,k_1,j_1}}{\partial c_{s_2,k_2,j_2}}(\mathbf{0})&=-\left((\alpha-k_1r-j_1)^2-p|\omega_0|^{p-1}-s_1^2n^2\right)\frac{\partial c_{s_1,k_1,j_1}}{\partial c_{s_2,k_2,j_2}} &(41)\\ &\quad+\sum_{q=0}^{j_1-1}(c^*_{j_1-q}+\sigma_{j_1-q}p|\omega_0|^{p-1})\frac{\partial c_{s_1,k_1,q}}{\partial c_{s_2,k_2,j_2}}. &(42)\\ &=I_{(41)}+I_{(42)}.\end{aligned}$$

The term $I_{(41)}$ is zero if $(s_1,k_1,j_1)\neq(s_2,k_2,j_2)$. If $(s_1,k_1,j_1)=(s_2,k_2,j_2)$ it follows that

$$I_{(41)}=-\left((\alpha-k_1r-j_1)^2-p|\omega_0|^{p-1}-s_1^2n^2\right).$$

Also $I_{(42)}$ is zero if $(s_1,k_1,q)\neq(s_2,k_2,j_2)$. Suppose $(s_1,k_1,q)=(s_2,k_2,j_2)$. Since $q\leqq j_1-1$, it follows $(s_1,k_1,q)\prec(s_1,k_1,j_1)$. Then $(s_2,k_2,j_2)\prec(s_1,k_1,j_1)$. This is a contradiction. Hence $(s_1,k_1,q)\neq(s_2,k_2,j_2)$. Therefore, $I_{(42)}$ is zero. The proof is complete. □

LEMMA 4.9. *Let* (s_1,k_1,j_1) *and* c_{s_2,k_2,j_2} *be* $(s_1,k_1,j_1)\in W_1(Int)$ *and* $c_{s_2,k_2,j_2}\in\mathbf{S}_1$, *respectively. Then the determinant of the Jacobian matrix*

$$\left(\frac{\partial(D_{s_1,k_1,j_1})}{\partial(c_{s_2,k_2,j_2})}(\mathbf{0})\right)$$

is not zero.

Proof. From Lemma 4.8, the Jacobian matrix is a lower triangular matrix. Then

$$\left(\frac{\partial(D_{s_1,k_1,j_1})}{\partial(c_{s_2,k_2,j_2})}(\mathbf{0})\right)=\prod_{(s_1,k_1,j_1)\in W_1(Int)}\left(p|\omega_0|^{p-1}+s_1^2n^2-(\alpha-k_1r-j_1)^2\right).$$

Hence Lemma 4.5 yields that the determinant is not zero. □

LEMMA 4.10. *Consider the recursive system of equations*

$$\{D_{s,k,j}(\mathbf{S})=0;\,(s,k,j)\in W_1(Int)\},$$

then there exist a positive constant $\delta > 0$ *and functions*

$$\{\phi_{s_1,k_1,j_1} \in C^1([-\delta,\delta]);\, (s_1,k_1,j_1) \in W_1(Int)\}$$

which satisfy the following properties:

$$\begin{cases} |c_{1,1,0}| \leqq \delta, \\ c_{s_1,k_1,j_1} = \phi_{s_1,k_1,j_1}(c_{1,1,0}) \in \mathbf{S}_1 \ \textit{for any}\ (s_1,k_1,j_1) \in W_1(Int), \\ D_{s,k,j}(c_{1,1,0}, \phi_{1,1,1}(c_{1,1,0}), \ldots, \phi_{s_1,k_1,j_1}(c_{1,1,0}), \ldots) = 0, \\ \textit{for any}\ (s,k,j) \in W_1(Int). \end{cases} \tag{43}$$

Proof. From Lemmas 4.6 and 4.9, the implicit function theorem implies the assertion. □

We construct a function $w(t,\theta)$ satisfying (4). Let c_{s_1,k_1,j_1} be constants satisfying (43) in Lemma 4.10. Define functions $\{h_s(t)\}_{s=0,\ldots,M}$ and $w(t,\theta)$ by

$$h_0(t) = \sum_{k=2}^{M} \sum_{j=0}^{[r_0-kr]} c_{0,k,j} t^{-kr-j},$$

$$h_s(t) = \sum_{k=s}^{M} \sum_{j=0}^{[r_0-kr]} c_{s,k,j} t^{-kr-j}$$

and

$$w(t,\theta) = h_{-1}(t) + \sum_{s=0}^{M} h_s(t) \cos sn\theta.$$

By combining (27)-(33), (36)-(39) and Lemma 4.10, it follows that $\{$the left hand side of (4)$\} = O(t^{-r_\varepsilon-2})$. Thus, the following proposition holds.

PROPOSITION 4.2. *It holds* $\widetilde{\mathscr{L}}w(t,\theta) = O(t^{-r_\varepsilon-2})$ *for* $t \to \infty$.

5. Super- and sub-solutions

In this section we show the existence of the super- and sub-solution of (1). Let D be a large positive number and t be also sufficiently large. Denote $u_+(t,\theta)$ by

$$u_+(t,\theta) = t^{\alpha}\left((\omega_0 + w(t,\theta)) + Dt^{-r_\varepsilon}\right).$$

Then,

$$\begin{aligned} g(t)^{-1} t^{-\alpha} \mathscr{L} u_+(t,\theta) &= \widetilde{\mathscr{L}}(w(t,\theta) + Dt^{-r_\varepsilon}) \\ &= \widetilde{\mathscr{L}} w(t,\theta) \end{aligned}$$

$$
\begin{aligned}
&+Dt^{-r\varepsilon-2}\left(2\varepsilon\sqrt{p|\omega_0|^{p-1}}-\varepsilon^2\right)\\
&+\Big(\sum_{i=0}^{N}\sigma_i t^{-i}\Big)t^{-2}\\
&\quad\times\big((\omega_0+w(t,\theta)+Dt^{-r\varepsilon})^p-(\omega_0+w(t,\theta))^p-p|\omega_0|^{p-1}Dt^{-r\varepsilon}\big)\\
&+\Big(\sum_{i=1}^{N}\sigma_i p|\omega_0|^{p-1}t^{-i}\Big)Dt^{-r\varepsilon-2}\\
&=V_1+V_2+V_3+V_4
\end{aligned}
$$

If $\varepsilon > 0$ is sufficiently small, then it follows

$$
V_2 \geqq \varepsilon\sqrt{p|\omega_0|^{p-1}}Dt^{-r\varepsilon-2}. \tag{44}
$$

Proposition 4.2 implies that there exist large numbers $T_{2,1}$ and M_1 such that

$$
V_1 \geqq -M_1 t^{-r\varepsilon-2}
$$

for any $t \geqq T_{2,1}$. Let D be a large number satisfying

$$
\varepsilon\sqrt{p|\omega_0|^{p-1}}D > 4M_1.
$$

Then it follows that

$$
|V_1| \leqq \frac{\varepsilon\sqrt{p|\omega_0|^{p-1}}}{4}Dt^{-r\varepsilon-2}. \tag{45}
$$

On the other hand the Taylor expansion yields

$$
|V_3| \leqq C(|w|+Dt^{-r\varepsilon})Dt^{-r\varepsilon-2}
$$

for some constant $C > 0$. From $|w(t,\theta)| \leqq C_1(t^{-1}+t^{-r}+t^{-r\varepsilon})$, it follows that

$$
|w+Dt^{-r\varepsilon}| \leqq C_2(t^{-1}+t^{-r}) \quad \text{for any } t \geqq T_{2,2},
$$

where C_2 is a constant independent of t. Then there exists $T_{2,3} \geqq T_{2,2}$ satisfying

$$
|V_3| \leqq \frac{\varepsilon\sqrt{p|\omega_0|^{p-1}}}{4}Dt^{-r\varepsilon-2} \tag{46}
$$

for $t \geqq T_{2,3}$. Also there exists $T_{2,4}$ such that

$$
|V_4| \leqq \frac{\varepsilon\sqrt{p|\omega_0|^{p-1}}}{4}Dt^{-r\varepsilon-2} \quad \text{for any } t \geqq T_{2,4}. \tag{47}
$$

Let $T_2 = \max(T_{2,1}, T_{2,3}, T_{2,4})$. The inequalities (44), (45), (46) and (47) yield

$$
V_1+V_2+V_3+V_4 \geqq \frac{\varepsilon\sqrt{p|\omega_0|^{p-1}}}{4}Dt^{-r\varepsilon-2} \quad \text{for any } t > T_2.
$$

Then it follows $\mathscr{L}u_+(t,\theta) \geqq 0$ for any $t > T_2$. Thus, $u_+(t,\theta)$ is the super-solution of (1). Let $u_-(t,\theta)$ be $u_-(t,\theta) = t^{\alpha}(\omega_0 + w(t,\theta) - Dt^{-r\varepsilon})$. If the similar method of the proof of the existence of the super-solution is used, it is shown that the function $u_-(t,\theta)$ is the sub-solution of (1) for any $t > T_2$. □

6. Proof of the main theorem

In this section we complete the proof of the theorem by applying the similar method of section 4 in [4]. If we construct super- and sub-solution of (1), the existence of the non-radially symmetric solution of (1) is proved by using the following proposition due to E. S. Noussair and C. A. Swanson [7].

PROPOSITION 6.1. *Let $\widetilde{u}_+$, $\widetilde{u}_-$ be super- and sub-solution of* (1) *such that*

$$\widetilde{u}_+(t,\theta) \geqq \widetilde{u}_-(t,\theta) \text{ and } \widetilde{u}_\pm \in C^2(\mathbb{R}^2 \backslash \overline{B_{T_0}}) \cap C(\mathbb{R}^2 \backslash B_{T_0}).$$

Then there exists a solution $u(t,\theta) \in C^2(\mathbb{R}^2 \backslash \overline{B_{T_0}}) \cap C(\mathbb{R}^2 \backslash B_{T_0})$ of (1) *satisfying*

$$\widetilde{u}_+(t,\theta) \geqq u(t,\theta) \geqq \widetilde{u}_-(t,\theta). \tag{48}$$

We construct the super- and sub-solutions $\widetilde{u}_\pm(t,\theta) \in C^2(\mathbb{R}^2 \backslash \overline{B_{T_0}}) \cap C(\mathbb{R}^2 \backslash B_{T_0})$ satisfying $\widetilde{u}_+(T_0,\theta) = \widetilde{u}_-(T_0,\theta)$. By the thory of viscosity solutions (see [1], [2]) there exists a continuous viscosity solution in $\mathbb{R}^2 \backslash B_{T_0}$. We can get the regurarity for this solution in any compact interval of t by the thoery of semilinear elliptic equations (see [3]). As mentioned in the Remark 2.3, the proof of the Theorem 2.1 is completed by these construction. Let $u_\pm(t,\theta)$ be the super- and sub-solution, respectively, which are constructed in section 5. Then,

$$\begin{aligned}
&\overline{\lim_{t\to\infty}} \frac{u_+(t,\theta) - t^\alpha(\omega_0 + h_{-1}(t))}{t^{\sqrt{p|\omega_0|^{p-1}+n^2}}} \\
&\quad = \underline{\lim}_{t\to\infty} \frac{u_-(t,\theta) - t^\alpha(\omega_0 + h_{-1}(t))}{t^{\sqrt{p|\omega_0|^{p-1}+n^2}}} = c_{1,1,0}\cos n\theta.
\end{aligned} \tag{49}$$

Let $y_\pm$ be solutions of (1) satisfying $\lim_{t\to\infty} \frac{y_\pm(t)}{t^\alpha} = \omega_\pm$, respectively (See [6]). Define $\widetilde{u}_\pm(t,\theta)$ by

$$\widetilde{u}_\pm(t,\theta) = c_2(t)u_\pm(t,\theta) + c_1(t)y_\pm(t) \pm A\log\frac{t}{T_0}.$$

Here c_1, $c_2 \in C^2([T_0,\infty))$ are functions satisfying

$$c_1(t) = \begin{cases} 1 & \text{for } T_0 < t < T_1, \\ 0 & \text{for } T_1 + 1 < t, \end{cases} \qquad c_2(t) = \begin{cases} 0 & \text{for } T_0 < t < T_2, \\ 1 & \text{for } T_2 + 1 < t, \end{cases}$$

and A is a positive number.

By applying the similar method in section 4 in [4], $\widetilde{u}_\pm(t,\theta)$ become the super- and sub-solutions, respectively. By noting (48) and (49), we obtain the existence of the solution $u(t,\theta)$ satisfying

$$\lim_{t\to\infty} \frac{u(t,\theta) - t^\alpha(\omega_0 + h_{-1}(t))}{t^{\sqrt{p|\omega_0|^{p-1}+n^2}}} = c_{1,1,0}\cos n\theta.$$

Remark that by the same method in section 5, $y_{\pm}(t) = t^{\alpha}\{(\omega_0 + h_1(t)) \pm Dt^{-r_\varepsilon}\}$ become super- and sub-solution, respectively. Hence there exists a radially symmetric solution $y(t)$ of (1) satisfying $y(t) - t^{\alpha}(\omega_0 + h_1(t)) = O(t^{-r_\varepsilon})$. Let $y_1(t)$ and $y_2(t)$ be solutions of the equation (1) satisfying $y_1(t) > y_2(t)$. Define $w_i(t)$ by $y_i(t) = t^{\alpha}(\omega_0 + w_i(t))$ for $i = 1,2$. Then it follows that there exists a sufficiently large number t_1 such that $2\max(|w_1(t)|,|w_2(t)|) < \varepsilon$ for any $t > t_1$, where ε is a small positive number. Since it holds

$$\frac{t^{\ell}}{g(t)} p|\omega_0 + \tau w_1(t) + (1-\tau)w_2(t)|^{p-1} \leqslant (p|\omega_0|^{p-1} + \varepsilon),$$

for some $0 < \tau < 1$ which depends on the Taylor expansion of $y_1(t)^p - y_2(t)^p$, it follows

$$\begin{aligned} y_1(t) - y_2(t) \leqq y_1(t_1) - y_2(t_1) + t_1 \frac{d(y_1 - y_2)}{dt}(t_1)\log\frac{t}{t_1} \\ + (p|\omega_0|^{p-1} + \varepsilon)\int_{t_1}^{t} \frac{1}{s}\int_{t_1}^{s} \frac{1}{\xi}(y_1(\xi) - y_2(\xi))d\xi. \end{aligned}$$

Let $Z(t)$ be a solution satisfying:

$$\begin{aligned} &\frac{d^2Z}{dt^2}(t) + \frac{1}{t}\frac{dZ}{dt}(t) = \frac{p|w_0|^{p-1} + \varepsilon}{t^2} Z(t), \\ &\frac{dZ}{dt}(t_1) = \frac{d(y_1 - y_2)}{dt}(t_1), \\ &Z(t_1) = (y_1 - y_2)(t_1). \end{aligned}$$

Then $(y_1 - y_2)(t) \leqslant Z(t)$ for any $t \geqslant t_1$. Therefore,

$$(y_1 - y_2)(t) \leqq Ct^{\sqrt{p|\omega_0|^{p-1}+\varepsilon}}.$$

Hence,

$$\lim_{t\to\infty} \frac{y_1(t) - y_2(t)}{t^{\sqrt{p|w_0|^{p-1}+n^2}}} = 0,$$

where $y(t)$ is any radially symmetric solution of (1). By putting $u_n(t,\theta) = u(t,\theta)$, the proof of the theorem is complete. □

REMARK 6.1. We proceed the proof of the theorem without (H-3). Assume that there exist $k,\ j \geqq 0$ satisfying $kr_n + j = r_0$.

Define $h_{-1}(t)$, $h_0(t)$ and $h_s(t)$ in (6), (12) by

$$\begin{aligned} h_{-1}(t) &= \sum_{j=1}^{N-1} c_{-1,j}t^{-j} + c_{-1,N}t^{-N}\log t \\ h_0(t) &= \sum_{2\leqq k\leqq M,\ 0\leqq j\leqq [r_0 - kr],\ (k,j)\notin J_0} c_{0,k,j}t^{-kr-j} + \sum_{(k,j)\in J_0} c_{0,k,j}t^{-kr-j}\log t, \end{aligned}$$

$$h_s(t) = \sum_{1 \leqq k \leqq M,\ s \leqq j \leqq [r_0 - kr],\ (k,j) \notin J_0} c_{s,k,j} t^{-kr-j} + \sum_{(k,j) \in J_0} c_{s,k,j} t^{-kr-j} \log t,$$

where $J_0 = \{(k, j);\, 1 \leqq k,\ j = r_0 - kr\}$. Then we can construct $w(t, \theta)$ by using these functions. It follows that

$$-\left(\frac{\partial^2}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial}{\partial t} + \frac{\alpha^2 - p|\omega_0|^{p-1}}{t^2}\frac{\partial^2}{\partial \theta^2}\right) t^{-r_0} \log t$$
$$= -(2(\alpha - r_0) + \alpha^2 - p|\omega_0|^{p-1}) t^{-r_0-2}.$$

Note that the coefficient of the right hand side is not 0. Next, by using the estimate $t^{-r_0} \log t (t^{-r} + t^{-1} + t^{-r_0}) = O(t^{-r_\varepsilon})$, we can take in all terms that contain $t^{-r_0} \log t$ in the expression

$$\frac{1}{t^2}\left(\sum_{i=0}^{N} \sigma_i t^{-i}\right)\{(\omega_0 + w)^p - \omega_0^p - p|\omega_0|^{p-1}\}$$
$$+ \frac{1}{t^2}\sum_{i=1}^{N}(\sigma_i p|\omega_0|^{p-1} w) + \frac{1}{t^2}\sum_{i=1}^{N}(\sigma_i \omega_0^p - f_i) t^{-i},$$

into the order term $O(t^{-r_\varepsilon})$. Therefore

$$\widetilde{\mathscr{L}} w(t, \theta) = I_{(27)} + I_{(28)} + \cdots + I_{(33)} + O(t^{-r_\varepsilon})$$

without the assumption (H-3). This completes the proof.

REMARK 6.2. We can replace the assumption (H-2) by the following:

$$\begin{cases} f(t) = t^{\alpha p}(1 + O((t^{-\alpha} \log t)^p)), \\ g(t) = t^{\ell}(1 + O((t^{-\alpha} \log t)^p)). \end{cases}$$

In this case, we assume assumption (H-1). By combining the result of [4], theorem 2.1 and remark 6.1, we obtain that there exist $2m$ kinds of sets of non-radially symmetric solutions of (1).

7. Appendix

We continue the proof of Lemma 4.1.

Suppose that our assertion is true in case of $q \geqq 2$. By using the similar method of the proof of $q = 2$ we show our assertion for $q + 1$. Let $G(q)$ be $G(q) = (h_0(t) + \sum_{s=1}^{M} h_s(t) \cos sn\theta)^q$. By applying the assumption of the induction to $G(q)$ and noting $G(1) = h_0(t) + \sum_{s=1}^{M} h_s(t) \cos sn\theta$ it holds

$$G(q+1) = G(1)G(q)$$
$$= h_0(t)\left(\sum_{k_2=q}^{M} \sum_{j_2=0}^{[r_0-k_2 r]} E_{0,k_2,j_2,q}(\mathbf{S}) t^{-k_2 r - j_2}\right)$$

$$
\begin{aligned}
&+ h_0(t)\Big(\sum_{s_2=1}^{M}\sum_{k_2=s_2\vee q}^{M}\sum_{j_2=0}^{[r_0-k_2r]} E_{s_2,k_2,j_2,q}(\mathbf{S})t^{-k_2r-j_2}\cos s_2n\theta\Big)\\
&+\Big(\sum_{s_1=1}^{M} h_{s_1}(t)\cos s_1n\theta\Big)\Big(\sum_{k_2=q}^{M}\sum_{j_2=0}^{[r_0-kr]} E_{0,k_2,j_2,q}(\mathbf{S})t^{-k_2r-j_2}\Big)\\
&+\Big(\sum_{s_1=1}^{M} h_{s_1}(t)\cos s_1n\theta\Big)\Big(\sum_{s_2=1}^{M}\sum_{k_2=s_2\vee q}^{M}\sum_{j_2=0}^{[r_0-k_2r]} E_{s_2,k_2,j_2,q}(\mathbf{S})\\
&\quad\times t^{-k_2r-j_2}\cos s_2n\theta\Big)+O(t^{-r\varepsilon})\\
=\;& I_{q,1}+I_{q,2}+I_{q,3}+I_{q,4}+O(t^{-r\varepsilon}).
\end{aligned}
$$

By replacing c_{0,k_2,j_2} in (13) by $E_{0,k_2,j_2,q}(\mathbf{S})$ in $I_{q,1}$ and using the similar change of variables of (14) for $I_{q,1}$, it follows

$$
\begin{aligned}
I_{q,1} &= \sum_{k=2+q}^{M}\sum_{j=0}^{[r_0-kr]}\Big(\sum_{k_2=q}^{k-2}\sum_{j_2=0}^{j} c_{0,k-k_2,j-j_2}E_{0,k_2,j_2,q}(\mathbf{S})\Big)t^{-rk-j}+O(t^{-r\varepsilon})\\
&= \sum_{k=q+1}^{M}\sum_{j=0}^{[r_0-kr]} E^1_{0,k,j,q+1}(\mathbf{S})t^{-rk-j}+O(t^{-r\varepsilon}).
\end{aligned}
\tag{50}
$$

Here $E^1_{0,k,j,q+1}(\mathbf{S})$ are denoted by

$$
E^1_{0,k,j,q+1}(\mathbf{S}) = \begin{cases} \displaystyle\sum_{k_2=q}^{k-2}\sum_{j_2=0}^{j} c_{0,k-k_2,j-j_2}E_{0,k_2,j_2,q}(\mathbf{S}) & \text{if } k \geqq q+2\\ 0 & \text{if } k=q+1. \end{cases}
$$

Thus $E^1_{0,k,j,q+1}(\mathbf{S})$ belong to $F_{q+1}(\mathbf{S})$ from $c_{0,k-k_2,j-j_2}F_q(\mathbf{S}) \subset F_{q+1}(\mathbf{S})$.

By regarding $2c_{s,k_2,j_2}$ in (15) as $E_{s_1,k_2,j_2,q}(\mathbf{S})$ in $I_{q,2}$, by the similar method for I_2, there exists $E^2_{s,k,j,q+1}(\mathbf{S}) \in F_{q+1}(\mathbf{S})$ satisfying

$$
\begin{aligned}
&I_{q,2}\\
&= \sum_{k_1=2}^{M}\sum_{j_1=0}^{[r_0-k_1r]}\sum_{s_2=1}^{M}\sum_{k_2=s_2\vee q}^{M}\sum_{j_2=0}^{[r_0-k_2r]} c_{0,k_1,j_1}E_{s_2,k_2,j_2,q}(\mathbf{S})t^{-(k_1+k_2)r-(j_1+j_2)}\cos s_2n\theta\\
&\quad + O(t^{-r\varepsilon})\\
&= \sum_{s=1}^{M}\sum_{k=2+s\vee q}^{M}\sum_{j=0}^{[r_0-k_1r]}\Big(\sum_{k_2=s\vee q}^{k-2}\sum_{j_2=0}^{j} c_{0,k-k_2,j-j_2}E_{s,k_2,j_2,q}(\mathbf{S})\Big)t^{-kr-j}\cos sn\theta\\
&\quad + O(t^{-r\varepsilon}).
\end{aligned}
$$

Note that $2+s\vee q=(q+2)\vee(s+2)>(q+1)\vee s$. By setting

$$
E^2_{s,k,j,q+1}(\mathbf{S}) = \begin{cases} \displaystyle\sum_{k_2=q}^{k-2}\sum_{j_2=0}^{j} c_{0,k-k_2,j-j_2}E_{s,k_2,j_2,q}(\mathbf{S}) & \text{if } k \geqq s\vee q+2,\\ 0 & \text{if } s\vee(q+1) \leqq k < s\vee q+2, \end{cases}
$$

$E^2_{s,k,j,q+1}$ satisfies

$$I_{q,2} = \sum_{s=1}^{M} \sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-k_1r]} E^2_{s,k,j,q+1}(\mathbf{S}) t^{-kr-j} \cos sn\theta + O(t^{-r_\varepsilon}), \tag{51}$$

where $E^2_{s,k,j,q+1} \in F_{q+1}(\mathbf{S})$.

By using the similar method for $I_{q,2}$ to $I_{q,3}$, it holds

$$\begin{aligned} I_{q,3} &= \sum_{s_1=1}^{M} \sum_{k_1=s_1}^{M} \sum_{j_1=0}^{[r_0-k_1r]} \sum_{k_2=q}^{M} \sum_{j_2=0}^{[r_0-k_2r]} c_{s_1,k_1,j_1} E_{0,k_2,j_2,q}(\mathbf{S}) t^{-(k_1+k_2)r-(j_1+j_2)} \cos s_1 n\theta \\ &\quad + O(t^{-r_\varepsilon}) \\ &= \sum_{s=1}^{M} \sum_{k=s+q}^{M} \sum_{j=0}^{[r_0-k_1r]} \Big(\sum_{k_2=q}^{k-s} \sum_{j_2=0}^{j} c_{s,k-k_2,j-j_2} E_{0,k_2,j_2,q}(\mathbf{S}) \Big) t^{-kr-j} \cos sn\theta \\ &\quad + O(t^{-r_\varepsilon}). \end{aligned}$$

Since $s+q \geqq s\vee(q+1)$ from $s \geqq 1$, we can find $E^3_{s,k,j,q+1}(\mathbf{S}) \in F_{q+1}(\mathbf{S})$ satisfying

$$I_{q,3} = \sum_{s=1}^{M} \sum_{k=(q+1)\vee s}^{M} \sum_{j=0}^{[r_0-kr]} E^3_{s,k,j,q+1}(\mathbf{S}) t^{-kr-j} \cos sn\theta + O(t^{-r_\varepsilon}), \tag{52}$$

where $E^3_{s,k,j,q+1}(\mathbf{S}) = 0$ if $s\vee(q+1) \leqq k < s+q$. Moreover, it holds $E^3_{s,k,j,q+1}(\mathbf{S}) \in F_{q+1}(\mathbf{S})$.

By replacing c_{s_2,k_2,j_2} in (16) by $E_{s_2,k_2,j_2,q}$ in $I_{q,4}$, it follows

$$I_{q,4} = \sum_{s_1=1}^{M} \sum_{s_2=1}^{M} (I_{q,4,1}) \cos s_1 n\theta \cos s_2 n\theta,$$

where

$$I_{q,4,1} = \sum_{k_1=s_1}^{M} \sum_{j_1=0}^{[r_0-k_1r]} \sum_{k_2=s_2\vee q}^{M} \sum_{j_2=0}^{[r_0-k_2r]} c_{s_1,k_1,j_1} E_{s_2,k_2,j_2,q} t^{-r(k_1+k_2)-(j_1+j_2)}.$$

By the change of variables of (14) we can find $E^4_{s_1\vee s_2,k,j,q+1}$ satisfying

$$\begin{aligned} I_{q,4,1} &= \sum_{k=s_1+s_2\vee q}^{M} \sum_{j=0}^{[r_0-kr]} \Big(\sum_{k_2=q}^{k-s_1} \sum_{j_2=0}^{j} c_{s_1,k_1,j_1} E_{s_2,k_2,j_2,q}(\mathbf{S}) \Big) t^{-r(k_1+k_2)-(j_1+j_2)} + O(t^{-r_\varepsilon}) \\ &= \sum_{k=s_1+s_2\vee q}^{M} \sum_{j=0}^{[r_0-kr]} 2E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} + O(t^{-r_\varepsilon}). \end{aligned}$$

Then it follows that $E^4_{s_1,s_2,k,j,q+1}(\mathbf{S})$ belong to $F_{q+1}(\mathbf{S})$.

Let $\mathbf{V_0} = \{(s_1, s_2);\ 1 \leqq s_1 \leqq M,\ 1 \leqq s_2 \leqq M$ and $s_1 + s_2 \leqq M.\}$

From $(s_1, s_2) \in \mathbf{V_0}$ it follows $s_1 + s_2 \vee q \geqq (s_1 + s_2) \vee (q+1)$. Set $E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) = 0$ for $(s_1 + s_2) \vee (q+1) \leqq k < s_1 + s_2 \vee q$. Then $I_{q,4}$ is rewritten as:

$$\begin{aligned} I_{q,4} &= \sum_{\mathbf{V_0}} \sum_{k=(s_1+s_2)\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} \cos(s_1+s_2)n\theta \\ &\quad + \sum_{\mathbf{V_0}} \sum_{k=(s_1+s_2)\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} \cos(s_1-s_2)n\theta + O(t^{-r_\varepsilon}) \\ &= I_{q,4,2} + I_{q,4,3} + O(t^{-r_\varepsilon}). \end{aligned} \tag{53}$$

By using the same change of variables as (17) we can find $E^{4,2}_{s,k,j,q+1}(\mathbf{S})$ satisfying

$$\begin{aligned} I_{q,4,2} &= \sum_{s=2}^{M} \Big(\sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} \Big(\sum_{s_2=1}^{s-1} E^4_{s-s_2,s_2,k,j,q+1}(\mathbf{S}) \Big) t^{-rk-j} \Big) \cos sn\theta + O(t^{-r_\varepsilon}) \\ &= \sum_{s=1}^{M} \Big(\sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^{4,2}_{s,k,j,q+1}(\mathbf{S}) t^{-rk-j} \Big) \cos sn\theta + O(t^{-r_\varepsilon}), \end{aligned} \tag{54}$$

where $E^{4,2}_{s,k,j,q+1}(\mathbf{S}) = 0$ if $s = 1$. Here, we see that $E^{4,2}_{s,k,j,q+1}(\mathbf{S})$ belong to $F_{q+1}(\mathbf{S})$ since $E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) \in F_{q+1}(\mathbf{S})$.

In order to study $I_{q,4,3}$, redefine

$$E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) = 0 \text{ if } |s_1 - s_2| \vee (q+1) \leqq k < (s_1+s_2) \vee (q+1)$$

and divide the set $\mathbf{V_0}$ as follows:

$$\begin{aligned} \mathbf{V_0} &= \{(s_1,s_2);\ s_1 - s_2 > 0,\ s_2 \geqq 1,\ s_1 + s_2 \leqq M\} \\ &\quad \cup \{(s_1,s_2);\ s_1 - s_2 = 0,\ s_2 \geqq 1,\ s_1 + s_2 \leqq M\} \\ &\quad \cup \{(s_1,s_2);\ s_1 - s_2 < 0,\ s_1 \geqq 1,\ s_1 + s_2 \leqq M\} \\ &= \mathbf{V_1} \cup \mathbf{V_2} \cup \mathbf{V_3}. \end{aligned}$$

Then $I_{q,4,3}$ is rewritten as

$$\begin{aligned} &I_{q,4,3} \\ &= \sum_{i=1}^{3} \sum_{\mathbf{V_i}} \Big(\sum_{k=|s_1-s_2|\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{s_1,s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} \Big) \cos(s_1-s_2)n\theta \\ &\quad + O(t^{-r_\varepsilon}) \\ &= I_{q,4,3,1} + I_{q,4,3,2} + I_{q,4,3,3} + O(t^{-r_\varepsilon}). \end{aligned} \tag{55}$$

By the change of variables of (17) there exists $E^{4,3,1}_{s,k,j,q+1}(\mathbf{S}) \in F_{q+1}(\mathbf{S})$ satisfying

$$\begin{aligned} I_{q,4,3,1} &= \sum_{s=1}^{M-2} \sum_{s_2=1}^{M_s} \Big(\sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{s+2s_2,s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} \Big) \cos sn\theta \\ &= \sum_{s=1}^{M-2} \sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} \Big(\sum_{s_2=1}^{M_s} E^4_{s+2s_2,s_2,k,j,q+1}(\mathbf{S}) \Big) t^{-rk-j} \cos sn\theta \\ &= \sum_{s=1}^{M} \sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^{4,3,1}_{s,k,j,q+1}(\mathbf{S}) t^{-rk-j} \cos sn\theta, \end{aligned} \tag{56}$$

where

$$\begin{cases} E^{4,3,1}_{s,k,j,q+1}(\mathbf{S}) = \sum_{s_2=q}^{M_s} E^4_{s+2s_2,s_2,k,j,q+1}(\mathbf{S}) & \text{if } 1 \leqq s \leqq M-2 \\ E^{4,3,1}_{s,k,j,q+1}(\mathbf{S}) = 0 & \text{if } s > M-2. \end{cases}$$

For $I_{q,4,3,2}$, let $E^4_{2s_2,s_2,k,j,q+1} = 0$ if $q+1 \leqq k < 2s_2 \vee (q+1)$. Then there exists $E^{4,3,2}_{0,k,j,q+1}(\mathbf{S}) \in F_{q+1}(\mathbf{S})$ satisfying

$$\begin{aligned} I_{q,4,3,2} &= \sum_{s_2=1}^{M_0} \Big(\sum_{k=q+1}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{2s_2,s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} \Big) \\ &= \sum_{k=q+1}^{M} \sum_{j=0}^{[r_0-kr]} \Big(\sum_{s_2=1}^{M_0} E^4_{2s_2,s_2,k,j,q+1}(\mathbf{S}) \Big) t^{-rk-j} \\ &= \sum_{k=q+1}^{M} \sum_{j=0}^{[r_0-kr]} E^{4,3,2}_{0,k,j,q+1}(\mathbf{S}) t^{-rk-j}, \end{aligned} \tag{57}$$

where $M_0 = [M/2]$.

Next consider $I_{q,4,3,3}$. By the change of variables of (17) from (s_1, s_2) to (s_2, s_1), it follows

$$I_{q,4,3,3} = \sum_{s=1}^{M-2} \sum_{s_2=1}^{M_s} \Big(\sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{s_2,s+2s_2,k,j,q+1}(\mathbf{S}) t^{-rk-j} \Big) \cos sn\theta.$$

By using the similar method as (54) it follows that there exists $E^{4,3,3}_{s,k,j,q+1}(\mathbf{S})$ satisfying

$$I_{q,4,3,3} = \sum_{s=1}^{M} \sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^{4,3,3}_{s,k,j,q+1}(\mathbf{S}) t^{-rk-j} \cos sn\theta. \tag{58}$$

Define $E^4_{s,k,j,q+1}(\mathbf{S})$ by

$$E^4_{s,k,j,q+1}(\mathbf{S}) = E^{4,2}_{s,k,j,q+1}(\mathbf{S}) + E^{4,3,1}_{s,k,j,q+1}(\mathbf{S}) + E^{4,3,3}_{s,k,j,q+1}(\mathbf{S})$$

$$E^4_{0,k,j,q+1}(\mathbf{S}) = E^{4,3,2}_{0,k,j,q+1}(\mathbf{S}).$$

Then by combining (53)–(58) it holds

$$I_{q,4} = \sum_{s=1}^{M} \sum_{k=s\vee(q+1)}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{s,k,j,q+1}(\mathbf{S}) t^{-kr-j} \cos sn\theta \tag{59}$$

$$+ \sum_{k=q+1}^{M} \sum_{j=0}^{[r_0-kr]} E^4_{0,k,j,q+1}(\mathbf{S}) t^{-kr-j} + O(t^{-r_\varepsilon}).$$

Define $E_{s,k,j,q+1}(\mathbf{S}) = \sum_{i=1}^4 E^i_{s,k,j,q+1}$ for $s = 0,1,\ldots,M$. From (50)-(52) and (59), it follows

$$\Big(h_0(t) + \sum_{s=1}^{M} h_s(t)\cos sn\theta\Big)^{q+1} = \sum_{k=q+1}^{M} \sum_{j=0}^{[r_0-kr]} E_{0,k,j,q+1}(\mathbf{S}) t^{-rk-j}$$

$$+ \sum_{s=1}^{M} \sum_{k=s\vee(q+1)}^{M} \Big(\sum_{j=0}^{[r_0-kr]} E_{s,k,j,q+1}(\mathbf{S}) t^{-rk-j}\Big) \cos sn\theta + O(t^{-r_\varepsilon}).$$

This is nothing but the case $q+1$. Therefore the proof is complete. □

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