

## ON A CLASS OF NONLOCAL ELLIPTIC PROBLEMS WITH CRITICAL GROWTH

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*Abstract.* This paper is concerned with the existence of positive solutions to the class of nonlocal boundary value problems of the Kirchhoff type

$$-\left[M\left(\int_{\Omega} |\nabla u|^2 dx\right)\right] \Delta u = \lambda f(x, u) + u^5 \text{ in } \Omega, u(x) > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

where  $\Omega \subset \mathbb{R}^N$ , for  $N=1,2$  and  $3$ , is a bounded smooth domain,  $M$  and  $f$  are continuous functions and  $\lambda$  is a positive parameter. Our approach is based on the variational method.

### 1. Introduction

The purpose of this article is to investigate the existence of positive solutions to the class of nonlocal boundary value problems of the Kirchhoff type

$$(P_{\lambda}) \quad \begin{cases} -\left[M\left(\int_{\Omega} |\nabla u|^2 dx\right)\right] \Delta u = \lambda f(x, u) + u^5 \text{ in } \Omega, \\ u(x) > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where, through this work,  $\Omega \subset \mathbb{R}^N$ , for  $N=1,2$  and  $3$ , is a bounded smooth domain,  $\lambda$  is a positive parameter and  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions that satisfy some conditions which will be stated later on.

Problem  $(P_{\lambda})$  is called nonlocal because of the presence of the term  $M(\int_{\Omega} |\nabla u|^2 dx)$  which implies that the equation in  $(P_{\lambda})$  is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Besides of this, we have its physical motivation. Indeed, the operator  $M(\int_{\Omega} |\nabla u|^2 dx) \Delta u$  appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$(1.1) \quad \begin{cases} u_{tt} - M(\int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u) \text{ in } \Omega \times (0, T) \\ u = 0 \text{ on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \end{cases}$$

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Such a hyperbolic equation is a general version of the Kirchhoff equation

$$(1.2) \quad \rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff [9]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (1.2) have the following meanings:  $L$  is the length of the string,  $h$  is the area of cross-section,  $E$  is the Young modulus of the material,  $\rho$  is the mass density and  $P_0$  is the initial tension.

Problem (1.1) began to call attention of several researchers mainly after the work of Lions [10], where a functional analysis approach was proposed to attack it.

We have to point out that nonlocal problems also appear in other fields as, for example, biological systems where  $u$  describes a process which depends on the average of itself (for example, population density). See, for example, [3] and its references.

The reader may consult [2], [3], [7] [8], [11] and [13] and the references therein, for more informations on  $(P_\lambda)$ .

We will always work in the three-dimensional space  $\mathbb{R}^3$ , because in the another dimensions  $N=1$  and 2 everything follows by making standard modifications. Note that, in the case that we will consider we have  $2^* = 6$ , which is the well known critical Sobolev exponent.

The hypotheses on the function  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are the following.

There exists  $m_0 > 0$  such that

$$(M_1) \quad M(t) \geq m_0 \text{ for all } t \in \mathbb{R}^+.$$

There exists  $\theta > 0$  such that the function  $\frac{1}{2}\widehat{M}(t^2) - \frac{1}{\theta}M(t^2)t^2$  is nonnegative in  $[0, +\infty)$ , that is,

$$(M_2) \quad \frac{1}{2}\widehat{M}(t^2) - \frac{1}{\theta}M(t^2)t^2 \geq 0 \text{ for all } t \geq 0$$

and

$$(M_3) \quad \frac{1}{t} \left[ \frac{1}{2}\widehat{M}(t^2) - \frac{1}{\theta}M(t^2)t^2 \right] \rightarrow +\infty \text{ as } t \rightarrow +\infty,$$

with  $2(1 - H(b)) + 4H(b) < \theta < 6$ , where  $H(t) = 1$  if  $t > 0$  and  $H(t) = 0$  if  $t \leq 0$  and  $\widehat{M}(t) = \int_0^t M(s)ds$ .

There exists  $b \geq 0$  such that

$$(M_4) \quad \frac{M(t)}{t} \rightarrow b \text{ as } t \rightarrow +\infty.$$

Note that in the hypothesis  $(M_3)$ , when  $b = 0$  we derive

$$2(1 - H(b)) + 4H(b) = 2$$

and in the case  $b > 0$ , we have

$$2(1 - H(b)) + 4H(b) = 4.$$

Moreover, from  $(M_4)$ , for all  $t \geq 0$ , there exists  $K > 0$  such that

$$(1.3) \quad M(t) \leq K(m_0 + bt).$$

A typical example of a function satisfying the conditions  $(M_1) - (M_4)$  is given by

$$M(t) = m_0 + bt$$

with  $b \geq 0$  and for all  $t \geq 0$ , which is the one considered in the Kirchhoff equation (1.2).

The hypotheses on function  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are the following.

$$(f_1) \quad \lim_{t \rightarrow 0} \frac{f(x, t)}{t} = 0 \text{ for all } x \in \Omega,$$

uniformly on  $x \in \Omega$ .

There exists  $q \in (2(1 - H(b)) + 4H(b), 6)$  verifying

$$(f_2) \quad \lim_{t \rightarrow +\infty} \frac{f(x, t)}{t^{q-1}} = 0 \text{ for all } x \in \Omega$$

and the well known Ambrosetti-Rabinowitz superlinear condition, that is,

$$(f_3) \quad 0 < \theta F(x, t) = \theta \int_0^t f(x, s) ds \leq t f(x, t) \text{ for all } x \in \Omega \text{ and } t > 0,$$

where  $\theta$  is given in  $(M_3)$ .

A typical example of a function satisfying the conditions  $(f_1) - (f_3)$  is given by

$$f(x, t) = \sum_{i=1}^k C_i(x) t_+^{q_i-1}$$

with  $k \in \mathbb{N}$ ,  $2(1 - H(b)) + 4H(b) < q_i < 6$ ,  $C_i \in L^\infty(\Omega)$  and  $t_+ = \max\{t, 0\}$ .

The main result of this paper is:

**THEOREM 1.1.** *Assume that conditions  $(M_1) - (M_4)$ ,  $(f_1) - (f_3)$  hold. Then, there exists  $\lambda^* > 0$ , such that problem  $(P_\lambda)$  has a weak positive solution in  $H_0^1(\Omega)$ , for all  $\lambda \geq \lambda^*$ .*

We have to point out that these results are new, at least to our knowledge, because we are treating with the Kirchhoff equation where the nonlinearity has a critical growth and we believe that this is the first paper that the condition  $(M_3)$  appears. We

would like to detach that in [7] the authors have considered a class of nonlocal problems with supercritical growth by using variational methods combined with Moser's iteration method for  $\lambda$  small enough. In that article, the smallness of  $\lambda$  helps to overcome the difficulty provoked by supercritical growth. However, in the present article the parameter  $\lambda$  is multiplying the term with subcritical growth. Hence we are working with a different class of problem and the methods used in [7] can not repeat for this class of problem. Beside of this, we have also considered classes of functions  $M$ , which are not treated in [7], but still including the one considered by Kirchhoff [9]. See, for example, conditions  $(M_2)$  and  $(M_3)$ .

## 2. Preliminary results

Since we intend to find positive solutions, in all this paper let us assume that

$$f(x, t) = 0, \quad \forall x \in \Omega \text{ and } t \geq 0.$$

We recall that  $u \in H_0^1(\Omega)$  is a weak solution of the problem  $(P_\lambda)$  if it verifies

$$M(\|u\|^2) \int_{\Omega} \nabla u \nabla \phi \, dx - \lambda \int_{\Omega} f(x, u) \phi \, dx - \int_{\Omega} u_+^5 \phi \, dx = 0$$

for all  $\phi \in H_0^1(\Omega)$ , where  $\|u\|^2 = \int_{\Omega} |\nabla u|^2 \, dx$ .

We will look for solutions of  $(P_\lambda)$  by finding critical points of the  $C^1$ -functional  $I: H_0^1(\Omega) \rightarrow \mathbb{R}$  given by

$$I(u) = \frac{1}{2} \widehat{M}(\|u\|^2) - \lambda \int_{\Omega} F(x, u) \, dx - \frac{1}{6} \int_{\Omega} u_+^6 \, dx.$$

Note that

$$I'(u)\phi = M(\|u\|^2) \int_{\Omega} \nabla u \nabla \phi \, dx - \lambda \int_{\Omega} f(x, u) \phi \, dx - \int_{\Omega} u_+^5 \phi \, dx,$$

for all  $\phi \in H_0^1(\Omega)$ , hence critical points of  $I$  are weak solutions for  $(P_\lambda)$ . Moreover, if the critical point is nontrivial, by maximum principle, we conclude that is a positive solution for  $(P_\lambda)$ .

In order to use Variational Methods, we first derive some results related to the Palais-Smale compactness condition.

We say that a sequence  $(u_n)$  is a Palais-Smale sequence for the functional  $I$  at the level  $d \in \mathbb{R}$  if

$$I(u_n) \rightarrow d \text{ and } \|I'(u_n)\| \rightarrow 0 \text{ in } (H_0^1(\Omega))'.$$

If every Palais-Smale sequence of  $I$  has a strong convergent subsequence, then one says that  $I$  satisfies the Palais-Smale condition ((PS) for short).

In the sequel, we prove that the functional  $I$  has the Mountain Pass Geometry. This fact is proved in the next lemmas:

LEMMA 2.1. Assume that conditions  $(M_1)$ ,  $(f_1)$  and  $(f_2)$  hold. There exist positive numbers  $\rho$  and  $\alpha$  such that,

$$I(u) \geq \alpha > 0, \forall u \in H_0^1(\Omega) : \|u\| = \rho.$$

*Proof.* It follows from  $(M_1)$ ,  $(f_1)$  and  $(f_2)$  that

$$I(u) \geq m_0 \frac{1}{2} \|u\|^2 - C_\varepsilon \lambda \int_\Omega |u|^q dx - \frac{1}{6} \int_\Omega u_+^6 dx.$$

So, using Sobolev Embedding Theorem, there is a positive constant  $C > 0$  such that

$$I(u) \geq \frac{m_0}{2} \|u\|^2 - \lambda C \|u\|^q - C \|u\|^6.$$

Since  $2(1 - H(b)) + 4H(b) < q < 6$ , the result follows by choosing  $\rho > 0$  small enough.  $\square$

LEMMA 2.2. Assume that conditions  $(M_3)$  and  $(f_3)$  hold. For all  $\lambda > 0$ , there exists  $e \in H_0^1(\Omega)$  with  $I(e) < 0$  and  $\|e\| > \rho$ .

*Proof.* Fix  $v_0 \in C_0^\infty(\Omega) \setminus \{0\}$  with  $v_0 \geq 0$  in  $\Omega$  and  $\|v_0\| = 1$ . Combining  $(M_4)$  and  $(f_3)$ , we get

$$I(tv_0) \leq \frac{K}{2} [m_0 t^2 + \frac{b}{2} t^4] - t^\theta \lambda \int_\Omega v_0^\theta dx + |\text{supp } v_0| - \frac{t^6}{6} \int_\Omega v_0^6 dx.$$

Since  $2(1 - H(b)) + 4H(b) < \theta < 6$ , the result follows by considering  $e = t_* v_0$  for some  $t_* > 0$  large enough.  $\square$

Using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [4], without  $(PS)$  condition (see [14, p.12]), there exists a sequence  $(u_n) \subset H_0^1(\Omega)$  satisfying

$$I(u_n) \rightarrow c_* \text{ and } I'(u_n) \rightarrow 0,$$

where

$$c_* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0$$

and

$$\Gamma := \{ \gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0, I(\gamma(1)) < 0 \}.$$

In the following, we devote by  $S$  the best constant of the Sobolev embedding  $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$  given by

$$S := \inf \left\{ \int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega |u|^6 dx = 1 \right\}.$$

As in [6] and arguing as in [1], we are able to compare the minimax level  $c_*$  with a suitable number which involves the constant  $S$ .

LEMMA 2.3. *If the conditions  $(M_1)$ - $(M_4)$  and  $(f_1)$ - $(f_3)$  hold, then there exist  $\lambda_* > 0$  such that  $c_*$  belongs to the interval  $(0, (\frac{1}{\theta} - \frac{1}{6})(m_0S)^{3/2})$ , for all  $\lambda \geq \lambda_*$ .*

*Proof.* If  $v_0$  is the function given by Lemma 2.2, it follows that there exists  $t_\lambda > 0$  verifying  $I(t_\lambda v_0) = \max_{t \geq 0} I(tv_0)$ . Hence

$$t_\lambda^2 M(t_\lambda^2 \|v_0\|^2) \|v_0\|^2 = \lambda \int_\Omega f(x, t_\lambda v_0) t_\lambda v_0 \, dx + t_\lambda^6 \int_\Omega v_0^6 \, dx. \tag{2.1}$$

From (1.3),

$$K m_0 \|v_0\|^2 + t_\lambda^2 K \frac{b}{2} \|v_0\|^4 \geq t_\lambda^4 \int_\Omega v_0^6 \, dx,$$

which implies that  $(t_\lambda)$  is bounded. Thus, there exists a sequence  $\lambda_n \rightarrow +\infty$  and  $t_0 \geq 0$  such that  $t_{\lambda_n} \rightarrow t_0$  as  $n \rightarrow +\infty$ . Consequently, there is  $D > 0$  such that

$$t_{\lambda_n}^2 M(t_{\lambda_n}^2 \|v_0\|^2) \|v_0\|^2 \leq D, \quad \forall n \in \mathbb{N},$$

and so

$$\lambda_n \int_\Omega f(x, t_{\lambda_n} v_0) t_{\lambda_n} v_0 \, dx + t_{\lambda_n}^6 \int_\Omega v_0^6 \leq D, \quad \forall n \in \mathbb{N}.$$

If  $t_0 > 0$ , the last inequality leads to

$$\lim_{n \rightarrow \infty} \lambda_n \int_\Omega f(x, t_{\lambda_n} v_0) t_{\lambda_n} v_0 \, dx + t_{\lambda_n}^6 \int_\Omega v_0^6 = +\infty,$$

which is an absurd. Thus, we conclude that  $t_0 = 0$ . Now, let us consider the path  $\gamma_*(t) = te$  for  $t \in [0, 1]$ , which belongs to  $\Gamma_*$  to get the following estimate

$$0 < c_* \leq \max_{t \in [0, 1]} I(\gamma_*(t)) = I(t_\lambda v_0) \leq \frac{1}{2} \widehat{M}(t_\lambda^2 \|v_0\|^2).$$

In this way, if  $\lambda$  is large enough, we derive  $\frac{1}{2} \widehat{M}(t_\lambda^2 \|v_0\|^2) < (\frac{1}{\theta} - \frac{1}{6})(m_0S)^{3/2}$ , which leads to

$$0 < c_* < (\frac{1}{\theta} - \frac{1}{6})(m_0S)^{3/2}.$$

□

### 2.1. Proof of Theorem 1.1

*Proof.* From Lemmas 2.1, 2.2 and 2.3, there exists a sequence  $(u_n) \subset H_0^1(\Omega)$  verifying

$$I(u_n) \rightarrow c_* \text{ and } I'(u_n) \rightarrow 0$$

with  $c_* \in (0, (\frac{1}{\theta} - \frac{1}{6})(m_0S)^{3/2})$  for all  $\lambda \geq \lambda_*$ .

From  $(f_3)$ , there is  $C > 0$  such that

$$C + \|u_n\| \geq I(u_n) - \frac{1}{\theta} I'(u_n) u_n \geq \frac{1}{2} \widehat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2) \|u_n\|^2, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Assuming, by contradiction, that  $(u_n)$  is not bounded in  $H_0^1(\Omega)$ , up to a subsequence, it follows from  $(M_3)$

$$\frac{C}{\|u_n\|} + 1 \geq \frac{1}{\|u_n\|} \left[ \frac{1}{2} \widehat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2) \|u_n\|^2 \right] \rightarrow +\infty,$$

which is an absurd, because the sequence on the left side of the inequality is bounded. Hence  $(u_n)$  is bounded in  $H_0^1(\Omega)$  and, up to a subsequence, there is  $t_0 \geq 0$  such that

$$\|u_n\| \rightarrow t_0. \tag{2.3}$$

Since  $M$  is a continuous function, we reach

$$M(\|u_n\|^2) \rightarrow M(t_0^2). \tag{2.4}$$

We claim that  $\|u_n\|^2 \rightarrow \|u\|^2$  as  $n \rightarrow \infty$ , which will imply that  $u_n \rightarrow u$  in  $H_0^1(\Omega)$ . Indeed, by using the fact that  $I$  is  $C^1$ , we obtain

$$I(u_n) \rightarrow I(u) \text{ and } I'(u_n) \rightarrow I'(u)$$

and so

$$I(u) = c_* \text{ and } I'(u) = 0.$$

Thus, the result follows.

In order to prove the claim, taking a subsequence, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \mu \text{ and } |u_n|^6 \rightharpoonup |u|^6 + \nu \text{ (weak }^* \text{-sense of measures).}$$

Using the concentration compactness-principle due to Lions (cf. [12, Lemma 1.2]), we obtain an at most countable index set  $\Lambda$ , sequences  $(x_i) \subset \mathbb{R}^3$ ,  $(\mu_i), (\nu_i) \subset [0, \infty)$ , such that

$$\nu = \sum_{i \in \Lambda} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i \in \Lambda} \mu_i \delta_{x_i} \text{ and } S \nu_i^{1/3} \leq \mu_i, \tag{2.5}$$

for all  $i \in \Lambda$ , where  $\delta_{x_i}$  is the Dirac mass at  $x_i \in \Omega$ .

Now, for every  $\rho > 0$ , we set  $\psi_\rho(x) := \psi((x - x_i)/\rho)$  where  $\psi \in C_0^\infty(\Omega, [0, 1])$  is such that  $\psi \equiv 1$  on  $B_1(0)$ ,  $\psi \equiv 0$  on  $\Omega \setminus B_2(0)$  and  $|\nabla \psi|_\infty \leq 2$ . Since  $(\psi_\rho u_n)$  is bounded,  $I'(\psi_\rho u_n) \rightarrow 0$ , that is,

$$\begin{aligned} M(\|u_n\|^2) \int_\Omega \nabla u_n \cdot \nabla \psi_\rho \, dx &= -M(\|u_n\|^2) \int_\Omega \psi_\rho |\nabla v_n|^p \, dx \\ &\quad + \lambda \int_\Omega f(x, u_n) \psi_\rho u_n \, dx + \int_\Omega \psi_\rho |u_n|^6 \, dx + o_n(1). \end{aligned}$$

Recalling that  $M(\|u_n\|^2)$  converges to  $M(t_0^2)$ , we can argue as in [5] to show that

$$\lim_{\varepsilon \rightarrow 0} [\lim_{n \rightarrow \infty} M(\|u_n\|^2) \int_\Omega \nabla u_n \cdot \nabla \psi_\rho \, dx] = 0.$$

Moreover, since  $u_n \rightarrow u$  in  $L^s(\Omega)$  for all  $1 \leq s < 6$  and  $\psi_\rho$  has compact support, we can let  $n \rightarrow \infty$  in the above expression to obtain

$$\int_{\Omega} \psi_\rho \, d\nu \geq \int_{\Omega} M(t_0^2) \psi_\rho \, d\mu.$$

Letting  $\rho \rightarrow 0$  we conclude that  $v_i \geq M(t_0^2) \mu_i \geq m_0 \mu_i$ . It follows from (2.5) that

$$v_i \geq (m_0 S)^{3/2} \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) (m_0 S)^{3/2}. \tag{2.6}$$

Now we shall prove that the above expression cannot occur, and therefore the set  $\Lambda$  is empty. Indeed, arguing by contradiction, let us suppose that  $v_i \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) (m_0 S)^{3/2}$  for some  $i \in \Lambda$ . Once that  $(u_n)$  is a  $(PS)_{c_*}$ , we have

$$c_* = I(u_n) - \frac{1}{\theta} I'(u_n) u_n + o_n(1).$$

From  $(M_2)$ , we derive that  $\frac{1}{2} [\widehat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2) \|u_n\|^2] \geq 0$  for all  $n \in \mathbb{N}$ , hence

$$c_* \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) \int_{\Omega} |u_n|^6 \, dx + o_n(1) \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) \int_{\Omega} \psi_\rho |u_n|^6 \, dx + o_n(1).$$

Letting  $n \rightarrow \infty$ , we get

$$c_* \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) \sum_{i \in \Lambda} \psi_\rho(x_i) v_i = \left(\frac{1}{\theta} - \frac{1}{6}\right) \sum_{i \in \Lambda} v_i \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) (m_0 S)^{3/2},$$

which does not make sense. Thus  $\Lambda$  is empty and it follows that  $u_n \rightarrow u$  in  $L^6(\Omega)$ . Since  $u_n \rightarrow u$  in  $L^q(\Omega)$ , we conclude

$$\lim_{n \rightarrow \infty} M(\|u_n\|^2) \|u_n\|^2 = \lambda \int_{\Omega} f(x, u) u \, dx + \int_{\Omega} u_+^6 \, dx.$$

On the other hand, by using well known arguments, we reach

$$M(t_0^2) \int_{\Omega} \nabla u \nabla \phi \, dx = \lambda \int_{\Omega} f(x, u) \phi \, dx + \int_{\Omega} u^5 \phi \, dx, \quad \forall \phi \in H_0^1(\Omega)$$

and so

$$M(t_0^2) \|u\|^2 = \lambda \int_{\Omega} f(x, u) u \, dx + \int_{\Omega} u^6 \, dx$$

from where it follows that

$$M(\|u_n\|^2) \|u_n\|^2 \rightarrow M(t_0^2) \|u\|^2.$$

From (2.4), the proof of claim is complete.  $\square$

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