Differential
Equations
& Applications
Volume 2, Number 4 (2010), 447–464

ASYMPTOTIC PROPERTY OF SOLUTIONS ON NONAUTONOMOUS LOTKA–VOLTERRA MODEL FOR N–COMPETING SPECIES

KUNIHIKO TANIGUCHI

(*Communicated by M. Gyllenberg and J. Sugie*)

Abstract. We consider nonautonomous N-dimensional generalized Lotka-Volterra competition systems. Under certain conditions we show that the set of values of solutions starting from any compact set tends to a set of measure zero. Our results give generalizations of previous ones.

1. Introduction and Statements of the main results

In this paper we consider the system of differential equations:

$$u_i' = u_i\left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i,u_j)\right], i = 1,\dots,N, N \geqslant 2, \tag{GLV}$$

where the functions $a_i(t)$, $1 \leqslant i \leqslant N$, and $b_{ij}(t)$, $1 \leqslant i,j \leqslant N$, are assumed to be continuous and nonnegative on $\mathbb{R}$. Furthermore, let the functions $f_{ij}(x,y)$, $1 \leqslant i,j \leqslant N$, be continuously differentiable on $\mathbb{R}_+^2 = (0,\infty)^2$, and we impose the following conditions on f_{ij}':

$$\begin{cases} f_{ii}(x,y) \text{ is continuously differentiable on } [0,\infty)\times[0,\infty),\ 1 \leqslant i \leqslant N;\\ f_{ij}(x,y) > 0,\ (x,y) \in \mathbb{R}_+^2, 1 \leqslant i,j \leqslant N;\\ \dfrac{d}{dx}(f_{ii}(x,x)) = (D_1 f_{ii} + D_2 f_{ii})(x,x) > 0,\ x \in \mathbb{R}_+, 1 \leqslant i \leqslant N;\\ D_1 f_{ij}(x,y) \geqslant 0,\ (x,y) \in \mathbb{R}_+^2, 1 \leqslant i \leqslant N;\\ D_2 f_{ij}(x,y) \geqslant 0,\ (x,y) \in \mathbb{R}_+^2, 1 \leqslant j \leqslant N;\\ f_{ii}(0,0) = 0,\ 1 \leqslant i \leqslant N;\\ \lim\limits_{x\to\infty} f_{ii}(x,x) = \infty,\ \ 1 \leqslant i \leqslant N, \end{cases} \tag{1.1}$$

where D_i, $i = 1,2$, denotes the differentiation with respect to the i-th variable.

Mathematics subject classification (2010): 34D05, 92D25.
Keywords and phrases: Lotka-Volterra systems, Nonautonomous system, competition system, permanence.

Paper DEA-02-27

System (GLV) is a generalization of the following nonautonomous N-dimensional Lotka-Volterra competition system which S. Ahmad and A. C. Lazer [1] considered:

$$u_i' = u_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) u_j \right], i = 1, \ldots, N, N \geqslant 2. \tag{LV}$$

An prototype of system (LV), as well as (GLV), is the classical Lotka-Volterra competition model for two species:

$$\begin{cases} u_1' = u_1(a_1 - b_{11}u_1 - b_{12}u_2), \\ u_2' = u_2(a_2 - b_{21}u_1 - b_{22}u_2), \end{cases} \tag{1.2}$$

where a_i, $i = 1,2$, and b_{ij}, $i,j = 1,2$, are positive constants. When the growth rates a_i, $i = 1,2$, and the interaction coefficients b_{ij}, $i,j = 1,2$, satisfy

$$a_1 - b_{12}\left(\frac{a_2}{b_{22}}\right) > 0, \tag{1.3}$$

$$a_2 - b_{21}\left(\frac{a_1}{b_{11}}\right) > 0, \tag{1.4}$$

there exists a unique equilibrium point $(u_1^*, u_2^*) \in \mathbb{R}_+^2$ that attracts any solution curve $(u_1(t), u_2(t))$ of system (1.2) with $(u_1(t_0), u_2(t_0)) \in \mathbb{R}_+^2$, i.e.

$$u_1(t) \to u_1^* \text{ and } u_2(t) \to u_2^* \quad \text{as } t \to \infty.$$

In [1]-[6] it is shown that analogous results still hold for the nonautonomous equation (LV), as seen below. In this paper we intend to generalize such results further.

We introduce notation. Put $c_M := \sup_{t \in \mathbb{R}} c(t)$ for bounded functions $c(t)$ on $\mathbb{R}$. For $i = 1, \ldots, N$, we put

$$\tilde{f}_{ii}(x) = f_{ii}(x,x), \; x \in \mathbb{R}_+.$$

By assumption (1.1) $\tilde{f}_{ii}$, $i = 1, \ldots, N$, have the inverse function $\tilde{f}_{ii}^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. The assumptions employed in the paper will be sellected from the following list:

(A1) $b_{ii}(t) > 0, \; t \in \mathbb{R}, 1 \leqslant i \leqslant N$;

(A2) $\displaystyle\int_0^\infty b_{ii}(s)ds = \infty, \; 1 \leqslant i \leqslant N$;

(A3) $\displaystyle\left(\frac{a_i}{b_{ii}}\right)_M < \infty, \; 1 \leqslant i \leqslant N$;

(A4) $\displaystyle\inf_{t \in \mathbb{R}} \frac{a_i(t) - \sum_{j \neq i} b_{ij}(t)(a_j/b_{jj})_M}{b_{ii}(t)} > 0, \; 1 \leqslant i \leqslant N$;

(A5) $\displaystyle\inf_{t \in \mathbb{R}} \frac{a_i(t) - \sum_{j \neq i} b_{ij}(t) f_{ij}(\tilde{f}_{ii}^{-1}((a_i/b_{ii})_M), \tilde{f}_{jj}^{-1}((a_j/b_{jj})_M))}{b_{ii}(t)} > 0, \; 1 \leqslant i \leqslant N$;

(A6) $f_{ij}(x,y) \leqslant \tilde{f}_{jj}(y),\ (x,y) \in \mathbb{R}^2_+, 1 \leqslant i,j \leqslant N, i \neq j$;
(A7) for any $s > 1$ sufficiently close to 1,

$$f_{ij}(\tilde{f}_{ii}^{-1}(sx), \tilde{f}_{jj}^{-1}(sy)) \leqslant s f_{ij}(\tilde{f}_{ii}^{-1}(x), \tilde{f}_{jj}^{-1}(y)),\ (x,y) \in \mathbb{R}^2_+, 1 \leqslant i,j \leqslant N.$$

REMARK 1.1. As in the case of (LV) and (1.2), if $f_{ij}(x,y)$, $1 \leqslant i,j \leqslant N$, are independent of x, (A6) is satisfied. In fact, for (LV) we can take $f_{ij}(x,y) = y$, $1 \leqslant i,j \leqslant N$, which satisfy (A6) and (A7).

REMARK 1.2. Let

$$f_{ij}(x,y) = \begin{cases} \dfrac{x^{\alpha_{ij}}}{1+x^{\alpha_{ij}}} y^{\beta_{ij}},\ i \neq j, \\ x^{\alpha_{ij}} y^{\beta_{ij}},\ i = j, \end{cases}$$

where α_{ij}, $\beta_{ij} \in \mathbb{R}_+$. If for $i \neq j$, $\beta_{ij} = \alpha_{jj} + \beta_{jj}$, then the functions f_{ij}, $1 \leqslant i,j \leqslant N$, satisfy (A6).

REMARK 1.3. Let

$$f_{ij}(x,y) = x^{\alpha_{ij}} y^{\beta_{ij}},\ (x,y) \in \mathbb{R}^2, 1 \leqslant i,j \leqslant N,$$

where α_{ij}, $\beta_{ij} \in \mathbb{R}_+$. If $\alpha_{ij} + \beta_{ij} \leqslant \min\{\alpha_{ii} + \beta_{ii}, \alpha_{jj} + \beta_{jj}\}$, then the functions f_{ij}, $1 \leqslant i,j \leqslant N$, satisfy (A7).

S. Ahmad and A. C. Lazer [1] supposed that the functions $a_i(t)$, $1 \leqslant i \leqslant N$ and $b_{ij}(t)$, $1 \leqslant i,j \leqslant N$, satisfy conditions (A1)-(A3) and (A4). Under these conditions they have shown the following [1]:

(I) If $u = (u_i, \ldots, u_N)$ is *a* solution of (LV) with $u_i(t_0) > 0$, $1 \leqslant i \leqslant N$, $t_0 \in \mathbb{R}$, then

$$0 < \inf_{t \geqslant t_0} u_i(t) \leqslant \sup_{t \geqslant t_0} u_i(t) < \infty \ \text{ for } 1 \leqslant i \leqslant N.$$

(II) If A is *a* compact subset of $\mathbb{R}^N_+$, then the Lebesgue measure of the set $\{u(t) \mid u$ is a solution of (LV) satisfying $u(t_0) \in A\}$ tends to 0 as $t \to \infty$.

Our main aim is to show that (I) and (II) are still valid for (GLV). To state the results we introduce the following. For compact subset A of $\mathbb{R}^N_+$ and $t_0 \in \mathbb{R}$ we set

$$u(t,t_0,A) = \{u(t) \mid u \text{ is a solution of } (GLV) \text{ satisfying } u(t_0) \in A\}.$$

By $\mu(\cdot)$ we denote the Lebesgue measure of measurable sets in $\mathbb{R}^N_+$. We can show the following:

THEOREM 1.4. *Let conditions* (A1)-(A3), (A4), *and* (A6) *hold. Let A be a compact subset of $\mathbb{R}^N_+$ and let $t_0 \in \mathbb{R}$. Then,*

$$\mu(u(t,t_0,A)) \to 0 \quad \text{as } t \to \infty.$$

THEOREM 1.5. *Let conditions* $(A1)$-$(A3)$, $(A5)$, *and* $(A7)$ *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$. *Then,*

$$\mu(u(t,t_0,A)) \to 0 \quad \text{as } t \to \infty.$$

We give examples of systems (GLV) for which above conditions hold.

EXAMPLE 1.6. We consider system (GLV) for two species

$$u_1' = u_1\left[(\cos t + 7) - (\sin t + 7)\cdot u_1^2 - (\sin t + 1)\cdot\left(\frac{u_1^3}{1+u_1^3}\cdot u_2^2\right)\right],$$
$$u_2' = u_2\left[(\cos t + 9) - (\sin t + 2)\cdot\left(\frac{u_2^4}{1+u_2^4}\cdot u_1^3\right) - (\sin t + 9)\cdot u_2^3\right].$$

Obviously (A6) holds. We have

$$a_1(t) - b_{12}(t)\left(\frac{a_2}{b_{22}}\right)_M > \cos t + 7 - (\sin t + 1)\cdot\frac{10}{8} > 2,$$
$$a_2(t) - b_{21}(t)\left(\frac{a_1}{b_{11}}\right)_M > \cos t + 9 - (\sin t + 2)\cdot\frac{8}{6} > 2.$$

So conditions (A1)-(A3) and (A4) hold. Of course condition (1.1) hold.

EXAMPLE 1.7. We consider system (GLV) for two-species

$$u_1' = u_1[(\cos t + 7) - (\sin t + 7)\cdot u_1^4 - (\sin t + 1)\cdot u_1u_2^2],$$
$$u_2' = u_2\left[(\cos t + 9) - (\sin t + 2)\cdot u_2^2u_1^2 - (\sin t + 9)\cdot u_2^6\right].$$

Obviously (A7) holds. We have

$$a_1(t) - b_{12}(t)f_{12}\left(\tilde{f}_{11}^{-1}\left(\left(\frac{a_1}{b_{11}}\right)_M\right), \tilde{f}_{22}^{-1}\left(\left(\frac{a_2}{b_{22}}\right)_M\right)\right)$$
$$> \cos t + 7 - (\sin t + 1)\cdot\left(\frac{8}{6}\right)^{1/4}\cdot\left(\frac{10}{8}\right)^{2/6} > 2,$$

$$a_2(t) - b_{21}(t)f_{21}\left(\tilde{f}_{22}^{-1}\left(\left(\frac{a_2}{b_{22}}\right)_M\right), \tilde{f}_{11}^{-1}\left(\left(\frac{a_1}{b_{11}}\right)_M\right)\right)$$
$$> \cos t + 9 - (\sin t + 2)\cdot\left(\frac{10}{8}\right)^{2/6}\cdot\left(\frac{4}{3}\right)^{2/4} > 2.$$

So conditions (A1)-(A3), (A5) hold. Of course condition (1.1) hold.

The rest of this paper is organized as follows. In Section 2 we give an important proposition which are employed in proving Theorems 1.4 and 1.5. The proof of Theorems 1.4 and 1.5 are given in Sections 3 and 4, separately. Related results are found, for example, in [2]-[5].

2. Squeezing theorem

In this section we consider system (GLV) on which we impose the following conditions:

$$b_{ij}(t) \geqslant 0,\ t \in \mathbb{R}, 1 \leqslant i,j \leqslant N, \tag{2.1}$$

$$b_{ii}(t) > 0,\ t \in \mathbb{R}, 1 \leqslant i \leqslant N, \tag{2.2}$$

$$\left(\frac{a_i}{b_{ii}}\right)_M < \infty,\ 1 \leqslant i \leqslant N, \tag{2.3}$$

$$\int_0^\infty \sum_{i=1}^N b_{ii}(s)ds = \infty. \tag{2.4}$$

We note that these conditions are weaker than conditions (A1)-(A3) for system (GLV). Then we can generalize the results due to S. Ahmad and A. C. Lazer [1, Theorem 2.1] as seen below:

PROPOSITION 2.1. *Let conditions* (2.1)-(2.4) *hold. Let A be a bounded measurable subset of $\mathbb{R}^N_+$ and let $t_0 \in \mathbb{R}$. Let $u = (u_1,\dots,u_N)$ be a solution of (GLV) with $u(t_0) \in A$. If there exists a number $\delta_A > 0$ such that*

$$u_i(t) \geqslant \delta_A,\ t \geqslant t_0,\ u(t_0) \in A,\ 1 \leqslant i \leqslant N, \tag{2.5}$$

then

$$\mu(u(t,t_0,A)) \to 0 \ \text{ as } t \to \infty.$$

REMARK 2.2. By Proposition 2.1, in order to prove Theorems 1.4 and 1.5, it is sufficient to prove the existence of δ_A satisfying (2.5). In Sections 3 and 4 we shall prove the existence of such a δ_A (Lemmas 3.1 and 4.1).

2.1. Preliminary lemmas for Proposition 2.1

In this subsection we prove lemmas needed later. As a first step, we show that every solutions u of (GLV) with $u(t_0) \in \mathbb{R}^N_+$ remains here as long as it exists. To see this we rewrite system (GLV) in the form

$$u_i'(t) = p_i(t)u_i(t),\ i = 1,2,\dots,N,$$

where the functions $p_i(t)$, $1 \leqslant i \leqslant N$, are given by

$$p_i(t) = a_i(t) - \sum_{j=1}^N b_{ij}(t) f_{ij}(u_i(t),u_j(t)).$$

Since p_i, $1 \leqslant i \leqslant N$, is continuous on the domain of u, for t in the domain of u we obtain

$$u_i(t) = u_i(t_0)\exp\int_{t_0}^t p_i(s)ds > 0.$$

Hence $u(t) \in \mathbb{R}^N_+$.

LEMMA 2.3. *Let conditions* (2.1)-(2.4) *hold. Let* $t_0 \in \mathbb{R}$ *and* u *a local solution of* (GLV) *with* $u(t_0) \in \mathbb{R}^N_+$. *Then the following statements hold:*

(i) *Let* $r_i > 0$ *be a number such that*

$$\left(\frac{a_i}{b_{ii}}\right)_M < r_i.$$

Then

$$\tilde{f}_{ii}(u_i(t)) \leqslant \max\{\tilde{f}_{ii}(u_i(t_0)), r_i\}. \tag{2.6}$$

(ii) *Let* $A \subset \mathbb{R}^N_+$ *be a bounded subset. Then there exists a number* $M_A > 0$ *such that*

$$u_i(t) \leqslant M_A,\ 1 \leqslant i \leqslant N, \tag{2.7}$$

for any solution of (GLV) *with* $u(t_0) \in A$.

Proof. (i) *Step* 1. First, we prove the following claim.
Claim. *If there exists some* $T \geqslant t_0$ *and some number* $i \in \{1,\dots,N\}$ *such that*

$$\tilde{f}_{ii}(u_i(T)) \geqslant r_i > \left(\frac{a_i}{b_{ii}}\right)_M > 0,$$

then $u_i'(T) \leqslant 0$.

In fact, from the assumption of the above claim, (2.2) and (2.3), we have

$$\begin{aligned} u_i'(T) &= u_i(T)\left[a_i(T) - \sum_{j\neq i} b_{ij}(T) f_{ij}(u_i(T), u_j(T)) - b_{ii}(T)\tilde{f}_{ii}(u_i(T))\right] \\ &\leqslant u_i(T)[a_i(T) - b_{ii}(T)\tilde{f}_{ii}(u_i(T))] \\ &\leqslant u_i(T)[a_i(T) - r_i b_{ii}(T)] = u_i(T)b_{ii}(T)\left[\frac{a_i(T)}{b_{ii}(T)} - r_i\right] \leqslant 0. \end{aligned}$$

*Step*2. We will prove Lemma 2.3. The proof is divided into two cases.

Case 1. Let

$$\tilde{f}_{ii}(u_i(t_0)) \geqslant r_i.$$

It suffices to show that for $t \geqslant t_0$, $\tilde{f}_{ii}(u_i(t)) \leqslant \tilde{f}_{ii}(u_i(t_0))$.

Case 2. Let

$$\tilde{f}_{ii}(u_i(t_0)) < r_i.$$

It suffices to show that for $t \geqslant t_0$, $\tilde{f}_{ii}(u_i(t)) \leqslant r_i$.

In Case 1, we assume to the contrary that there exists some $\tilde{t} > t_0$ such that $\beta := \tilde{f}_{ii}(u_i(\tilde{t})) > \tilde{f}_{ii}(u_i(t_0)) =: \gamma$. We take t_1 satisfying:

$$t_1 := \inf\{t \in \mathbb{R} \mid t > t_0, \tilde{f}_{ii}(u_i(t)) = \beta\}.$$

Next we take t_2 satisfying:

$$t_2 := \sup\{t \in \mathbb{R} \mid t_0 \leqslant t < t_1, \tilde{f}_{ii}(u_i(t)) = \gamma\}.$$

Then $\tilde{f}_{ii}(u_i(t_1)) > \tilde{f}_{ii}(u_i(t_2))$ and $\tilde{f}_{ii}(u_i(t)) \geqslant r_i$ on $[t_2,t_1]$. From Step 1, we have $u_i' \leqslant 0$ on $[t_2,t_1]$. Consequently, we obtain $u_i(t_1) \leqslant u_i(t_2)$. This gives $\tilde{f}_{ii}(u_i(t_1)) \leqslant \tilde{f}_{ii}(u_i(t_2))$ by condition (1.1), which is a contradiction.

In Case 2, we assume that there exists some number $\check{t} > t_1$ such that

$$\eta := \tilde{f}_{ii}(u_i(\check{t})) > r_i.$$

We take t_3 satifying:

$$t_3 := \inf\{t \in \mathbb{R} \mid t > t_0, \tilde{f}_{ii}(u_i(t)) = \eta\}.$$

Next we take t_4 satisfying:

$$t_4 := \sup\{t \in \mathbb{R} \mid t_0 \leqslant t < t_3, \tilde{f}_{ii}(u_i(t)) = r_i\}.$$

Then $\tilde{f}_{ii}(u_i(t_3)) > \tilde{f}_{ii}(u_i(t_4))$ and $\tilde{f}_{ii}(u_i(t)) \geqslant r_i$ on $[t_4,t_3]$. Similarly to Case 1, we get $u_i(t_3) \leqslant u_i(t_4)$. This gives $\tilde{f}_{ii}(u_i(t_3)) \leqslant \tilde{f}_{ii}(u_i(t_4))$, a contradiction. The proof is then completed.

(ii) From (i), it follows that for all $t \geqslant t_0$

$$\tilde{f}_{ii}^{-1}(\tilde{f}_{ii}(u_i(t))) \leqslant \tilde{f}_{ii}^{-1}(\max\{\tilde{f}_{ii}(u_i(t_0)), r_i\}).$$

That is

$$u_i(t) \leqslant \max\{u_i(t_0), \tilde{f}_{ii}^{-1}(r_i)\}.$$

Therefore if we let M_A be

$$M_A = \max\left\{ \sup_{1\leqslant k\leqslant N} \{x_k \mid x \in A\}, \tilde{f}_{ii}^{-1}(r_i) \right\},$$

where $x = (x_1,\ldots,x_N) \in A$, then for $t \geqslant t_0$, $u_i(t) \leqslant M_A$. The proof is then completed.

REMARK 2.4. By (i) of Lemma 2.3, we see that every solution u of (GLV) with $u(t_0) \in \mathbb{R}_+^N$ exists on $[t_0,\infty)$ under conditions (2.1)-(2.4). So in this case (2.6) and (2.7) hold on $[t_0,\infty)$.

Now we rewrite system (GLV) in the form

$$u' = g(u,t),$$

where $u(t) = (u_1(t),\ldots,u_N(t)) \in \mathbb{R}^N$, and $g(u,t) = (g_1(u,t),\ldots,g_N(u,t))$ is given by

$$g_i(x,t) = x_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(x_i,x_j) \right], \ 1 \leqslant i \leqslant N,$$

for $x=(x_1,\dots,x_N)\in\mathbb{R}^N$. Since the functions a_i, $1\leqslant i\leqslant N$, and b_{ij}, $1\leqslant i,j\leqslant N$, are continuous on $\mathbb{R}$ and the functions f_{ij}, $1\leqslant i,j\leqslant N$, are continuousuly differentiable on $\mathbb{R}^2_+$, for every $\xi=(\xi_i)\in\mathbb{R}^N_+$ and $\tau\in\mathbb{R}$, there exists a unique solution $u(t)$ of (GLV) with $u(\tau)=\xi$. We denote it by $u(t,\tau,\xi)=(u_i(t,\tau,\xi))$. Recall that we have introduced the notation:

$$u(t,t_0,A)=\{u(t,t_0,\xi)\mid \xi\in A\}$$

for $A\subset\mathbb{R}^N_+$. Furthermore, since the functions $g_i(x,t)$, $1\leqslant i\leqslant N$, are continuously differentiable with respect to the components of $x\in\mathbb{R}^N$, $u(t,\tau,\xi)$ are continuously differentiable with respect to the components of $\xi\in\mathbb{R}^N$. Therefore we can introduce the following notations. We denote by $D_\xi(u(t,\tau,\xi))$ the $N\times N$ matrix with (i,j)th entry equal to $\partial u_i(t,\tau,\xi)/\partial\xi_j$:

$$D_\xi u(t,\tau,\xi)=\left[\frac{\partial u_i(t,\tau,\xi)}{\partial\xi_j}\right],$$

where $\xi\in\mathbb{R}^N_+$. Similarly we define $N\times N$ matrix $D_xg(x,t)$ by

$$D_xg(x,t)=\left[\frac{\partial g_i(x,t)}{\partial x_j}\right],$$

where $x\in\mathbb{R}^N_+$.

Now for $t\geqslant t_0$ and $\xi_0\in\mathbb{R}^N_+$, we set $u_0(t)=u(t,t_0,\xi_0)$. Then it is well known [4] that

$$X'(t)=A(t)X(t),\ X(t_0)=I,$$

where

$$X(t)=D_\xi u(t,t_0,\xi_0),\ A(t)=D_xg(u_0(t),t),$$

and I is the $N\times N$ identity matrix. Furthermore we know that

$$\det X(t)=\exp\int_{t_0}^t \mathrm{tr}A(s)ds.$$

Therefore, we have

$$\det D_\xi u(t,t_0,\xi_0)=\exp\int_{t_0}^t\sum_{i=1}^N\frac{\partial g_i}{\partial x_i}(u_0(s),s)ds. \tag{2.8}$$

2.2. Proof of Proposition 2.1

In this section we prove Proposition 2.1.

PROOF OF PROPOSITION 2.1. First, from assumptions of Proposition 2.1 and Lemma 2.3, we note that there exist some numbers M_A and δ_A such that

$$0<\delta_A\leqslant u_i(t)\leqslant M_A,\ t\geqslant t_0, 1\leqslant i\leqslant N. \tag{2.9}$$

For $t \in \mathbb{R},\ i = 1, \ldots, N$, we have

$$\begin{aligned}\frac{\partial g_i}{\partial x_i}(u(t),t) &= a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i(t),u_j(t)) \\ &\quad - u_i(t)\sum_{j\neq i} b_{ij}(t) D_1 f_{ij}(u_i(t),u_j(t)) \\ &\quad - u_i(t) b_{ii}(t)(D_1 f_{ii} + D_2 f_{ii})(u_i(t),u_i(t)) \\ &\leqslant \frac{u_i'(t)}{u_i(t)} - u_i(t) b_{ii}(t)(D_1 f_{ii} + D_2 f_{ii})(u_i(t),u_i(t)).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{t_0}^{t}\sum_{i=1}^{N}\frac{\partial g_i}{\partial x_i}(u(s),s)ds &\leqslant \int_{t_0}^{t}\sum_{i=1}^{N}\Bigg[\frac{u_i'(s)}{u_i(s)} \\ &\qquad - u_i(s) b_{ii}(s)(D_1 f_{ii} + D_2 f_{ii})(u_i(s),u_i(s))\Bigg] ds \\ &= \sum_{i=1}^{N}[\log u_i(s)]_{t_0}^{t} \\ &\qquad - \int_{t_0}^{t}\sum_{i=1}^{N} u_i(s) b_{ii}(s)(D_1 f_{ii} + D_2 f_{ii})(u_i(s),u_i(s))ds.\end{aligned}$$

Here, from condition (1.1), if we set

$$\delta_A' := \min_{1\leqslant i\leqslant N}\{(D_1 f_{ii} + D_2 f_{ii})(\delta_A,\delta_A)\} > 0,$$

then we obtain from (2.9)

$$\begin{aligned}\int_{t_0}^{t}\sum_{i=1}^{N}\frac{\partial g_i}{\partial x_i}(u(s),s)ds &\leqslant \sum_{i=1}^{N}\log\frac{M_A}{\delta_A} - \delta_A\delta_A'\int_{t_0}^{t}\sum_{i=1}^{N} b_{ii}(s)ds \\ &= N\log\frac{M_A}{\delta_A} - \delta_A\delta_A'\int_{t_0}^{t}\sum_{i=1}^{N} b_{ii}(s)ds.\end{aligned}$$

Hence by (2.4),

$$\int_{t_0}^{t}\sum_{i=1}^{N}\frac{\partial g_i}{\partial x_i}(u(s),s)ds \to -\infty \quad \text{as } t\to\infty \tag{2.10}$$

uniformly with respect to $\xi_0 \in A$. Thus it follows from (2.8) that

$$\det D_\xi u(t,t_0,\xi_0) = \exp\int_{t_0}^{t}\sum_{i=1}^{N}\frac{\partial g_i}{\partial x_i}(u(s),s)ds \to 0 \quad \text{as } t\to\infty \tag{2.11}$$

uniformly with respect to $\xi_0 \in A$. Since, from (2.8),

$$\det D_\xi u(t,t_0,\xi_0) > 0,\ t \geqslant t_0,$$

it follows from the change of variables formula [3] that

$$\mu(u(t,t_0,A)) = \int_{u(t,t_0,A)} dx = \int_A \det D_\xi u(t,t_0,\xi_0) d\xi_0.$$

Hence from (2.11)

$$\mu(u(t,t_0,A)) \to 0 \quad \text{as } t \to \infty.$$

This completes the proof.

3. Proof of Theorem 1.4

From Remark 2.2, in order to prove Theorem 1.4, it is sufficient to prove the following lemma.

LEMMA 3.1. *Let conditions* $(A1)$-$(A3)$, $(A4)$ *and* $(A6)$ *hold. Let* A *be a compact subset of* $\mathbb{R}^N_+$ *and let* $t_0 \in \mathbb{R}$. *Then there exist some numbers* $\delta_A > 0$, $r > 0$ *and* $t_A \geqslant t_0$ *such that*

$$\delta_A < u_i(t) < r,\ t \geqslant t_A,\ 1 \leqslant i \leqslant N$$

for any solution u *of* (GLV) *with* $u(t_0) \in A$.

Before we prove Lemma 3.1, we give several lemmas which are employed in proving Lemma 3.1. Firstly from (A4), there exists some number $\varepsilon > 0$ such that

$$a_i(t) - \sum_{j \neq i} b_{ij}(t) \left(\frac{a_j}{b_{jj}} \right)_M \geqslant \varepsilon b_{ii}(t),\ t \geqslant t_0,\ 1 \leqslant i \leqslant N. \tag{3.1}$$

Therefore we have the following proposition.

PROPOSITION 3.2. *There exists some number* $s > 1$ *such that*

$$s\varepsilon - (s-1) \left(\frac{a_i}{b_{ii}} \right)_M > 0,\ 1 \leqslant i \leqslant N. \tag{3.2}$$

Furthermore

$$a_i(t) - \sum_{j \neq i} b_{ij}(t) s \left(\frac{a_j}{b_{jj}} \right)_M \geqslant \alpha b_{ii}(t),\ t \geqslant t_0, 1 \leqslant i \leqslant N, \tag{3.3}$$

where

$$\alpha \equiv \min_{1 \leqslant i \leqslant N} \left\{ s\varepsilon - (s-1) \left(\frac{a_i}{b_{ii}} \right)_M \right\} > 0. \tag{3.4}$$

Proof. Since, for all $t \geqslant t_0$ and $i = 1, \ldots, N$,

$$s\varepsilon - (s-1) \left(\frac{a_i}{b_{ii}} \right)_M \to \varepsilon > 0 \text{ as } s \to 1+0,$$

there exists some number $s > 1$ such that (3.2) holds. Therefore if α is defined by (3.4), then for $t \geqslant t_0$, $i = 1, \dots, N$ we have from (3.1)

$$\begin{aligned} a_i(t) - \sum_{j \neq i} b_{ij}(t) s \left(\frac{a_j}{b_{jj}} \right)_M &= s \left[a_i(t) - \sum_{j \neq i} b_{ij}(t) \left(\frac{a_j}{b_{jj}} \right)_M \right] - (s-1) a_i(t) \\ &\geqslant s\varepsilon b_{ii}(t) - (s-1) a_i(t) \\ &= \left[s\varepsilon - (s-1) \frac{a_i(t)}{b_{ii}(t)} \right] b_{ii}(t) \geqslant \alpha b_{ii}(t). \end{aligned}$$

This completes the proof.

Henceforth let s and α denote the numbers given in Proposition 3.2. Then it leads to the following lemma.

LEMMA 3.3. *Let conditions* $(A1)$ - $(A3)$ *and* $(A4)$ *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$. *Let* $u(t, t_0, \xi_0)$ *be a solution of* (GLV) *with* $\xi_0 \in A$. *Then there exists some number* $\hat{t} = \hat{t}_A \geqslant t^*$ *such that*

$$\tilde{f}_{ii}(u_i(t, t_0, \xi_0)) < s \left(\frac{a_i}{b_{ii}} \right)_M, \; t \geqslant \hat{t}, \xi_0 \in A, 1 \leqslant i \leqslant N. \tag{3.5}$$

Proof. Step 1. First, we prove the following claim.
CLAIM. *For* $\xi_0 \in A$, *there exist some* $t^* = t^*_{\xi_0} \geqslant t_0$ *such that*

$$\tilde{f}_{ii}(u_i(t^*, t_0, \xi_0)) < s \left(\frac{a_i}{b_{ii}} \right)_M, \; 1 \leqslant i \leqslant N. \tag{3.6}$$

We assume to the contrary that there exist some number $\xi_0 \in A$ and $i \in \{1, \dots, N\}$ such that

$$\tilde{f}_{ii}(u_i(t, t_0, \xi_0)) \geqslant s \left(\frac{a_i}{b_{ii}} \right)_M > \left(\frac{a_i}{b_{ii}} \right)_M > 0, \; t \geqslant t_0. \tag{3.7}$$

To simplify the notation we put $u(t) = u(t, t_0, \xi_0)$. For $t \geqslant t_0$, we have

$$\begin{aligned} u_i'(t) &= u_i(t) \left[a_i(t) - \sum_{j \neq i} b_{ij}(t) f_{ij}(u_i(t), u_j(t)) - b_{ii} \tilde{f}_{ii}(u_i(t)) \right] \\ &\leqslant u_i(t)[a_i(t) - b_{ii}(t) \tilde{f}_{ii}(u_i(t))] \leqslant u_i(t) \left[a_i(t) - b_{ii}(t) s \left(\frac{a_i}{b_{ii}} \right)_M \right]. \end{aligned}$$

Here, noting that

$$a_i(t) = \frac{a_i(t)}{b_{ii}(t)} \cdot b_{ii}(t) \leqslant \left(\frac{a_i}{b_{ii}} \right)_M \cdot b_{ii}(t),$$

we have

$$u_i'(t) \leqslant u_i(t) \left[\left(\frac{a_i}{b_{ii}} \right)_M b_{ii}(t) - s \left(\frac{a_i}{b_{ii}} \right)_M b_{ii}(t) \right] = -(s-1) \left(\frac{a_i}{b_{ii}} \right)_M b_{ii}(t) u_i(t).$$

From (3.7), we have

$$u_i'(t) \leqslant -(s-1)\tilde{f}_{ii}^{-1}\left(\left(\frac{a_i}{b_{ii}}\right)_M\right)\left(\frac{a_i}{b_{ii}}\right)_M b_{ii}(t) < 0,\ t \geqslant t_0.$$

Integrating the both sides on $[t_0,t]$, it follows that

$$u_i(t,t_0,\xi_0) \leqslant -(s-1)\tilde{f}_{ii}^{-1}\left(\left(\frac{a_i}{b_{ii}}\right)_M\right)\left(\frac{a_i}{b_{ii}}\right)_M \int_{t_0}^{t} b_{ii}(s)ds - \xi_0.$$

From (A2), and the fact that $s>1$ and $(a_i/b_{ii})_M > 0$, we have

$$u_i(t,t_0,\xi_0) \to -\infty \quad \text{as } t \to \infty;$$

but this is an obvious contradiction. That proves the claim.

Step 2. We fix $\xi \in A$. From Step 1, there exists some $t^* = t_\xi \geqslant t_0$ which satisfy (3.6). Since $s>1$, we choose r_i so that

$$s\left(\frac{a_i}{b_{ii}}\right)_M > r_i > \left(\frac{a_i}{b_{ii}}\right)_M,\ 1 \leqslant i \leqslant N.$$

Then by replacing t_0 of Lemma 2.3 by t^*, we have

$$\tilde{f}_{ii}(u_i(t,t_0,\xi)) < s\left(\frac{a_i}{b_{ii}}\right)_M,\ t \geqslant t_\xi^*, 1 \leqslant i \leqslant N.$$

By continuity with respect to initial conditions, there exists a neighborhood V_ξ of ξ such that

$$\tilde{f}_{ii}(u_i(t_\xi^*,t_0,\eta)) < s\left(\frac{a_i}{b_{ii}}\right)_M,\ \eta \in V_\xi, 1 \leqslant i \leqslant N.$$

Thus, it follows that

$$\tilde{f}_{ii}(u_i(t,t_0,\eta)) < s\left(\frac{a_i}{b_{ii}}\right)_M,\ t \geqslant t_\xi^*, \eta \in V_\xi, 1 \leqslant i \leqslant N.$$

Since $A \subset \bigcup_{\xi \in A} V_\xi$ by compactness of A, there exist some points $\xi_1,\ldots,\xi_L \in A$ such that

$$A \subset \bigcup_{k=1}^{L} V_{\xi_k}.$$

Hence, (3.5) holds by setting $\hat{t}_A = \max_{1\leqslant k\leqslant L}\{t_{\xi_k}^*\}$. Then the proof is completed.

Henceforth let $\hat{t}$ denote the number given by (3.5) in Lemma 3.3. Then it immediately leads to the following corollary from (A6).

COROLLARY 3.4. *Let conditions* (A1)-(A3), (A4) *and* (A6) *hold. Let A be a compact subset of* $\mathbb{R}_+^N$ *and* $t_0 \in \mathbb{R}$, $\xi_0 \in A$. *Then, for all i,j=1,..., N,*

$$f_{ij}(u_i(t,t_0,\xi_0),u_j(t,t_0,\xi_0)) < s\left(\frac{a_j}{b_{jj}}\right)_M,\ t \geqslant \hat{t}, \xi_0 \in A. \tag{3.8}$$

It leads to the following lemma.

LEMMA 3.5. *Let conditions* (A1)-(A3), (A4) *and* (A6) *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$, $\xi_0 \in A$. *Then, for all* $i = 1,\dots,N$,

$$\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) \geqslant \min\{\tilde{f}_{ii}(u_i(\hat{t},t_0,\xi_0)),\alpha\},\ t \geqslant \hat{t}, \xi_0 \in A, \tag{3.9}$$

where α *is the number defined by* (3.4).

Proof. Step 1. First, we shall prove the following claim.
CLAIM. *If there exist some* $T \geqslant \hat{t}$, $\xi_0 \in A$ *and* $i = 1,\dots,N$ *such that*

$$0 < \tilde{f}_{ii}(u_i(T,t_0,\xi_0)) \leqslant \alpha, \tag{3.10}$$

then $u_i'(T,t_0,\xi_0) \geqslant 0$.

In fact, from (3.3), (3.8) and (3.10), we have

$$\begin{aligned} u_i'(T) &= u_i(T)\left[a_i(T) - \sum_{j\neq i} b_{ij}(T) f_{ij}(u_i(T),u_j(T)) - b_{ii}(T)\tilde{f}_{ii}(u_i(T))\right] \\ &\geqslant u_i(T)\left[a_i(T) - \sum_{j\neq i} b_{ij}(T) s\left(\frac{a_j}{b_{jj}}\right)_M - \alpha b_{ii}(T)\right] \\ &\geqslant u_i(T)(\alpha b_{ii}(T) - \alpha b_{ii}(T)) \geqslant 0, \end{aligned}$$

where $u_i(T) = u_i(T,t_0,\xi_0)$, $1 \leqslant i \leqslant N$. That proves the claim.

Step 2. We prove Lemma 3.5. The proof is devided into two cases.

Case 1. Let

$$0 < \tilde{f}_{ii}(u_i(\hat{t},t_0,\xi_0)) \leqslant \alpha.$$

It suffices to show that for all $t \geqslant \hat{t}$, $\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) \geqslant \tilde{f}_{ii}(u_i(\hat{t},t_0,\xi_0))$.

Case 2. Let

$$\tilde{f}_{ii}(u_i(\hat{t},t,\xi_0)) \geqslant \alpha.$$

It sufficies to show that for all $t \geqslant \hat{t}$, $\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) \geqslant \alpha$.

In Case 1, we assume to the contrary that there exists some $\tilde{t} > \hat{t}$ such that $\beta := \tilde{f}_{ii}(u_i(\tilde{t},t_0,\xi_0)) < \tilde{f}_{ii}(u_i(\hat{t},t_0,\xi_0)) =: \gamma$. We take t_1 satisfying:

$$t_1 := \inf\{t \in \mathbb{R} \mid t > \hat{t}, \tilde{f}_{ii}(u_i(t,t_0,\xi_0)) = \beta\}.$$

Next we take t_2 satisfying:

$$t_2 := \sup\{t \in \mathbb{R} \mid \hat{t} \leqslant t < t_1, \tilde{f}_{ii}(u_i(t,t_0,\xi_0)) = \gamma\}.$$

Since it follows that $\tilde{f}_{ii}(u_i(t_1,t_0,\xi_0)) < \tilde{f}_{ii}(u_i(t_2,t_0,\xi_0))$ and $\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) \leqslant \alpha$ on $[t_2,t_1]$, from Step 1 $u_i'(t,t_0,\xi_0) \geqslant 0$ on $[t_2,t_1]$. Hence, we have $u_i(t_1,t_0,\xi_0) \geqslant u_i(t_2,t_0,\xi_0)$. By (1.1) $\tilde{f}_{ii}(u_i(t_1,t_0,\xi_0)) \geqslant \tilde{f}_{ii}(u_i(t_2,t_0,\xi_0))$, which is a contradiction.

In Case 2, we assume to the contrary that there exists some $\check{t} > \hat{t}$ such that $\eta := \tilde{f}_{ii}(u_i(\check{t}, t_0, \xi_0)) < \alpha$. We take t_3 satisfying:

$$t_3 := \inf\{t \in \mathbb{R} \mid t > \hat{t}, \tilde{f}_{ii}(u_i(t, t_0, \xi_0)) = \eta\}.$$

Next we take t_4 satisfying:

$$t_4 := \sup\{t \in \mathbb{R} \mid \hat{t} \leqslant t < t_3, \tilde{f}_{ii}(u_i(t, t_0, \xi_0)) = \alpha\}.$$

Then $\tilde{f}_{ii}(u_i(t_3, t_0, \xi_0)) < \tilde{f}_{ii}(u_i(t_4, t_0, \xi_0))$ similarly to Case 1, we can get $u_i(t_3, t_0, \xi_0) \geqslant u_i(t_4, t_0, \xi_0)$. By (1.1) $\tilde{f}_{ii}(u_i(t_3, t_0, \xi_0)) \geqslant \tilde{f}_{ii}(u_i(t_4, t_0, \xi_0))$, which is a contradiction. This completes the proof.

From Lemma 3.5, we have an important lemma which will be employed to prove Lemma 3.1.

LEMMA 3.6. *Let conditions* (A1)-(A3), (A4) *and* (A6) *hold. Let A be a compact subset of $\mathbb{R}^N_+$ and let $t_0 \in \mathbb{R}$. Let $\delta > 0$ be a number such that $\delta < \alpha$. Then there exists some $t' = t'_A \geqslant \hat{t}$ such that,*

$$\tilde{f}_{ii}(u_i(t, t_0, \ \xi_0)) > \delta > 0, \ t \geqslant t', \ \xi_0 \in A, \ 1 \leqslant i \leqslant N.$$

Proof. Step 1. First, we prove the following claim.

CLAIM. *For $\xi_0 \in A$ and $i = 1, \ldots, N$, there exists some $t'_i = t'_{i,\xi_0}$ such that*

$$\tilde{f}_{ii}(u_i(t'_i, t_0, \xi_0)) > \delta. \tag{3.11}$$

We assume to the contrary that there exists some $\xi_0 \in A$ and $i = 1, \ldots, N$ such that

$$\tilde{f}_{ii}(u_i(t, t_0, \xi_0)) \leqslant \delta < \alpha, \ t \geqslant \hat{t}.$$

For simplicity, we put $u(t) = u(t, t_0, \xi_0)$. Then, by (3.3) for $t \geqslant \hat{t}$

$$\begin{aligned} u'_i(t) &= u_i(t)\left[a_i(t) - \sum_{j \neq i} b_{ij}(t) f_{ij}(u_i(t), u_j(t)) - b_{ii}(t)\tilde{f}_{ii}(u_i(t))\right] \\ &\geqslant u_i(t)\left[a_i(t) - \sum_{j \neq i} b_{ij}(t) s\left(\frac{a_j}{b_{jj}}\right)_M - \delta b_{ii}(t)\right] \\ &\geqslant u_i(t)(\alpha b_{ii}(t) - \delta b_{ii}(t)) = (\alpha - \delta) u_i(t) b_{ii}(t). \end{aligned}$$

Since it follows from the above inequality that $u'_i \geqslant 0$ on $[\hat{t}, \infty)$, we get

$$u_i(t) \geqslant u_i(\hat{t}) =: C_+ > 0, \ t \geqslant \hat{t}.$$

Thus, since $\delta < \alpha$,

$$u'_i(t) \geqslant C_+(\alpha - \delta) b_{ii}(t) > 0, \ t \geqslant \hat{t},$$

integrating both of sides on $[\hat{t},t]$, gives

$$u_i(t) \geqslant C_+(\alpha-\delta)\int_{\hat{t}}^{t} b_{ii}(s)ds - C_+.$$

From (A2), it follows that $u_i(t,t_0,\xi_0) \to \infty$ as $t \to \infty$, which is a contradiction. That proves the claim.

Step 2. We fix $\xi_0 \in A$ and $i = 1,\dots,N$. From Step 1, there exists some $t_i' = t_{i,\xi_0}' \geqslant \hat{t}$ satisfy (3.11). Since $\delta < \min\{\tilde{f}_{ii}(u_i(t_i',t_0,\xi_0)),\alpha\}$ and $t_i' \geqslant \hat{t}$, we can claim the following:

CLAIM. *For* $t \geqslant t_i'$,

$$\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) > \delta.$$

In fact, by $t_i' \geqslant \hat{t}$, we have (3.8) for $t \geqslant t_i'$. Therefore similarly to the proof of Lemma 3.5, we have

$$\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) \geqslant \min\{\tilde{f}_{ii}(u_i(t_i',t_0,\xi_0)),\alpha\} > \delta$$

for $t \geqslant t_i'$. That proves the claim. Here we set $t'(\xi_0) = \max_{1\leqslant i\leqslant n}\{t_{i,\xi_0}'\}$. Then

$$\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) > \delta > 0,\ t \geqslant t'(\xi_0), 1 \leqslant i \leqslant N.$$

Therefore similarly to the proof of Lemma 3.3, from the compactness of A, there exists $t_A' \geqslant \hat{t}$ such that

$$\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) > \delta,\ t \geqslant t_A', \xi_0 \in A, 1 \leqslant i \leqslant N.$$

Then the proof is completed.

By employing Lemma 3.6, we shall prove Lemma 3.1.

PROOF OF LEMMA 3.1 Firstly, we take the number t_A given in Lemma 3.6. Let us set

$$r := \max_{1\leqslant i\leqslant N}\left\{s\left(\frac{a_i}{b_{ii}}\right)_M\right\}.$$

Then

$$0 < \delta < \tilde{f}_{ii}(u_i(t,t_0,\xi_0)) < r,\ t \geqslant t_A, \xi \in A, 1 \leqslant i \leqslant N,$$

where s, δ are the numbers appearing in Proposition 3.2 and Lemma 3.6. Then it follows that

$$0 < \tilde{f}_{ii}^{-1}(\delta) < u_i(t,t_0,\xi_0) < \tilde{f}_{ii}^{-1}(r),\ t \geqslant t_A, \xi_0 \in A, 1 \leqslant i \leqslant N.$$

Hence we have

$$0 < \delta' < u_i(t,t_0,\xi_0) < r'\ t \geqslant t_A, \xi_0 \in A, 1 \leqslant i \leqslant N$$

by setting

$$\delta' := \min_{1\leqslant i\leqslant N}\{\tilde{f}_{ii}^{-1}(\delta)\},\ r' := \max_{1\leqslant i\leqslant N}\{\tilde{f}_{ii}^{-1}(r)\}.$$

This completes the proof. $\square$

From Proposition 2.1 and Lemma 3.1, we can prove Theorem 1.4.

4. Proof of Theorem 1.5

From Remark 2.2, in order to prove Theorem 1.5, it is sufficient to prove the following lemma.

LEMMA 4.1. *Let conditions* $(A1)$-$(A3)$, $(A5)$ *and* $(A7)$ *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$. *Then there exist some numbers* $\delta_A > 0$, $r > 0$ *and* $t_A \geqslant t_0$ *such that*

$$\delta_A < u_i(t) < r,\ t \geqslant t_A,\ 1 \leqslant i \leqslant N.$$

We give several lemmas which are employed in proving Lemma 4.1. Firstly from (A5), there exists some number $\varepsilon > 0$ such that for all $t \geqslant t_0$ and $i = 1, \ldots, N$,

$$a_i(t) - \sum_{j \neq i} b_{ij}(t) f_{ij}\left(\tilde{f}_{ii}^{-1}\left(\left(\frac{a_i}{b_{ii}}\right)_M\right), \tilde{f}_{jj}^{-1}\left(\left(\frac{a_j}{b_{jj}}\right)_M\right)\right) \geqslant \varepsilon b_{ii}(t). \tag{4.1}$$

Therefore we have the following proposition.

PROPOSITION 4.2. *There exists some number* $s > 1$ *such that*

$$s\varepsilon - (s-1)\left(\frac{a_i}{b_{ii}}\right)_M > 0,\ 1 \leqslant i \leqslant N.$$

Therefore, if we put

$$\alpha \equiv \min_{1 \leqslant i \leqslant N}\left\{s\varepsilon - (s-1)\left(\frac{a_i}{b_{ii}}\right)_M\right\} > 0, \tag{4.2}$$

then for all $t \geqslant t_0$ *and* $i = 1, \ldots, N$,

$$a_i(t) - \sum_{j \neq i} b_{ij}(t) s f_{ij}\left(\tilde{f}_{ii}^{-1}\left(\left(\frac{a_i}{b_{ii}}\right)_M\right), \tilde{f}_{jj}^{-1}\left(\left(\frac{a_j}{b_{jj}}\right)_M\right)\right) \geqslant \alpha b_{ii}(t). \tag{4.3}$$

Proof. Similarly to Proposition 3.2, we can prove the proposition. □

Henceforth let s and α denote the numbers given in Proposition 4.2. Inequality (4.3) corresponds to inequality (3.3). Now we recall that conditions (A1)-(A3) and (A4) imply Lemma 3.3. Hence under conditions (A1)-(A3) and (A5) we can obtain the same results as in Lemma 3.3. That is,

LEMMA 4.3. *Let conditions* $(A1)$-$(A3)$ *and* $(A5)$ *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$. *Let* $u(t, t_0, \xi_0)$ *be a solution of* (GLV) *with* $\xi_0 \in A$. *Then Lemma* 3.3 *are still valid, where* s *is the number given by Proposition* 4.2.

Henceforth let $\hat{t} = \hat{t}_A$ denote the number satisfying

$$\tilde{f}_{ii}(u_i(t, t_0, \xi_0)) < s\left(\frac{a_i}{b_{ii}}\right)_M,\ t \geqslant \hat{t},\ \xi_0 \in A,\ 1 \leqslant i \leqslant N,$$

where A is compact. Then it immediately leads to the following corollary from (A7).

COROLLARY 4.4. *Let conditions* (A1)-(A3), (A5) *and* (A7) *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and* $t_0 \in \mathbb{R}$, $\xi_0 \in A$. *Then for all* $t \geqslant \hat{t}$, $\xi_0 \in A$ *and* $i, j = 1, \dots, N$,

$$f_{ij}(u_i(t,t_0,\xi_0), u_j(t,t_0,\xi_0)) < s f_{ij}\left(\tilde{f}_{ii}^{-1}\left(\frac{a_i}{b_{ii}}\right)_M, \tilde{f}_{jj}^{-1}\left(\frac{a_j}{b_{jj}}\right)_M\right). \tag{4.4}$$

Proof. We fix $\xi_0 \in A$ and $j = 1, \dots, N$. Then it follows, from Lemma 4.3, that

$$u_j(t,t_0,\xi_0) < \tilde{f}_{jj}^{-1}\left(s\left(\frac{a_j}{b_{jj}}\right)_M\right), \ t \geqslant \hat{t}.$$

This completes the proof by (A7).

Corollary 4.4 corresponds to Corollary 3.4. We recall that Corollary 3.4 implies Lemmas 3.5 and 3.6. Hence, under conditions (A1)-(A3), (A5) and (A7), we can obtain the same results as in Lemmas 3.5 and 3.6, as seen below:

LEMMA 4.5. *Let conditions* (A1)-(A3), (A5) *and* (A7) *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$, $\xi_0 \in A$. *Then for all* $i = 1, \dots, N$, (3.9) *is still valid, where* α *is the number defined by* (4.2).

Proof. First we will prove the following claim.
CLAIM. *If there are* $T \geqslant \hat{t}$, $\xi_0 \in A$ *and* $i = 1, 2 \dots, N$ *satisfying* (3.10), *then*

$$u_i'(T,t_0,\xi_0) \geqslant 0.$$

In fact, from (4.3), (4.4) and (3.10), we have

$$\begin{aligned} u_i'(T) &= u_i(T)\left[a_i(T) - \sum_{j \neq i} b_{ij}(T) f_{ij}(u_i(T), u_j(T)) - b_{ii}(T)\tilde{f}_{ii}(u_i(T))\right] \\ &\geqslant u_i(T)\left[a_i(T) - \sum_{j \neq i} b_{ij}(T) s f_{ij}\left(\tilde{f}_{ii}^{-1}\left(\frac{a_i}{b_{ii}}\right)_M, \tilde{f}_{jj}^{-1}\left(\frac{a_j}{b_{jj}}\right)_M\right) - \alpha b_{ii}(T)\right] \\ &\geqslant u_i(T)(\alpha b_{ii}(T) - \alpha b_{ii}(T)) \geqslant 0, \end{aligned}$$

where $u_i(T) = u_i(T,t_0,\xi_0)$, $1 \leqslant i \leqslant N$. That proves the claim.
The remainder of the proof proceeds as in the proof of Lemma 3.5.

LEMMA 4.6. *Let conditions* (A1)-(A3), (A5) *and* (A7) *hold. Let* A *be a compact subset of* $\mathbb{R}_+^N$ *and let* $t_0 \in \mathbb{R}$. *Let* $\delta > 0$ *be a number such that* $\delta < \alpha$. *Then Lemma* 3.6 *are still valid.*

Proof. We only give the sketch of the proof of (i).
To show our statement by contradiction suppose to the contrary that

$$\tilde{f}_{ii}(u_i(t,t_0,\xi_0)) \leqslant \delta < \alpha, \ t \geqslant \hat{t}$$

for some $\xi_0 \in A$ and $i = 1,\ldots,N$. For simplicity we put $u(t) = u(t,t_0,\xi_0)$. Then, by (4.3) and (4.4)

$$\begin{aligned} u_i'(t) &= u_i(t)\left[a_i(t) - \sum_{j\neq i} b_{ij}(t) f_{ij}(u_i(t),u_j(t)) - b_{ii}(t)\tilde{f}_{ii}(u_i(t))\right] \\ &\geqslant u_i(t)\left[a_i(t) - \sum_{j\neq i} b_{ij}(t) s f_{ij}\left(\tilde{f}_{ii}^{-1}\left(\frac{a_i}{b_{ii}}\right)_M, \tilde{f}_{jj}^{-1}\left(\frac{a_j}{b_{jj}}\right)_M\right) - \delta b_{ii}(t)\right] \\ &\geqslant u_i(t)(\alpha b_{ii}(t) - \delta b_{ii}(t)) = (\alpha-\delta)u_i(t)b_{ii}(t),\ t \geqslant \hat{t}. \end{aligned}$$

Hence we can get a contradiction as in the proof of Lemma 3.6.

By the above lemmas we can prove Lemma 4.1, and so we can prove Theorem 1.5.

Acknowledgements. The author would like to express his appreciation to Professor Hiroyuki Usami of Hiroshima University for his useful comments and suggestions, and to the referees for many valuable comments and suggestions

REFERENCES

[1] S. AHMAD, A. C. LAZER, *On a property of nonautonomous Lotka-Volterra competition model*, Nonlinear Anal., **37** (1999), 603–611.
[2] S. AHMAD, A. C. LAZER, *On the nonautonomous N-competing species problem*, Appl. Anal., **57** (1995), 309–323.
[3] P. BILLINGSLEY, *Probability and measure*, John Wiley & Sons, New York, 1979.
[4] W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*, P. C. Heath and Company, Boston, 1965.
[5] J. HOFBAUER, K. SIGMUND, *Evolutionary Games and Population Dynamics*, Cambridge University Press, New York, 1998.
[6] R. REDHEFFER, *Nonautonomous Lotka-Volterra system I*, J. Differential Equations, **127** (1996), 519–540.

(Received December 9, 2009)
(Revised August 20, 2010)

Kunihiko Taniguchi
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan
e-mail: d092134@hiroshima-u.ac.jp