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ON THE SOLVABILITY OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. The focus of this paper is the study of nonlinear differential equations subject to general boundary conditions. We formulate sufficient conditions for the existence of solutions based on the dimension of the solution space of the corresponding linear, homogeneous equation and the "size" of the nonlinear terms. Our approach is based on the Lyapunov-Schmidt Procedure (Alternative Method).

1. Introduction

In this paper, we consider boundary value problems of the form

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(y(t))+(Gy)(t),\ \ 0\leqslant t\leqslant 1, \tag{1}$$

subject to

$$\begin{aligned}
&b_{11}y(0)+\cdots+b_{1n}y^{(n-1)}(0)+d_{11}y(1)+\cdots+d_{1n}y^{(n-1)}(1)=0,\\
&b_{21}y(0)+\cdots+b_{2n}y^{(n-1)}(0)+d_{21}y(1)+\cdots+d_{2n}y^{(n-1)}(1)=0,\\
&\quad\vdots\\
&b_{n1}y(0)+\cdots+b_{nn}y^{(n-1)}(0)+d_{n1}y(1)+\cdots+d_{nn}y^{(n-1)}(1)=0.
\end{aligned} \tag{2}$$

We assume that $f:\mathbb{R}\to\mathbb{R}$ is continuous and that the limits $f(\infty)$ and $f(-\infty)$ exist. The map $G:(\mathscr{C}([0,1],\mathbb{R}),\|\cdot\|_\infty)\to(\mathscr{C}([0,1],\mathbb{R}),\|\cdot\|_\infty)$ is nonlinear and continuous. We concern ourselves with problems where the corresponding linear, homogeneous boundary value problem

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=0 \tag{3}$$

subject to (2) has a one dimensional solution space. For such problems, we provide sufficient conditions for the existence of solutions to (1)-(2). These conditions are based

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on the limiting behavior of the real valued function f, the properties of the solution space of the linear homogeneous boundary value problem (3)-(2), and the ”size” of the nonlinear map G. It is significant to observe that the results we obtain may be applied to boundary value problems for integro-differential equations of the form

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(y(t))+\int_0^1 k(t,s)g(t,y(s))ds,\ \ 0\leqslant t\leqslant 1,$$

subject to (2) as well as to classical boundary value problems of the form

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(y(t))+g(t,y(t)),\ \ 0\leqslant t\leqslant 1,$$

subject to (2).

Our approach is based on the Lyapunov-Schmidt Procedure (Alternative Method). The results we present here allow us to establish the solvability of boundary value problems that do not fall within the scope of the results previously obtained by Rodríguez and Taylor [20]. Ideas and techniques similar to the ones we use in this paper have been successfully applied to the study of periodic behavior in discrete and continuous dynamical systems [3], [5], [6], [8], [10], [21] boundary value problems for differential and difference equations [1], [7], [12], [13], [15]-[20], and more general systems [2], [22].

2. Preliminaries

In order to analyze the boundary value problem (1)-(2), we formulate it in system form. The matrix $A(t)$ is defined by

$$A(t)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & & \ddots & & \vdots\\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t)\end{bmatrix}.$$

The vector

$$x=\begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix}$$

is given by $x_1=y,\ x_2=y',\ \cdots,\ x_n=y^{(n-1)}$ and the boundary matrices B, D are

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n}\\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n}\\ \vdots & & \ddots & & \vdots\\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn}\end{bmatrix} \quad\text{and}\quad D=\begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n}\\ d_{21} & d_{22} & d_{23} & \cdots & d_{2n}\\ \vdots & & \ddots & & \vdots\\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nn}\end{bmatrix}.$$

It is clear that the boundary value problem (1)-(2) is equivalent to

$$\dot{x}(t) = A(t)x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x_1(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ G(x_1(t)) \end{bmatrix}, 0 \leqslant t \leqslant 1, \tag{4}$$

subject to

$$Bx(0) + Dx(1) = 0. \tag{5}$$

Throughout the paper we will assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that it has finite limits at ∞ and $-\infty$. We write

$$f(\infty) = \lim_{s \to \infty} f(s)$$

and

$$f(-\infty) = \lim_{s \to -\infty} f(s).$$

For any integer $p \geqslant 1$ the space $(\mathscr{C}([0,1], \mathbb{R}^p), \|\cdot\|_\infty)$ will denote $\{\phi : [0,1] \to \mathbb{R}^p : \phi \text{ is continuous}\}$. The norm used on this space is the sup norm; this is,

$$\|\phi\|_\infty = \sup\{|\phi(t)| : 0 \leqslant t \leqslant 1\},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^p$.

The map $G : (\mathscr{C}([0,1], \mathbb{R}), \|\cdot\|_\infty) \to (\mathscr{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)$ is continuous and there exists an M such that for any $\phi \in (\mathscr{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)$,

$$\sup\{|G(\phi(t))| : 0 \leqslant t \leqslant 1\} \leqslant M < \infty.$$

So as to be able to use functional analytic ideas we introduce the following notation. The space

$$X = \{x \in (\mathscr{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty) : Bx(0) + Dx(1) = 0\}$$

and $\mathscr{F} : X \to (\mathscr{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$ is defined by

$$(\mathscr{F}x)(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x_1(t)) \end{bmatrix},$$

and $\mathscr{G} : X \to (\mathscr{C}([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$ is given by

$$(\mathscr{G}x)(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ G(x_1(t)) \end{bmatrix}.$$

It is obvious that $\mathscr{F}$ and $\mathscr{G}$ are continuous maps from X into $(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ and that $\sup\{\|\mathscr{F}(x)\|_\infty : x\in X\}$ and $\sup\{\|\mathscr{G}(x)\|_\infty : x\in X\}$ are both finite.

We define the operator $L: D(L)\to(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ by $(Lx)(t)=\dot{x}(t)-A(t)x(t)$ where $D(L)$ consists of the continuously differentiable functions in X. It is evident that the boundary value problem (1)-(2) is equivalent to

$$Lx=\mathscr{F}(x)+\mathscr{G}(x). \tag{6}$$

Since the properties of the solution space of the linear homogeneous boundary value problem (3)-(2) play a role in the solvability of (1)-(2), we must first consider the linear problem $Lx=0$.

PROPOSITION 2.1. *$Lx=0$ if and only if $x(t)=\Gamma(t)v$, where $\Gamma(t)$ is the principal matrix solution of $\dot{x}(t)=A(t)x(t)$ and $v\in\ker(B+D\Gamma(1))$.*

Proof. $Lx=0$ if and only if

$$\dot{x}(t)-A(t)x(t)=0 \text{ and } Bx(0)+Dx(1)=0$$

if and only if

$$x(t)=\Gamma(t)C \text{ for some } C \text{ and } B\Gamma(0)C+D\Gamma(1)C=0$$

if and only if $[B+D\Gamma(1)]C=0$ if and only if $C\in\ker(B+D\Gamma(1))$. $\square$

COROLLARY 2.2. *$\ker(B+D\Gamma(1))$ and $\ker(L)$ have the same dimension.*

It is well documented that solutions of

$$\dot{x}(t)=A(t)x(t)+h(t)$$

are given by the variation of constants formula

$$x(t)=\Gamma(t)x(0)+\Gamma(t)\int_0^t\Gamma^{-1}(s)h(s)ds.$$

PROPOSITION 2.3. *$Lx=h$ if and only if x is given by the variation of constants formula above, where $x(0)$ must satisfy*

$$[B+D\Gamma(1)]x(0)=-D\Gamma(1)\int_0^1\Gamma^{-1}(s)h(s)ds.$$

Proof. $Lx=h$ if and only if

$$x(t)=\Gamma(t)x(0)+\Gamma(t)\int_0^t\Gamma^{-1}(s)h(s)ds$$

and

$$Bx(0)+Dx(1)=0$$

if and only if

$$Bx(0)+D[\Gamma(1)x(0)+\Gamma(1)\int_0^1 \Gamma^{-1}(s)h(s)ds]=0$$

if and only if

$$[B+D\Gamma(1)]x(0)=-D\Gamma(1)\int_0^1 \Gamma^{-1}(s)h(s)ds. \quad \square$$

COROLLARY 2.4. *L is a bijection on $D(L)$ if and only if $(B+D\Gamma(1))$ is invertible.*

3. The Case of Invertible L

It should be observed that if L is invertible and the nonlinearities f and G are bounded, it is straightforward to establish the existence of solutions of (1)-(2). In fact, (1)-(2) is solvable if and only if the operator $L^{-1}(\mathscr{F}+\mathscr{G})$ has a fixed point. The existence of such a fixed point is an immediate consequence of Schauder's Theorem once we observe that $L^{-1}(\mathscr{F}+\mathscr{G})$ is compact.

4. The Case of Singular L

Since the existence of solutions is relatively straightforward when L is invertible, the more interesting case is when $\ker(L)$ or, equivalently, $\ker(B+D\Gamma(1))$ is nontrival. In this paper, we consider the case when the dimension of $\ker(L)$ is one. For the reader's convenience, we offer a self-contained presentation of the basic ideas of the Lyapunov-Schmidt reduction. These ideas have been applied to a large class of problems in differential and difference equations [3], [6], [7], [8], [12]-[20]. For an abstract formulation and for a vast number of applications, we refer the reader to [4], [5], [9].

We know that $Lx=0$ if and only if $x(t)=\Gamma(t)v$, where $v\in\ker(B+D\Gamma(1))$. We now wish to examine when $Lx=h$ has a solution. According to Proposition 2.3, $h\in \text{Im}(L)$ if and only if there is some $x_0\in\mathbb{R}^n$ such that

$$[B+D\Gamma(1)]x_0=-D\Gamma(1)\int_0^1 \Gamma^{-1}(s)h(s)ds;$$

that is, if and only if

$$\int_0^1 D\Gamma(1)\Gamma^{-1}(s)h(s)ds\in \text{Im}(B+D\Gamma(1)).$$

Since $\text{Im}(B+D\Gamma(1))=[\ker(B+D\Gamma(1))^T]^{\perp}$, $h\in\text{Im}(L)$ if and only if

$$W^T\int_0^1 D\Gamma(1)\Gamma^{-1}(s)h(s)ds=0,$$

where the columns of the n by n matrix W form a basis for $\ker(B+D\Gamma(1))^T$.

We define

$$\Psi^T(t) = W^T D\Gamma(1)\Gamma^{-1}(t).$$

By the argument outlined above, $Lx = h$ if and only if $\int_0^1 \Psi^T(t)h(t)dt = 0$.

Since L is not invertible, we can't apply the Schauder Fixed Point Theorem directly. We will use the splittings of $D(L)$ and $(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ typically used in the Lyapunov-Schmidt procedure.

We find projections, P, of $D(L)$ onto $\ker(L)$, and E, of $(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)$ onto $\mathrm{Im}(L)$, so that we may write

$$D(L) = \ker(L) \oplus \mathrm{Im}(I-P) \text{ and } (\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty) = \mathrm{Im}(L) \oplus \mathrm{Im}(I-E).$$

Let $\Phi(t) = \Gamma(t)V$ where the vector V forms a basis for $\ker(B+D\Gamma(1))$. Let

$$C_1 = \int_0^1 \Phi^T(t)\Phi(t)dt$$

and

$$C_2 = \int_0^1 \Psi^T(t)\Psi(t)dt.$$

PROPOSITION 4.1. *C_1 is invertible and C_2 is invertible when $[B:D]$ has full rank.*

Proof. To show C_1 is invertible, assume $C_1 a = 0$ and define $q(t) = \Phi(t)a$. Then $a^T C_1 a = \int_0^1 q^T(t)q(t)dt = 0$ which implies $q(t) = 0$ for all $t \in [0,1]$. This implies $a = 0$ because $\Phi(t)$ is a nonzero vector.

To show C_2 is invertible, we need to show the columns of $\Psi^T(t)$ are linearly independent. Let $\Psi_j^T(t)$ be the jth column of $\Psi^T(t)$. If $[B:D]$ has full rank, $a \in \ker(B^T)$ and $a \in \ker(D^T)$ implies $a = 0$. Now,

$$c_1\Psi_1^T(t) + c_2\Psi_2^T(t) = 0$$

if and only if

$$(c_1,c_2)D = 0$$

if and only if

$$(c_1,c_2)^T \in \ker(D^T).$$

Since

$$(c_1,c_2)^T \in \ker[B+D\Gamma(1)]^T,\ (c_1,c_2)^T \in \ker(B^T)$$

and hence $(c_1,c_2)^T = (0,0)$.

Let

$$(I-E)x(t) = \Psi(t)C_2^{-1}\int_0^1 \Psi^T(s)x(s)ds,$$

and

$$Px(t) = \Phi(t)C_1^{-1}\int_0^1 \Phi^T(s)x(s)ds.$$

For the reader's convenience, we have presented a detailed construction of the projections onto the $\ker(L)$ and $\operatorname{Im}(L)$. For the case of periodic boundary conditions, we refer the reader to D.C. Lewis [11]; for discrete boundary value problems, we suggest Rodríguez [14]. Although the projections we have constructed here are a special case of those that appear in Spealman and Sweet [22] and Rodríguez and Taylor [20], we have chosen present this construction due to the fact that we do not need to appeal to the full generality of the results mentioned previously.

We now use the standard techniques of the Lyapunov-Schmidt method to analyze $Lx = \mathscr{F}(x) + \mathscr{G}(x)$.

REMARK 4.2. If $\tilde{L}$ is the restriction of L to $\operatorname{Im}(I-P)$ then $\operatorname{Im}(\tilde{L}) = \operatorname{Im}(L)$. $\tilde{L}$, viewed as a map from $\operatorname{Im}(I-P)$ into $\operatorname{Im}(L)$ is invertible. We denote $(\tilde{L})^{-1}$ by M. From this, it follows that $MLx = (I-P)x$. Later, we will use the obvious fact that M is compact.

PROPOSITION 4.3. *$Lx = \mathscr{F}(x) + \mathscr{G}(x)$ is equivalent to*

$$\begin{cases} x = Px + ME\mathscr{F}(x) + ME\mathscr{G}(x) \\ \text{and} \\ (I-E)\mathscr{F}(Px + ME(\mathscr{F}(x)+\mathscr{G}(x))) + (I-E)\mathscr{G}(Px + ME(\mathscr{F}(x)+\mathscr{G}(x))) = 0. \end{cases}$$

Proof. We have $Lx = \mathscr{F}(x) + \mathscr{G}(x)$ if and only if

$$\begin{cases} E(Lx - (\mathscr{F}(x)+\mathscr{G}(x))) = 0 \\ \text{and} \\ (I-E)(Lx - (\mathscr{F}(x)+\mathscr{G}(x))) = 0, \end{cases}$$

if and only if

$$\begin{cases} Lx = E(\mathscr{F}(x)+\mathscr{G}(x)) \\ \text{and} \\ (I-E)(\mathscr{F}(x)+\mathscr{G}(x)) = 0, \end{cases}$$

if and only if

$$\begin{cases} (I-P)x = ME(\mathscr{F}(x)+\mathscr{G}(x)) \\ \text{and} \\ (I-E)(\mathscr{F}(x)+\mathscr{G}(x)) = 0, \end{cases}$$

if and only if

$$\begin{cases} x = Px + ME(\mathscr{F}(x)+\mathscr{G}(x)) \\ \text{and} \\ (I-E)(\mathscr{F}(Px + ME(\mathscr{F}(x)+\mathscr{G}(x))) + \mathscr{G}(Px + ME(\mathscr{F}(x)+\mathscr{G}(x)))) = 0. \end{cases}$$

We have limited our presentation of the Lyapunov-Schmidt Procedure to only those aspects necessary for the problem at hand. This approach, as well as its generalization, the Alternative Method, is well documented [2]-[5], [9], [12], [13], [15], [17]. For those interested in the study of periodicity, in either discrete or continuous dynamical systems, we suggest [3], [5], [6], [8], [10], [21]. For applications in the field of discrete boundary value problems, the reader may consult [7], [14], [16], [18], [19]. An abstract and very general presentation appears in [2], [5], [9].

5. Main Results

The conditions of 4.3 may be rewritten as

$$\begin{cases} x = \alpha\Phi(t) + ME\mathscr{F}(x) + ME\mathscr{G}(x) \\ \text{and} \\ 0 = \int_0^1 \Psi_2(t) f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt + \int_0^1 \Psi_2(t) G(\alpha\Phi_1(t) \\ \quad +[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt, \end{cases}$$

where $\Phi_i(t),\ \Psi_i(t)$, and $[ME(\mathscr{F}(x)+\mathscr{G}(x))]_i(t)$ are the ith entries of $\Phi(t),\ \Psi(t)$, and $\alpha\Phi(t)+ME(\mathscr{F}(x)+\mathscr{G}(x))(t)$, respectively.

We will assume that there are finite numbers, which we designate $f(\infty)$ and $f(-\infty)$, such that

$$\lim_{r\to\infty} f(r) = f(\infty)$$

and

$$\lim_{r\to-\infty} f(r) = f(-\infty).$$

We define J_1 and J_2 as

$$J_1 = f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}} \Psi_2(t)dt + f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}} \Psi_2(t)dt$$

and

$$J_2 = f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}} \Psi_2(t)dt + f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}} \Psi_2(t)dt.$$

THEOREM 5.1. *Suppose that:*

1. $\dim(\ker(B+D\Gamma(1))) = 1$ *where* $\Gamma(t)$ *is the principal matrix solution of* $\dot{x}(t) = A(t)x(t)$;

2. $[B:D]$ *has full rank;*

3. $f:\mathbb{R}\to\mathbb{R}$ *is continuous;*

4. $f(\infty)$ *and* $f(-\infty)$ *exist;*

5. $J_1J_2 < 0$;

6. $G:(\mathscr{C}([0,1],\mathbb{R}),\|\cdot\|_\infty)\to(\mathscr{C}([0,1],\mathbb{R}),\|\cdot\|_\infty)$ *is continuous and*

$$\sup\{\|G(w)\| : w\in(\mathscr{C}([0,1],\mathbb{R}),\|\cdot\|_\infty)\} \leqslant \min\{|J_1|,|J_2|\}.$$

Then there exists at least one solution of

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(y(t)) + (Gy)(t)$$

that satisfies

$$b_{11}y(0)+\cdots+b_{1n}y^{(n-1)}(0)+d_{11}y(1)+\cdots+d_{1n}y^{(n-1)}(1)=0,$$
$$b_{21}y(0)+\cdots+b_{2n}y^{(n-1)}(0)+d_{21}y(1)+\cdots+d_{2n}y^{(n-1)}(1)=0,$$
$$\vdots$$
$$b_{n1}y(0)+\cdots+b_{nn}y^{(n-1)}(0)+d_{n1}y(1)+\cdots+d_{nn}y^{(n-1)}(1)=0.$$

Proof. Let $J=\min\{|J_1|,|J_2|\}$. We define mappings:

$$\begin{aligned}
&H_1:\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}n),\|\cdot\|_\infty)\to(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty),\\
&H_2:\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)\to\mathbb{R},\\
&H:\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty)\to\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n),\|\cdot\|_\infty),
\end{aligned}$$

by

$$\begin{aligned}
H_1(\alpha,x)&=\alpha\Phi(t)+ME\mathscr{F}(x)+ME\mathscr{G}(x),\\
H_2(\alpha,x)&=\alpha-\Big(\int_0^1\Psi_2(t)f(\alpha\Phi_1+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt\\
&\qquad+\int_0^1\Psi_2(t)G(\alpha\Phi_1+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt\Big),
\end{aligned}$$

and

$$H(\alpha,x)=(H_1(\alpha,x),H_2(\alpha,x)).$$

Since $\{t:\Phi_1(t)=0\}$ has Lebesgue measure zero, it follows that

$$\begin{aligned}
&\int_0^1\Psi_2(t)f(\alpha\Phi_1(t)+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt\\
&\quad=\int_{\{t\in[0,1]:\Phi_1(t)>0\}}\Psi_2(t)f(\alpha\Phi_1+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt\\
&\qquad+\int_{\{t\in[0,1]:\Phi_1(t)<0\}}\Psi_2(t)f(\alpha\Phi_1+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt.
\end{aligned}$$

Since $ME(\mathscr{F}+\mathscr{G})$ is bounded, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{\alpha\to\infty}\int_0^1\Psi_2(t)f(\alpha\Phi_1(t)+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt\\
&\quad=f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}}\Psi_2(t)dt+f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}}\Psi_2(t)dt\\
&\quad=J_1.
\end{aligned}$$

Similarly,

$$\lim_{\alpha\to-\infty}\int_0^1\Psi_2(t)f(\alpha\Phi_1(t)+[ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt$$

$$= f(-\infty)\int_{\{t\in[0,1]:\Phi_1(t)>0\}}\Psi_2(t)dt + f(\infty)\int_{\{t\in[0,1]:\Phi_1(t)<0\}}\Psi_2(t)dt$$
$$= J_2.$$

Without loss of generality, we assume $J_2 < 0 < J_1$.

Assuming that $\Psi_2(t)$ is not identically zero, we can choose our basis for $\ker(B + D\Gamma(1))^T$ so that $\|\Psi\|_\infty \leqslant 1$. Therefore, there is some $\alpha_0 \geqslant m$ where $m = sup\{|f(t)| : t \in \mathbb{R}\}$ such that for all $\alpha \geqslant \alpha_0$,

$$\int_0^1 \Psi_2(t) f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))dt \geqslant J$$

and

$$\int_0^1 \Psi_2(t) f(-\alpha\Phi_1(t) + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))dt \leqslant -J.$$

Since

$$|G(\alpha\Phi_1 + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))| \leqslant J$$

for all $t \in \mathbb{R}$, for $\alpha \geqslant \alpha_0$ and $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$,

$$\begin{aligned} H_2(\alpha,x) = \alpha - \Big(&\int_0^1 \Psi_2(t) f(\alpha\Phi_1 + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))dt \\ &+ \int_0^1 \Psi_2(t) G(\alpha\Phi_1 + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))dt\Big) \\ \leqslant \alpha - (J - J) = \alpha.& \end{aligned}$$

Similarly, for $\alpha \geqslant \alpha_0$ and $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$, $H_2(-\alpha,x) \geqslant -\alpha$.

Letting $\delta = \alpha_0 + (m+J)$, define

$$\begin{aligned} \mathscr{B} = \{(\alpha,x) \in \mathbb{R} \times (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty) \\ : |\alpha| \leqslant \delta \text{ and } \|x\|_\infty \leqslant \delta\|\Phi\|_\infty + \|ME\|(m+J)\}. \end{aligned}$$

Here, $\|ME\|$ denotes the operator norm of the bounded, linear map ME.

Note that $\|ME\mathscr{F}(x)\|_\infty \leqslant \|ME\| m$ and $\|ME\mathscr{G}(x)\|_\infty \leqslant \|ME\| J$ for every $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$.

Now if $\alpha \in [\alpha_0, \delta]$, for all $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$, we have

$$\begin{aligned} H_2(\alpha,x) = \alpha - \Big(&\int_0^1 \Psi_2(t) f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t)\Big)dt \\ &+ \int_0^1 \Psi_2(t) G(\alpha\Phi_1(t) + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))dt) \\ \geqslant \alpha - \Big(&\int_0^1 |\Psi_2(t)||f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))|dt \\ &+ \int_0^1 |\Psi_2(t)||G(\alpha\Phi_1(t) + [ME(\mathscr{F}(x) + \mathscr{G}(x))]_1(t))|dt\Big) \\ \geqslant \alpha - (m+J) &\geqslant \alpha - \alpha_0 - J \geqslant -J \geqslant -\delta \end{aligned}$$

and

$$\begin{aligned} H_2(-\alpha,x) &= -\alpha - \Big(\int_0^1 \Psi_2(t) f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt \\ &\qquad + \int_0^1 \Psi_2(t) G(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))dt\Big) \\ &\leqslant -\alpha + \int_0^1 |\Psi_2(t)||f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))|dt \\ &\qquad + \int_0^1 |\Psi_2(t)||G(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))|dt \\ &\leqslant -\alpha + (m+J) \leqslant -\alpha + \alpha_0 + J \leqslant J \leqslant \delta. \end{aligned}$$

Thus, for all $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ and $\alpha \in [\alpha_0, \delta]$,

$$H_2(\alpha,x),\ H_2(-\alpha,x) \in [-\alpha,\alpha] \subseteq [-\delta,\delta].$$

Furthermore, if $0 \leqslant \alpha < \alpha_0$, for all $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$,

$$\begin{aligned} |H_2(\pm\alpha,x)| &\leqslant |\pm\alpha| + \int_0^1 |\Psi_2(t)||f(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))|dt \\ &\qquad + \int_0^1 |\Psi_2(t)||G(\alpha\Phi_1(t) + [ME(\mathscr{F}(x)+\mathscr{G}(x))]_1(t))|dt \\ &\leqslant \alpha_0 + (m+J) \leqslant \delta. \end{aligned}$$

We have shown that H_2 maps $[-\delta,\delta] \times (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ into $[-\delta,\delta]$ when $\Psi_2(t)$ is not identically zero. However, if $\Psi_2(t)$ is identically zero, $H_2(\alpha,x) = \alpha$ and so H_2 will map $[-\delta,\delta] \times (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ into $[-\delta,\delta]$. From this it follows that $H(\mathscr{B}) \subseteq \mathscr{B}$. For if $(\alpha,x) \in \mathscr{B}$, then $H_2(\alpha,x) \in [-\delta,\delta]$, while

$$\begin{aligned} |H_1(\alpha,x)| &\leqslant |\alpha||\Phi| + \|ME(\mathscr{F}(x)+\mathscr{G}(x))\|_\infty \\ &\leqslant \delta\|\Phi\|_\infty + \|ME\|m + \|ME\|J. \end{aligned}$$

Since M is compact and E, $\mathscr{F}$, and $\mathscr{G}$ are continuous and map bounded sets to bounded sets, H is completely continuous. So, the completely continuous function H maps the non-empty, closed, bounded, convex set $\mathscr{B}$ into itself. Hence, the Schauder Fixed Point Theorem guarantees existence of at least one fixed point, $\tilde{x}$, of H in $\mathscr{B}$. For each such $\tilde{x}$, $\tilde{y} = \tilde{x}_1$ is a solution of

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(y(t)) + (Gy)(t)$$

which satisfies

$$\begin{gathered} b_{11}y(0) + \cdots + b_{1n}y^{(n-1)}(0) + d_{11}y(1) + \cdots + d_{1n}y^{(n-1)}(1) = 0, \\ b_{21}y(0) + \cdots + b_{2n}y^{(n-1)}(0) + d_{21}y(1) + \cdots + d_{2n}y^{(n-1)}(1) = 0, \\ \vdots \\ b_{n1}y(0) + \cdots + b_{nn}y^{(n-1)}(0) + d_{n1}y(1) + \cdots + d_{nn}y^{(n-1)}(1) = 0. \end{gathered}$$

6. Final Remarks

In the case of a classical boundary value problem of the form

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(y(t))+g(t,y(t))$$

subject to (2), we can ensure solvability whenever

$$\sup\{|g(u,v)|:(u,v)\in\mathbb{R}^2\}\leqslant\min\{|J_1|,|J_2|\}.$$

Similarly, in the case of a integro-differential boundary value problem of the form

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(y(t))+\int_0^1 g(t,y(s))ds$$

subject to (2), we obtain the existence of solutions if

$$\sup\{|g(u,v)|:(u,v)\in\mathbb{R}^2\}\leqslant\min\{|J_1|,|J_2|\}.$$

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