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EXISTENCE OF POSITIVE SOLUTIONS OF A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS ON $\mathbb{R}^N$

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Abstract. In this paper we study the following problem: $-\Delta_p u + |u|^{p-2}u = k(x)f(u) + h(x)$, $x \in \mathbb{R}^N$, where $u \in W^{1,p}(\mathbb{R}^N)$, $u > 0$ in $\mathbb{R}^N$. Under appropriate assumptions on k, h and f, we prove that problem has at least two positive solutions.

1. Introduction

In this paper, we are concerned with the existence of positive solutions to the following nonhomogeneous quasilinear elliptic equations,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = k(x)f(u) + h(x), \ x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \ u > 0, \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < \infty$ and k is a positive bounded function, $h \in L^{p^*}(\mathbb{R}^N), h \geqslant 0, h \not\equiv 0$ and the function f satisfies the following conditions:

(H_1) $f \in C(\mathbb{R}, \mathbb{R}^+), f(0) = 0$ and $f(t) \equiv 0$ if $t < 0$;
(H_2) $\lim\limits_{t \to 0} f(t)/t^{p-1} = 0$;
(H_3) there exists $\delta \in (p-1, \frac{N+p}{N-p})$ such that $\lim\limits_{t \to \infty} f(t)/t^{\delta} = 0$;
(H_4) $\lim\limits_{t \to \infty} f(t)/t^{p-1} = l \leqslant +\infty$.

For $p = 2, h \not\equiv 0$, the existence of positive solutions to the following nonhomogeneous semilinear elliptic equations,

$$\begin{cases} -\Delta u + u = k(x)f(u) + h(x), \ x \in \mathbb{R}^N, \\ u \in H(\mathbb{R}^N), u > 0, \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \end{cases}$$

has been studied by many authors, see [11, 19, 22-23] and the references therein. For $p = 2$, $h \equiv 0$, that is the homogeneous case, has been studied extensively in the last

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decade, see [12, 15, 17] and the references therein. Still in [14] and [16], the authors consider the equations $-\Delta u = h(x)u^p + f(x,u)$ and $-\Delta u + V(x)u(x) = K(x)f(u)$ respectively, they all obtain the existence of the solution to the problem.

For $h \equiv 0$, $p > 1$, the existence and uniqueness of the positive solutions for the quasilinear elliptic equation with eigenvalue problems,

$$\begin{cases} \Delta_p u + \lambda f(u) = 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

with $\lambda > 0$, $p > 1$, $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ have been studied by many authors, see [1-2, 5-10, 20-21] and the references therein. When f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leqslant \alpha_1 + \alpha_2 s^\mu, 0 < \mu < p-1, \alpha_1, \alpha_2 > 0$, it was shown in [7] that there exists at least two positive solutions for Eqs (1.2) when λ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0$, $f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [8] that the problem (1.2) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [9] that problem (1.2) has a unique positive large solution and at least one positive small solution when λ is large if f is nondecreasing; there exists $\alpha_1, \alpha_2 > 0$ such that $f(s) \leqslant \alpha_1 + \alpha_2 s^\beta$, $0 < \beta < p-1$; $\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$, and there exists $T, Y > 0$ with $Y \geqslant T$ such that

$$(f(s)/s^{p-1})' > 0 \text{ for } s \in (0,T)$$

and

$$(f(s)/s^{p-1})' < 0 \text{ for } s > Y.$$

Recently, Hai [10] considered the case when Ω is an annular domain, and obtained the existence of positive large solutions for the problem (1.2) when λ sufficiently small. Xuan & Chen proved in [20] the singular problem (1.2) has a unique positive radial solution if f is a continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball).

Moreover, it was also shown in [21] that problem

$$\Delta_p u + q(x)u^{-\gamma} = 0, \ x \in \mathbb{R}^N$$

has a positive entire solution if $q \in C(\mathbb{R}^+), 0 \leqslant \gamma < p-1$,

$$\int_1^\infty r^{p+\varepsilon-1+[(N-p)|\gamma|/(p-1)]} q(r)dr < \infty,$$
$$0 < \varepsilon < (N-p)(p-1-|\gamma|)/(p-1),$$

and for $r \in (0,1)$, $\delta < 1$, $q(r) = O(r^{-\delta})$.

Still in [3], the authors studied the existence of nontrivial solutions for the problem

$$-\Delta_p u + |u|^{p-2}u = 0$$

in a bounded smooth domain $\Omega \subset R^N$ with a nonlinear boundary condition by variational and topological argument, and the authors in reference [18] obtained ground and bound state solution of quasilinear equation

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u)$$

with unbounded or decaying radial potential.

To the author's knowledge, it seems that there are few results for nonhomogeneous problem (1.1). Only more recently, in [13], Yun-Ho Kim studied the following boundary value problem with a nonhomogeneous principal part ϕ:

$$-\mathrm{div}(\phi(\nabla u)\nabla u) = \mu_0|u|^{p-2}u + q(\lambda, x, u, \nabla u) \quad \text{in } \Omega$$

with the dirichlet boundary condition under certain assumptions on ϕ and q when μ_0 is not an eigenvalue of the p-laplacian. The author showed a bifurcation result on a noncompact component of solutions for the above nonlinear equation. Motivated by the results of the above cited papers, we study the existence of two positive solutions of problem (1.1) under the condition (H_1)-(H_4) and some further assumption on h, the results of the semilinear equations are extended to the quasilinear ones. We modify the methods developed in [19, 23] and extend the results of [19] to a quasilinear elliptic equation (1.1).

We consider the following energy function $I : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by:

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + |u|^p)dx - \int_{\mathbb{R}^N} k(x)F(u)dx - \int_{\mathbb{R}^N} h(x)udx, \tag{1.3}$$

where $F(u) = \int_0^u f(s)ds$. It is known that a critical point of I is a weak solution of problem (1.1). By (H_1) and the strong maximum principle, we see that a nontrivial weak solution of problem (1.1) is indeed a positive solution. By the Ekeland's variational principle [4] it is not difficult to get a weak solution u_0 for $|h|_{L^{p^*}}$ suitably small. Moreover, u_0 is the local minimizer of I and $I(u_0) < 0$. However, under our assumptions which without the monotonicity of $f(t)$ and $f(t)/t^p$ it seems difficult to get a second solution (different from u_0 of (1.1) by applying the Mountain Pass Theorem as the mentioned reference), we have to find new ways to show that a (PS) sequence is bounded in $W^{1,p}(\mathbb{R}^N)$. On the other hand , once a (PS) sequence is bounded in $W^{1,p}(\mathbb{R}^N)$, the usual strategy is trying to show this sequence converges to a different solution from u_0, but this still seems not so easy under our assumption. Motivated by [19] here we study the problem (1.1) by the following two cases:

$$(A)\ \ l < +\infty; \quad \text{and} \quad (B)\ \ l = +\infty.$$

Noted that in both cases, we do not require that $\frac{f(t)}{t^{p-1}}$ and $f(t)$ are nondecreasing in $t \geqslant 0$.

In case (A), by using the fact $l < +\infty$, we can prove that the (PS) sequence obtained by mountain pass theorem converges strongly in $W^{1,p}(\mathbb{R}^N)$ to a solution u_1 of (1.1) with $I(u_1) > 0$ then it is clear that $u_1 \neq u_0$.

In case (B), the method for case (A) does not work any more. For proving a (PS) sequence converge to a different solution from u_0, we simply suppose that $k(x) \equiv 1$ and seek the solution in $W_r^{1,p}(\mathbb{R}^N)$, otherwise, we have to use the concentration compactness principle to show that the related (PS) sequence converges strongly in $W^{1,p}(\mathbb{R}^N)$ which is complicated and some more assumptions on k and f are required.

This paper is organized as follows. In section 2, we give the main results. In section 3, we give the proofs of the main theorems.

2. The main results of this paper

By modifying the methods developed in [19, 23], we obtain the next theorems.

THEOREM 2.1. *Suppose that* $h \in L^{p^*}(\mathbb{R}^N)$, $p^* = \frac{p}{p-1}$, $h \geqslant 0$, $h \not\equiv 0$ *and* $k \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ *satisfies,*

(K) *there exists* $R_0 > 0$ *such that*

$$\sup\{\frac{f(s)}{s^{p-1}} : s > 0\} < \inf\{\frac{1}{k(x)} : |x| \geqslant R_0\}. \tag{2.1}$$

Let (H_1)-(H_4) *hold and* $l \in (\mu^*, +\infty)$ *with*

$$\mu^* = \inf\Big\{ \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)dx \ : u \in W^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} k(x)u^p dx = 1\Big\}. \tag{2.2}$$

Then, there exists $m > 0$ *such that problem* (1.1) *has at least two positive solutions* $u_0, u_1 \in W^{1,p}(\mathbb{R}^N)$ *satisfying* $I(u_0) < 0$ *and* $I(u_1) > 0$ *if* $| h |_{L^{p^*}} < m$.

THEOREM 2.2. *Suppose that* $k(x) \equiv 1$, $h(x) = h(|x|) \in C^1(\mathbb{R}^N) \bigcap L^{p^*}(\mathbb{R}^N)$, $h(x) \geqslant (\not\equiv) 0$ *and satisfies,*

(G) *there exists* $\xi \in L^{p^*}(\mathbb{R}^N) \bigcap W^{1,\infty}(\mathbb{R}^N)$ *such that*

$$|\nabla h(x)||x| \leqslant |\xi|^{\frac{p}{p-1}}(x) \quad \textit{for all } x \in \mathbb{R}^N. \tag{2.3}$$

Let f *satisfies* (H_1)-(H_4) *with* $l = +\infty$, *then there exists* $m_1 > 0$ *such that problem* (1.1) *has two positive solutions* $v_0, v_1 \in W_r^{1,p}(\mathbb{R}^N)$ *with* $I(v_0) < 0$ *and* $I(v_1) > 0$ *for* $|h|_{L^{p^*}} < m_1$.

NOTATION. Through this paper, we denote by $\|\cdot\|$ the norm of $W^{1,p}(\mathbb{R}^N)$ and $C_1, C_2 \cdots$ are positive constants.

3. The proof of the main results

In this section, we consider the nonhomogeneous elliptic problem in two subsections.

3.1. Case A: $l < +\infty$.

In this part, we consider the following nonhomogeneous elliptic problem,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = k(x)f(u) + h(x), \ x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \ u > 0, \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \end{cases} \tag{3.1}$$

where $k(x)$ is a bounded positive function, $h \in L^{p^*}(\mathbb{R}^N)$, $h \geqslant 0$, $h \not\equiv 0$ and the function f satisfies (H_1)-(H_4) with $l < +\infty$, we have the following results for problem (3.1).

LEMMA 3.1.1. *Assume that* (H_1)-(H_4) *with* $l<+\infty$ *hold. Let* $h\in L^{p^*}(\mathbb{R}^N)$, k *satisfies* (K) *and* $\{u_n\}\subset W^{1,p}(\mathbb{R}^N)$ *be a bounded* (PS) *sequence of* I. *Then* $\{u_n\}$ *has a strongly convergent subsequence in* $W^{1,p}(\mathbb{R}^N)$.

Proof. It is sufficient to prove that for any $\varepsilon>0$ there exists $R(\varepsilon)>R_0$ (R_0 is given by (K)) and $n(\varepsilon)>0$ such that:

$$\int_{\mathbb{R}^N}(|\nabla u_n|^p+|u_n|^p)dx\leqslant\varepsilon \quad \text{for all } R\geqslant R(\varepsilon) \quad \text{and } n\geqslant n(\varepsilon). \tag{3.2}$$

Let $\xi_R:\mathbb{R}^N\to[0,1]$ be a smooth function such that

$$\xi_R(x)=\begin{cases}0, & 0\leqslant|x|\leqslant R,\\ 1, & |x|\geqslant 2R.\end{cases} \tag{3.3}$$

Moreover, there exists a constant C_0 independent of R such that

$$|\nabla\xi_R|\leqslant\frac{C_0}{R} \quad \text{for all } x\in\mathbb{R}^N. \tag{3.4}$$

Then for any $u\in W^{1,p}(\mathbb{R}^N)$, $u\neq0$ and all $R\geqslant1$ there exists a constant $C_1>0$ such that $\|\xi_R u\|\leqslant C_1\|u\|$. Since $I'(u_n)\to 0(n\to\infty)$ in $(W^{1,p}(\mathbb{R}^N))'$ and $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, we know that for any $\varepsilon>0$, there exists $n(\varepsilon)>0$ such that

$$\begin{aligned}\langle I'(u_n),\xi_R u_n\rangle&=\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(\xi_R u_n)+|u_n|^p\xi_R dx,\\ -\int_{\mathbb{R}^N}k(x)f(u_n)\xi_R u_n dx&-\int_{\mathbb{R}^N}h(x)\xi_R u_n dx\leqslant C_1\|I'(u_n)\|_{(W^{1,p}(\mathbb{R}^N))'}\|u_n\|\leqslant\frac{\varepsilon}{4},\end{aligned} \tag{3.5}$$

for $n\geqslant n(\varepsilon)$. That is, if $n\geqslant n(\varepsilon)$, then

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u_n|^p+|u_n|^p)\xi_R dx&+\int_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\xi_R dx\\ &\leqslant\int_{\mathbb{R}^N}k(x)f(u_n)\xi_R u_n dx+\int_{\mathbb{R}^N}h(x)\xi_R u_n dx+\frac{\varepsilon}{4}.\end{aligned} \tag{3.6}$$

It follows from (H_1) and (K) that there exists $0<\theta<1$ such that

$$k(x)f(u_n)u_n\leqslant\theta u_n^p \quad \text{for } |x|\geqslant R_0$$

since $h\in L^{p^*}(\mathbb{R}^N)$ and $\|u_n\|\leqslant C$ for some constant $C>0$, it follows from (3.3), there exists $R(\varepsilon)\geqslant R_0$ such that

$$\int_{\mathbb{R}^N}h(x)\xi_R u_n dx\leqslant|h(x)\xi_R|_{L^{p^*}}|u_n|_{L^p}\leqslant\frac{\varepsilon}{4} \quad \text{for } R\geqslant R(\varepsilon). \tag{3.7}$$

Then, for $R\geqslant R(\varepsilon)$ and $n>n(\varepsilon)$, combining (3.3)-(3.4) with (3.6), we deduce that

$$\int_{\mathbb{R}^N}(|\nabla u_n|^p+(1-\theta)|u_n|^p)\xi_R dx\leqslant\frac{C_0}{R}\|u_n\|+\frac{\varepsilon}{2}\leqslant\frac{C_2}{R}+\frac{\varepsilon}{2}. \tag{3.8}$$

Noting that the constant C_2 is independent on R, we can choose $R > 0$ large enough such that (3.2) holds, then we finish the proof of the lemma. $\square$

Now we give a property of the variational functional I defined by (1.3) which is required by using Ekeland's variational principle.

LEMMA 3.1.2. *If* (H_1)-(H_3) *hold,* $h \in L^{p^*}(\mathbb{R}^N)$ *and* $k \in L^\infty(\mathbb{R}^N)$, *then there exists* $\rho, \alpha, m > 0$ *such that* $I(u)\mid_{\|u\|=\rho} \geqslant \alpha > 0$ *for* $|h|_{L^{p^*}} < m$.

Proof. It follows from (H_1)-(H_3) that for any $\varepsilon > 0$ there exists $\delta \in (p-1, \frac{N+p}{N-p})$ and $A = A(\varepsilon, \delta) > 0$ such that for all

$$s > 0, F(s) \leqslant \varepsilon s^p + As^{\delta+1}. \tag{3.9}$$

By the Sobolev Embedding Theorem, we have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + |u|^p)dx - \int_{\mathbb{R}^N} k(x)F(u)dx - \int_{\mathbb{R}^N} h(x)udx \\ &\geqslant \frac{1}{p}\|u\| - \varepsilon\int_{\mathbb{R}^N} k(x)u^p - A\int_{\mathbb{R}^N} k(x)u^{\delta+1} - \int_{\mathbb{R}^N} h(x)udx \\ &\geqslant \frac{1}{p}\|u\| - C_3\varepsilon\|u\| - C_4(\varepsilon)\|u\|^{\delta+1} - |h|_{L^{p^*}}\|u\| \\ &= \|u\|\Big[(\frac{1}{p} - C_3\varepsilon)\|u\|^{p-1} - C_4(\varepsilon)\|u\|^{\delta} - |h|_{L^{p^*}}\Big]. \end{aligned} \tag{3.10}$$

Taking $\varepsilon = \frac{1}{2pC_3}$ and setting $g(t) = \frac{1}{2p}t^{p-1} - C_4 t^\delta$ for all $t \geqslant 0$. Since $\delta > p-1$ we see that there exists $\rho > 0$ such that $\max_{t\geqslant 0} g(t) = g(\rho) = m$. Then it follows from (3.10) that there exists $\alpha > 0$ such that

$$I(u)\mid_{\|u\|=\rho} \geqslant \alpha > 0 \text{ for } |h|_{L^{p^*}} < m.$$

For ρ given by Lemma 3.1.1, we denote

$$B_\rho = \{u \in W^{1,p}(\mathbb{R}^N) : \ \|u\|_{W^{1,p}} < \rho\}.$$

Then by Ekeland's variational principle and Lemma 3.1.2, we have the following lemma which show that I has a local minimum if $|h|_{L^{p^*}}$ is small.

LEMMA 3.1.3. *Assume that* (H_1)-(H_4) *with* $l < +\infty$ *hold,* $h \in L^{p^*}, h \geqslant (\not\equiv) 0$ *and* k *satisfies* (1.3), *if* $|h|_{L^{p^*}} < m, m$ *is given by Lemma 3.1.2, then there exists* $u_0 \in W^{1,p}(\mathbb{R}^N)$ *such that* $I(u_0) = \inf\{I(u) : u \in \overline{B_\rho}\} < 0$ *and* u_0 *is a positive solution of problem* (3.1).

Proof. Since $h \in L^{p^*}, h \geqslant 0$ and $h \not\equiv 0$, we can choose a function $\varphi \in W^{1,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} h(x)\varphi(x)dx > 0. \tag{3.11}$$

For $t > 0$, we have

$$I(t\varphi) = \frac{t^p}{p}\int_{\mathbb{R}^N}(|\nabla\varphi|^p + |\varphi|^p)dx - \int_{\mathbb{R}^N} k(x)F(t\varphi)dx - t\int_{\mathbb{R}^N} h(x)\varphi dx$$
$$\leqslant \frac{t^p}{p}\|\varphi\|^p - t\int_{\mathbb{R}^N} h(x)\varphi dx < 0 \tag{3.12}$$

for $t > 0$ small enough. Hence $c_0 = \inf\{I(u) : u \in \bar{B_\rho}\} < 0$. By the Ekeland's variational principle, there exists $\{u_n\} \subset \bar{B_\rho}$ such that

$$c_0 \leqslant I(u_n) < c_0 + \frac{1}{n} \quad \text{and} \quad I(w) \geqslant I(u_n) - \frac{1}{n}\|w - u_n\| \quad \text{for all } w \in \bar{B_\rho}.$$

Then by a standard procedure, we can show that $\{u_n\}$ is a bounded (PS) sequence of I, hence, Lemma 3.1.1 implies that there exists $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Next, we prove the problem (3.1) has a mountain pass type solution. For this purpose, we use a variant version of Mountain Pass Theorem which allows us to find a so-called Cerami type (PS) sequence. Let us recall this theorem, and the proof can be found in [3, Chapter IV]

PROPOSITION 3.1. (Mountain Pass Theorem) *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E,R)$ satisfies*

$$max\{I(0), I(e)\} \leqslant \mu < \eta \leqslant \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta$, $\rho > 0$ and $\|e\| > \rho$. Let $c \geqslant \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{0\leqslant\tau\leqslant 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geqslant \eta \quad and \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0.$$

The following lemma shows that I defined in (1.3) has the so-called mountain pass geometry.

LEMMA 3.1.4. *Suppose that (H_1)-(H_4) hold and $l \in (\mu^*, +\infty)$ with μ^* given by (2.2). Then there exists $v \in W^{1,p}(\mathbb{R}^N)$ with $\|v\| > \rho$ (ρ is given by Lemma 3.1.2), such that $I(v) < 0$.*

Proof. Since $l > \mu^*$, we can choose a nonnegative function $\phi \in W^{1,p}(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N} k(x)\phi^p dx = 1 \quad \text{such that} \quad \int_{\mathbb{R}^N}(|\nabla\phi|^p + \phi^p)dx < l. \tag{3.13}$$

Therefore, by (H_4) and Fatou's Lemma we deduce that

$$\lim_{t\to\infty}\frac{I(t\phi)}{t^p}=\frac{1}{p}\|\phi\|^p-\lim_{t\to\infty}\int_{\mathbb{R}^N}k(x)\frac{F(t\phi)}{t^p}dx-\lim_{t\to\infty}\frac{1}{t^{p-1}}\int_{\mathbb{R}^N}h(x)\phi dx$$
$$\leqslant\frac{1}{p}(\|\phi\|^p-l)<0 \tag{3.14}$$

and the lemma is proved by taking $v=t_0\phi$ with $t_0>0$ large enough.

From Lemma 3.1.2 and 3.1.4, there is a sequence $\{u_n\}\subset W^{1,p}(\mathbb{R}^N)$ such that

$$I(u_n)\to c \quad\text{and}\quad \|I'(u_n)\|_{(W^{1,p})'}(1+\|u_n\|)\to 0 \quad\text{as}\quad n\to\infty, \tag{3.15}$$

where $(W^{1,p})'$ denotes the dual space of $W^{1,p}(\mathbb{R}^N)$. For this sequence $\{u_n\}$, let

$$w_n=\frac{u_n}{\|u_n\|}. \tag{3.16}$$

Clearly, $\{w_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and there is a $w\in W^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence:

$$\begin{aligned}&w_n\rightharpoonup w \ \text{ weakly in } W^{1,p}(\mathbb{R}^N),\\&w_n\to w \ \text{ a.e. in } \mathbb{R}^N,\\&w_n\to w \ \text{ strongly in } L^p_{loc}(\mathbb{R}^N).\end{aligned} \tag{3.17}$$

For the above w we have the following lemma.

LEMMA 3.1.5. *Assume that (H_1)-(H_4) and (K) hold. Let $h\in L^{p^*}(\mathbb{R}^N)$ and $l\in(\mu^*,+\infty)$ for u^* given by (2.2). If $\|u_n\|\to+\infty$ as $n\to+\infty$, then w given by (3.17) is a nontrivial nonnegative solution of*

$$-\Delta_p u+|u|^{p-2}u=lk(x)u,\ \ u\in W^{1,p}(\mathbb{R}^N). \tag{3.18}$$

Proof. We prove this lemma through the following three steps.

Step 1: $w\not\equiv 0$. By contradiction, if $w\equiv 0$, the Sobolev Embedding Theorem implies that $w_n\to 0$ strongly in $L^p(B_{R_0})$ as $n\to+\infty$, R_0 is given by (K) and then by (H_1), (H_4) and $l<+\infty$, there exists $C_5>0$ such that

$$\frac{f(t)}{t^{p-1}}\leqslant C_5 \quad\text{for all } t\in\mathbb{R}. \tag{3.19}$$

Hence

$$\int_{|x|<R_0}k(x)\frac{f(u_n)}{u_n^{p-1}}|w_n|^p dx\leqslant C_5|k|_\infty\int_{|x|<R_0}|w_n|^p dx\to 0 \quad\text{as}\quad n\to+\infty \tag{3.20}$$

on the other hand by (K) there exists $\eta\in(0,1)$ such that

$$\sup\{\frac{f(s)}{s^{p-1}}:s>0\}<\eta\inf\{\frac{1}{k(x)}:|x|\geqslant R_0\} \tag{3.21}$$

and for all $n \in \mathbb{R}$,

$$\int_{|x|\geqslant R_0} k(x)\frac{f(u_n)}{u_n^{p-1}}|w_n|^p dx \leqslant \eta \int_{|x|\geqslant R_0} k(x)|w_n|^p dx \leqslant \eta < 1. \tag{3.22}$$

Therefore, from (3.20) and (3.22), we see that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{u_n^{p-1}}|w_n|^p dx < 1. \tag{3.23}$$

However, since $\|u_n\| \to \infty$, it follows from (3.15) that

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^p} = o(1), \tag{3.24}$$

that is

$$o(1) = \|w_n\|^p - \int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{u_n^{p-1}}|w_n|^p dx = 1 - \int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{u_n^{p-1}}|w_n|^p dx, \tag{3.25}$$

where and in what follows, $o(1)$ denotes a quality which goes to zero as $n \to +\infty$, clearly this contradicts (3.23), so $w \not\equiv 0$.

Step 2: $w \geqslant 0$. In this step, we show that w is nonnegative, that is $w \geqslant 0$. Let $w_n^-(x) = \max\{-w_n, 0\}$, and $\{w_n^-\}$ is also bounded in $W^{1,p}(\mathbb{R}^N)$, if $\|u_n\| \to \infty$, then

$$\frac{\langle I'(u_n), w_n^-\rangle}{\|u_n\|^{p-1}} = o(1). \tag{3.26}$$

That is

$$-\|w_n^-\|^p = \int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{\|u_n\|^{p-1}} w_n^- dx + o(1). \tag{3.27}$$

By (H_1), $f(t) \equiv 0$ for all $t \leqslant 0$, then (3.27) implies $\lim_{n\to\infty}\|w_n^-\| = 0$. Thus $w_n^- = 0$ a.e. $x \in \mathbb{R}^N$ and $w \geqslant 0$.

Step 3: w solves (3.18). By (3.15) and $\|u_n\| \to \infty$, we have

$$\frac{\langle I'(u_n), \phi\rangle}{\|u_n\|^{p-1}} = o(1) \quad \text{for any} \quad \phi \in C_0^\infty(\mathbb{R}^N), \tag{3.28}$$

that is

$$\int_{\mathbb{R}^N} |\nabla w_n|^{p-2}\nabla w_n \nabla\phi + |w_n|^{p-2} w_n \phi dx = \int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{u_n^{p-1}} w_n \phi dx + o(1). \tag{3.29}$$

That implies that

$$\int_{\mathbb{R}^N} |\nabla w|^{p-2}\nabla w \nabla\phi + |w|^{p-2} w \phi dx = \int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{u_n^{p-1}} w_n \phi dx + o(1) \tag{3.30}$$

since $w_n \rightharpoonup w$ weakly in $W^{1,p}(\mathbb{R}^N)$, hence to show w solves equation (3.18), it is sufficient to show that

$$\int_{\mathbb{R}^N} k(x)\frac{f(u_n)}{u_n^{p-1}}w_n\phi dx \to \int_{\mathbb{R}^N} lk(x)w(x)\phi dx \quad \text{as } n\to+\infty. \tag{3.31}$$

Let

$$\Omega_+ = \{x\in\mathbb{R}^N : w(x)>0\} \quad \text{and } \Omega_0 = \{x\in\mathbb{R}^N : w(x)=0\}, \tag{3.32}$$

then by (3.17) we have

$$w_n = \frac{u_n}{\|u_n\|} \to w \text{ a.e. in } \Omega_+ \text{ and } \|u_n\|\to+\infty,$$

so we get that $u_n(x)\to+\infty$ a.e. in Ω_+. Hence, (H_4) implies that

$$\frac{f(u_n)}{u_n^{p-1}}w_n(x)\to lw(x) \quad \text{a.e. in } \Omega_+$$

since $w_n\to 0$ a. e. in Ω_0, it follows from (3.19) that

$$\frac{f(u_n)}{u_n^{p-1}}w_n(x)\to 0\equiv lw(x) \quad \text{a.e. in } \Omega_0.$$

Hence, by (3.19) and $\|w_n\|=1$, we have

$$\frac{f(u_n)}{u_n^{p-1}}w_n(x)\rightharpoonup lw(x) \quad \text{weakly in } L^p(\mathbb{R}^N). \tag{3.33}$$

So, for $\phi\in C_0^\infty(\mathbb{R}^N)$ and $k\in L^\infty(\mathbb{R}^N)$ we have that

$$\int_{\mathbb{R}^N} |\nabla w|^{p-2}\nabla w\nabla\phi + |w|^{p-2}w\phi dx = \int_{\mathbb{R}^N} lk(x)w(x)\phi(x)dx,$$

so w is the solution of equation (3.18) in weak sense.

LEMMA 3.1.6. *Let us suppose that $k\in L^\infty(\mathbb{R}^N,\mathbb{R}^+)$ and let μ^* be defined by (2.2) with $l\in(\mu^*,+\infty)$. Then (3.18) has no any nontrivial nonnegative solution.*

Proof. Since $l>\mu^*$, there is a constant $\sigma>0$ such that $\mu^*<\mu^*+\sigma<l$. By the definition of μ^* in (2.2), there exists $v_\sigma\in W^{1,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} k(x)v_\sigma^p dx = 1 \quad \text{and} \quad \mu^*\leqslant \|v_\sigma\|_{W^{1,p}}^p < \mu^*+\sigma. \tag{3.34}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we may assume $v_\sigma\in C_0^\infty(\mathbb{R}^N)$. Let $R>0$ be such that $\mathrm{Supp} v_\sigma\subset B_R$ and define

$$\mu_R = \inf\Big\{\int_{B_R}(|\nabla u|^p+|u|^p)dx : \int_{B_R} k(x)u^p dx = 1,\ u\in W_0^{1,p}(B_R)\Big\}. \tag{3.35}$$

Then $v_\sigma \in W_0^{1,p}(B_R)$ and

$$\mu_R \leqslant \|v_\sigma\|^p < \mu^* + \sigma < l. \tag{3.36}$$

By the compactness of the embedding $W_0^{1,p}(B_R) \hookrightarrow L^q(B_R)$, it is not difficult to see that there exists $w_R \in W_0^{1,p}(B_R)\backslash\{0\}$ with $w_R \geqslant 0$ and $\int_{B_R} k(x) w_R^p(x)dx = 1$ such that

$$-\Delta_p w_R + |w_R|^{p-2} w_R = \mu_R k(x) w_R, \quad x \in B_R. \tag{3.37}$$

It follows from the strong maximum principle that

$$w_R(x) > 0,\ \forall x \in B_R,\ \ \frac{\partial w_R}{\partial \nu} < 0,\ \forall |x| = R. \tag{3.38}$$

Therefore, if $0 \neq u \in W^{1,p}(R^N)$ is a nonnegative solution, then

$$\begin{aligned}
\mu_R \int_{B_R} k(x) w_R u dx &= \int_{B_R} (-\Delta_p w_R + |w_R|^{p-2} w_R) u dx \\
&= \int_{B_R} |\nabla w_R|^{p-2} \nabla w_R \nabla u + \int_{B_R} |w_R|^{p-2} w_R u dx \\
&\quad - \int_{\partial B_R} |\nabla w_R|^{p-2} \frac{\partial w_R}{\partial \nu} u d\sigma \\
&= \int_{B_R} l k(x) u w_R u dx - \int_{\partial} B_R |\nabla w_R|^{p-2} \frac{\partial w_R}{\partial \nu} u d\sigma \\
&\geqslant l \int_{B_R} k(x) u w_R u dx
\end{aligned}$$

using $u \geqslant 0$ and $u \neq 0$, we may choose $R > 0$ large enough such that $\int_{B_R} k(x) u w_R u dx > 0$ so, the above inequality implies that $\mu_R \geqslant l$. This contradicts (3.36).

PROOF OF THEOREM 2.1. From the above Lemma 3.1.5 and 3.1.6, if $\|u_n\| \to +\infty$, we get a contradiction. Hence, $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Then by Lemma 3.1.1, we see that problem (3.1) has a positive solution $u_1 \in W^{1,p}(\mathbb{R}^N)$ with $I(u_1) > 0$. So, we finish the proof of Theorem 2.1.

3.2. Case B: $l = \infty$.

In this section, we consider the problem (1.1) with $k(x) \equiv 1$, that is,

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = f(u) + h(x),\ \ x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N),\ u > 0, \quad x \in\ \mathbb{R}^N,\ N \geqslant 3, \end{cases} \tag{3.39}$$

where $h(x) = h(|x|) \in C^1 \bigcap L^{p^*}(\mathbb{R}^N), h \geqslant (\not\equiv) 0$ and f satisfies (H_1)-(H_4) with $l = +\infty$. Since we assume that $k(x) \equiv 1$ and h is radial, it is known that the energy function I in (1.3) can be defined on $W_r^{1,p}(\mathbb{R}^N)$, the subspace of radial function of $W^{1,p}(\mathbb{R}^N)$. Moreover, a non-zero critical point of I is a solution of problem (3.39).

LEMMA 3.2.1. *Suppose that* $h(x) = h(|x|) \in L^{p^*}(\mathbb{R}^N)$, $h \geqslant (\not\equiv) 0$, *and conditions* (H_1)-(H_3) *hold. Then there exists* $m_1 > 0$ *and* $v_0 \in W_r^{1,p}(R^N)$ *such that* $I'(v_0) = 0$ *and* $I(v_0) < 0$ *if* $|h|_{L^{p^*}} < m_1$.

Proof. Similar to the proof of Lemma 3.1.3, by the Ekeland's variational principle, we can find a bounded (PS) sequence $\{v_n\} \subset W_r^{1,p}(\mathbb{R}^N)$ such that

$$I(v_n) \to \tilde{b_0} := \inf\{I(v) : v \in W_r^{1,p}(\mathbb{R}^N) \text{ and } \|v\| = \rho\} < 0, \tag{3.40}$$

where ρ is given by Lemma 3.1.2. Then from (H_1)-(H_3) and the compactness of the embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)(p < q < \frac{pN}{N-p})$, there exists $v_0 \in W_r^{1,p}(\mathbb{R}^N)$ such that $v_n \to v_0$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. Hence, we get $I(v_0) = \tilde{b_0} < 0$ and $I'(v_0) = 0$.

Next, we prove that problem (3.39) has a mountain pass type solution. In order to prove this, we use the following theorem which given in [9].

LEMMA 3.2.2. (see [9, Theorem 1.1]). *Let* X *be a Banach space equipped with a norm* $\|u\|_X$ *and let* $J \subset \mathbb{R}^+$ *be an interval. Consider a family* $(I_\lambda)_{\lambda \in J}$ *of* C^1 *function on* X *of the form*

$$I_\lambda(u) = A(u) - \lambda B(u) \quad \textit{for} \quad \lambda \in J,$$

such that $A(u) \to +\infty$ *as* $\|u\|_X \to +\infty$. *Suppose that there are two points* v_1, v_2 *in* X *such that*

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in(0,1)} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \textit{for } \lambda \in J,$$

where $\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$. *Then for almost every* $\lambda \in J$, *there is a sequence* $\{v_n\} \subset X$ *such that:*

(1) $\{v_n\}$ *is bounded in* X,
(2) $I_\lambda(v_n) \to c_\lambda$ *and*
(3) $I'_\lambda(v_n) \to 0$ *in* X^{-1} *as* $n \to +\infty$.

Moreover, the map $\lambda \to c_\lambda$ *is continuous from the left.*

For $\lambda \in [\frac{1}{p}, 1]$, we define the family of functionals $I_\lambda : W_r^{1,p}(\mathbb{R}^N) \to R$ by

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}_N} (|\nabla u|^p + |u|^p)dx - \lambda \int_{\mathbb{R}^N} (F(u) + h(x)u)dx.$$

LEMMA 3.2.3. *Assume that* (H_1)-(H_4) *with* $l = +\infty$ *hold, then:*
(1) *there exists* $\bar{v} \in W_r^{1,p}(\mathbb{R}^N)\backslash\{0\}$ *such that* $I_\lambda(\bar{v}) < 0$ *for all* $\lambda \in [\frac{1}{p}, 1]$;
(2) *for* $m_1 > 0$ *given in Lemma 3.2.1, if* $|h|_{L^{p^*}} < m_1$, *then*

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in(0,1)} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(\bar{v})\} \quad \textit{for all } \lambda \in [\frac{1}{p}, 1],$$

where $\Gamma = \{\gamma \in C([0,1], W_r^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = \bar{v}\}$.

Proof. (1) For $\sigma > 0$, there exists $v \in W_r^{1,p}(\mathbb{R}^N)\backslash\{0\}$ and $v \geqslant 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^p dx < \sigma \int_{\mathbb{R}^N} |v|^p dx.$$

Since

$$\inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^p dx : u \in W_r^{1,p}(\mathbb{R}^N) \quad \text{and} \quad |u|_{L^p} = 1 \Big\} = 0.$$

By (H_4) with $l = +\infty$ and Fatou's Lemma, we have

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} \frac{F(tv)}{t^p} dx \geqslant (1+\sigma) \int_{\mathbb{R}^N} |v|^p dx \tag{3.41}$$

and then for any $\lambda \in [\frac{1}{p}, 1]$,

$$\begin{aligned} \lim_{t\to+\infty} \int_{\mathbb{R}^N} \frac{I_\lambda(tv)}{t^p} dx &\leqslant \lim_{t\to+\infty} \int_{\mathbb{R}^N} \frac{I_{\frac{1}{p}}(tv)}{t^p} dx \\ &\leqslant \frac{1}{p}\Big(\int_{\mathbb{R}^N} |\nabla v|^p dx - \sigma \int_{\mathbb{R}^N} |v|^p dx\Big) < 0, \end{aligned} \tag{3.42}$$

so, we can choose $t_1 > 0$ large enough such that $I_{\frac{1}{p}}(tv) < 0$. Then taking $\bar{v} = t_1 v$, we see that $I_\lambda(\bar{v}) \leqslant I_{\frac{1}{p}}(\bar{v}) < 0$ and (1) holds.

(2) For any $\lambda \in [\frac{1}{p}, 1]$ and $u \in W_r^{1,p}(\mathbb{R}^N)$, we have

$$I_\lambda(u) \geqslant \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx - \int_{\mathbb{R}^N} (F(u) dx - |h|_{L^{p^*}} |u|_{L^p}) = J(u).$$

Similar to the proof of Lemma 3.1.2, if $|h|_{L^{p^*}} < m_1$ with m_1 given by Lemma 3.2.1, we deduce that $\inf_{\gamma\in\Gamma} \max_{t\in(0,1)} J(\gamma(t)) > 0$. Then

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in(0,1)} I_\lambda(\gamma(t)) \geqslant \inf_{\gamma\in\Gamma} \max_{t\in(0,1)} J(\gamma(t)) > \max\Big\{ I_\lambda(0), I_\lambda(\bar{v}) \quad \text{for all } \lambda \in [\frac{1}{p}, 1] \Big\}$$

and we complete the proof of the lemma. □

By Lemma 3.2.3 and Lemma 3.2.2, there exists $\{\lambda_j\} \subset [\frac{1}{p}, 1]$ such that:
(a) $\lambda_j \to 1$ as $j \to +\infty$, and
(b) I_{λ_j} has a bounded (PS) sequence $\{u_n^j\}$ at the level c_{λ_j}.

Since the embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) (p < q < \frac{Np}{N+p})$ is compact, we deduce that for each $j \in \mathbb{N}$, there exists $u_j \in W_r^{1,p}(\mathbb{R}^N)$ such that $u_n^j \to u_j$ strongly in $W_r^{1,p}(\mathbb{R}^N)$ and u_j is a positive solution of

$$-\Delta_p u + |u|^{p-2} u = \lambda_j (f(u) + h(x)) \quad \text{in } \mathbb{R}^N. \tag{3.43}$$

Next, we deduce the Pohozaev type identity which u_j is satisfies

$$\frac{N-p}{p}\int_{\mathbb{R}^N}|\nabla u_j|^p dx+\frac{N}{p}\int_{\mathbb{R}^N}|u_j|^p dx = N\lambda_j\int_{\mathbb{R}^N}(F(u_j)+hu_j)dx+\lambda_j\int_{\mathbb{R}^N}\nabla h(x)\cdot xu_j dx. \quad (3.44)$$

In order to prove that $\{u_j\}$ is a bounded (PS) sequence of I, we need the technique condition (K).

LEMMA 3.2.4. *Assume that* (H_1)-(H_4) *with* $l=+\infty$ *hold,* h *satisfies* (1.5) *and* $|h|_{L^{p^*}}<m_1$ *for* m_1 *given by Lemma 3.1.1. Then* $\{u_j\}\subset W_r^{1,p}(\mathbb{R}^N)$ *is bounded.*

Proof. Since the map $\lambda\to c_\lambda$ is continuous from the left by Lemma 3.2.2, then by Lemma 3.2.3 (b), we have

$$I_{\lambda_j}(u_j)=c_{\lambda_j}\to c_1>0 \quad \text{as } \lambda_j\to 1.$$

Thus, there exists $K>0$ such that $I_{\lambda_j}(u_j)\leqslant K$ for all $j\in N$. From this and (3.44) we deduce that

$$\int_{\mathbb{R}^N}|\nabla u_j|^p dx+\lambda_j\int_{\mathbb{R}^N}\nabla h\cdot xu_j dx=NI_{\lambda_j}(u_j)$$

so

$$\int_{\mathbb{R}^N}|\nabla u_j|^p dx\leqslant KN+\int_{\mathbb{R}^N}|\nabla h||x||u_j|dx. \quad (3.45)$$

Then by (G) we have

$$\int_{\mathbb{R}^N}|\nabla u_j|^p dx\leqslant KN+\int_{\mathbb{R}^N}|\xi|^{\frac{p}{p-1}}|u_j|dx \leqslant KN+\frac{1}{p^*}\int_{\mathbb{R}^N}|\xi|^{p^*}dx+\int_{\mathbb{R}^N}|\xi^{\frac{p}{p-1}}u_j|^p dx. \quad (3.46)$$

Since $\xi\in L^{p^*}(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$, we know that $\xi^{p^*}u_j\in W^{1,p}(\mathbb{R}^N)$. It follows from $\langle I'_{\lambda_j}(u_j),\xi^{p^*}u_j\rangle=0$ that

$$\int_{\mathbb{R}^N}|\nabla u_j|^{p-2}\nabla u_j\nabla(\xi^{\frac{p}{p-1}}u_j)+\xi^{\frac{p}{p-1}}u_j^p dx=\lambda_j\int_{\mathbb{R}^N}(f(u_j)+h)\xi^{\frac{p}{p-1}}u_j dx. \quad (3.47)$$

It follows from (3.45) that (denoting by $p^* = \frac{p}{p-1}$)

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_j|^{p-2}\nabla u_j \nabla(\xi^{\frac{p}{p-1}} u_j)dx \\
&\quad\leqslant \int_{\mathbb{R}^N} |\nabla u_j|^p \xi^{\frac{p}{p-1}} dx + \frac{p}{p-1}\int_{\mathbb{R}^N} |\nabla u_j|^{p-2}\nabla u_j \nabla\xi u_j \xi^{\frac{p}{p-1}} dx \\
&\quad\leqslant \int_{\mathbb{R}^N} |\nabla u_j|^p \xi^{\frac{p}{p-1}} dx + \int_{\mathbb{R}^N} (|\nabla u_j|^{p-2}(\nabla u_j \nabla \xi)^{\frac{p}{p-1}} \\
&\qquad + \frac{1}{p-1}\int_{\mathbb{R}^N} (u_j \xi^{\frac{p}{p-1}})^p dx \\
&\quad\leqslant (|\xi|_\infty^{p^*} + |\nabla\xi|_\infty^{p^*}) \int_{\mathbb{R}^N} |\nabla u_j|^p dx + \frac{1}{p-1}\int_{\mathbb{R}^N} (u_j \xi^{\frac{p}{p-1}})^p dx \\
&\quad\leqslant C_6 \int_{\mathbb{R}^N} |\nabla u_j|^p dx + C_7 \int_{\mathbb{R}^N} (u_j \xi^{\frac{p}{p-1}})^p dx.
\end{aligned} \tag{3.48}$$

By (H_4) with $l = +\infty$, for any $L > 0$, there exists $C(L) > 0$ such that

$$f(s)s \geqslant Ls^p + C(L) \quad \text{for all } s > 0, \tag{3.49}$$

so

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_j|^{p-2}\nabla u_j \nabla(\xi^{\frac{p}{p-1}} u_j)dx \\
&\qquad = \lambda_j \int_{\mathbb{R}^N} (f(u_j) + h(x))\xi^{\frac{p}{p-1}} u_j^p dx - \int_{\mathbb{R}^N} \xi^{\frac{p}{p-1}} u_j^p dx \\
&\qquad \geqslant \frac{1}{p}\int_{\mathbb{R}^N} (f(u_j) + h(x))\xi^{\frac{p}{p-1}} u_j dx - \int_{\mathbb{R}^N} \xi^{\frac{p}{p-1}} u_j^p dx \\
&\qquad \geqslant \frac{L}{p}\int_{\mathbb{R}^N} \xi^{\frac{p}{p-1}} u_j^p dx - \int_{R^N} \xi^{\frac{p}{p-1}} u_j^p dx + C_8.
\end{aligned} \tag{3.50}$$

From the above statements (3.46), (3.48), and (3.50), we obtain:

$$\begin{aligned}
&\frac{L}{p}\int_{\mathbb{R}^N} \xi^{\frac{p}{p-1}} u_j^p dx - \int_{R^N} \xi^{\frac{p}{p-1}} u_j^p dx + C_8 \\
&\qquad \leqslant C_6 \int_{\mathbb{R}^N} |\nabla u_j|^p dx + C_7 \int_{\mathbb{R}^N} (u_j \xi^{\frac{p}{p-1}})^p dx \\
&\qquad \leqslant C_6\Big(KN + \frac{1}{p^*}\int_{\mathbb{R}^N} |\xi|^{p^*} dx + \int_{\mathbb{R}^N} |\xi^{\frac{p}{p-1}} u_j|^p dx\Big) \\
&\qquad\qquad + C_7 \int_{\mathbb{R}^N} (u_j \xi^{\frac{p}{p-1}})^p dx \\
&\qquad \leqslant C_9 + C_{10}\int_{\mathbb{R}^N} |\xi^{\frac{p}{p-1}} u_j|^p dx.
\end{aligned} \tag{3.51}$$

That is

$$\frac{L}{p}\int_{\mathbb{R}^N} \xi^{\frac{p}{p-1}} u_j^p dx \leqslant C_{11} \int_{\mathbb{R}^N} \xi^{\frac{p}{p-1}} u_j^p dx + C_{12} \tag{3.52}$$

taking $L>0$ large enough, we get that $\int_{\mathbb{R}^N}\xi^{\frac{p}{p-1}}u_j^p dx$ is bounded, since (3.46), we easily know that $\int_{\mathbb{R}^N}|\nabla u_j|^p$ is bounded, ect $|\nabla u_j|_{L^p}$ is bounded. It follows from $I_{\lambda_j}(u_j)\leqslant K$ for all $j\in N$ that

$$\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u_j|^p dx+|u_j|^p dx-\lambda_j\int_{\mathbb{R}^N}(F(u_j)+h(x)u_j)dx\leqslant K. \tag{3.53}$$

By (H_2)-(H_3) there exists a constant $C>0$ such that

$$F(u_j)\leqslant\frac{1}{2p}|u_j|^p+Cu_j^{P^*},\ \ p^*=\frac{Np}{N-p}. \tag{3.54}$$

Then, substituting this inequality into (3.53) and by the Sobolev inequality we deduce that

$$\frac{1}{2p}|u_j|^p dx\leqslant C+|h|_{p^*}|u_j|_p.$$

Thus, $\{|u_j|_p\}$ is bounded and we complete the proof of the lemma.

LEMMA 3.2.5. *Under the assumptions of Lemma 3.2.4, the above sequence $\{u_j\}$ is also a (PS) sequence for I.*

Proof. From the definition of I and I_{λ_j}, we have

$$I(u_j)=I_{\lambda_j}(u_j)+(\lambda_j-1)\int_{\mathbb{R}^N}(F(u_j)+h(x)u_j)dx.$$

By Lemma 3.2.2, we know that $I_{\lambda_j}(u_j)=c_{\lambda_j}\to c_1>0$ as $\lambda_j\to 1$. Then from Lemma 3.2.4 we get $I(u_j)\to c_1>0$. Since $I'_{\lambda_j}(u_j)=0$, for any $\varphi\in W_r^{1,p}(\mathbb{R}^N)$, we have :

$$\langle I'(u_j),\varphi\rangle=\langle I'_{\lambda_j}(u_j),\varphi\rangle+(\lambda_j-1)\int_{\mathbb{R}^N}(f(u_j)+h(x))\varphi dx\to 0.$$

Thus $I'(u_j)\to 0$ in the dual space of $W_r^{1,p}(\mathbb{R}^N)$.

PROOF OF THEOREM 2.2. By Lemma 3.2.1, problem (3.39) has a positive solution $v_0\in W_r^{1,p}(\mathbb{R}^N)$ with $I(v_0)<0$. On the other hand, from Lemma 3.2.5 and the compactness of the embedding $W_r^{1,p}(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)(p<q<p^*)$, we know that problem (3.39) possess a second positive solution $v_1\in W_r^{1,p}(\mathbb{R}^N)$ with $I(v_1)=c_1>0$. Hence, $v_0\neq v_1$ and we complete the proof of Theorem 2.2.

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