Differential
Equations
& Applications
Volume 2, Number 4 (2010), 519–535

# SOME RESULTS ON WEIGHTED CRITICAL QUASILINEAR PROBLEMS

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(*Communicated by C. L. Tang*)

*Abstract.* In this paper, we are concerned with a kind of quasilinear elliptic problems, which involves the Caffarelli-Kohn-Nirenberg inequality and critical exponents. By employing variational methods and analytical techniques, the existence of sign-changing solutions to the problem is proved.

## 1. Introduction

In this paper, we investigate the following elliptic problem:

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big) - \mu\dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = \dfrac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} + \lambda\dfrac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \\ u \in W_{0,a}^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and

$$0 \in \Omega,\ N \geqslant 3,\ \lambda > 0,\ 1 < p < N,$$
$$0 \leqslant \mu < \bar{\mu},\ \bar{\mu} := \Big(\frac{N-p}{p} - a\Big)^p,$$
$$0 \leqslant a < \frac{N-p}{p},\ a \leqslant b,\ d < a+1,\ p \leqslant q < p^*(a,d),$$

where

$$p^*(a,b) := \frac{Np}{N-p(a+1-b)} \quad \text{and} \quad p^*(a,d) := \frac{Np}{N-p(a+1-d)}$$

are the critical Hardy-Sobolev exponents and $p^*(0,0) = p^* := Np/(N-p)$ is the critical Sobolev exponent. The space $W_{0,a}^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\big)^{1/p}$.

*Mathematics subject classification* (2010): 35J25, 35J60, 35B33.
*Keywords and phrases*: solution, elliptic problem, Caffarelli-Kohn-Nirenberg inequality, variational method.
This work is supported by the National Natural Science Foundation of China (10771219).


Paper DEA-02-31

By a solution to the problem (1.1), we mean a function $u \in W_{0,a}^{1,p}(\Omega)$ such that the following equality holds for all $v \in W_{0,a}^{1,p}(\Omega)$:

$$\int_\Omega \Big( \frac{|\nabla u|^{p-2}\nabla u \nabla v}{|x|^{ap}} - \mu \frac{|u|^{p-2}uv}{|x|^{p(a+1)}} - \frac{|u|^{p^*(a,b)-2}uv}{|x|^{bp^*(a,b)}} - \lambda \frac{|u|^{q-2}uv}{|x|^{dp^*(a,d)}} \Big) \mathrm{d}x = 0.$$

By the standard elliptic regularity argument, $u \in C^1(\Omega \setminus \{0\})$.

Problem (1.1) is related to the Caffarelli-Kohn-Nirenberg inequality [3]:

$$\left( \int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \mathrm{d}x \right)^{\frac{p}{p^*(a,b)}} \leqslant C \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}} \mathrm{d}x, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N), \tag{1.2}$$

which is also named as the (weighted or general) Hardy-Sobolev inequality. For the sharp constants and extremal functions, see [9]. If $b = a+1$, then $p^*(a,b) = p$ and the following Hardy inequality holds [1] [16]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} \mathrm{d}x \leqslant \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}} \mathrm{d}x, \ \forall u \in C_0^\infty(\mathbb{R}^N), \tag{1.3}$$

where $\bar{\mu} = (\frac{N-p}{p} - a)^p$.

We employ the following norm in the space $W_{0,a}^{1,p}(\Omega)$ for $\mu < \bar{\mu}$:

$$||u|| = ||u||_{W_{0,a}^{1,p}(\Omega)} := \left( \int_\Omega \left( \frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.4}$$

By (1.3) it is equivalent to the norm $(\int_\Omega |x|^{-ap}|\nabla u|^p \, \mathrm{d}x)^{1/p}$ of the space $W_{0,a}^{1,p}(\Omega)$. According to the Hardy inequality, the following best constant is well-defined:

$$S_{\mu,a,b} := \inf_{u \in D_a^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \left( \frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \mathrm{d}x}{\left( \displaystyle\int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \mathrm{d}x \right)^{\frac{p}{p^*(a,b)}}},$$

where the space $D_a^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \, \mathrm{d}x)^{1/p}$. By Lemma 2.1 of this paper, we can also define the following constant:

$$\Lambda_{\mu,a,d} := \inf_{u \in W_{0,a}^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left( \frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \mathrm{d}x}{\displaystyle\int_\Omega \frac{|u|^p}{|x|^{dp^*(a,d)}} \mathrm{d}x}.$$

The elliptic problems involving the Hardy and Hardy-Sobolev inequalities have been studied by many authors, either in bounded domain or in the whole space $\mathbb{R}^N$, see [2]-[6], [8]-[10], [15]-[19] and the references therein. Many important results were obtained in these publications and the results give us very good insight into these problems. In particular, when $p=2,\ a=0$ and $\mu<(\frac{N-2}{2})^2$, the problem (1.1) was investigated extensively. On the other hand, we know less about (1.1) when $p\neq 2,\ a\neq 0$ and $\mu<\bar{\mu}$. Thus it is meaningful for us to study (1.1) deeply.

The purpose of this paper is to investigate the existence of the sign-changing solutions to (1.1). In order to state clearly the conclusions of this paper, we need to explain some notations. $\alpha(\mu)$ and $\beta(\mu)$ $(\alpha(\mu)<\beta(\mu))$ are the zeroes of the function

$$f(t)=(p-1)t^p-\big(N-p(a+1)\big)t^{p-1}+\mu,\quad t\geqslant 0,\quad 0\leqslant\mu<\bar{\mu}.$$

The following constants are well defined and will be used in this paper:

$$\delta:=\tfrac{N-p}{p}-a,$$
$$\tau_0:=\tfrac{N-p(a+1)}{N-p(a+1-b)},$$
$$N':=p(a+1+\tfrac{p(a+1)-dp^*(a,d)}{p-1}),$$
$$N'':=p(a+1+(p-1)(p(a+1)-dp^*(a,d))),$$
$$q_1:=\max\{p,\ \tfrac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(p-1)(\delta-\alpha(\mu)}{\delta}\},$$
$$q_2:=\max\{p,\ \tfrac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(\delta-\alpha(\mu)}{\delta}\},$$
$$\mu_1:=\big(\delta+\tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta+dp^*(a,d)-p(a+1)),$$
$$\mu_2:=\big(\delta-\tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta+p(a+1)-dp^*(a,d)),$$
$$\mu_3:=\big(\delta+p(a+1)-dp^*(a,d)\big)^{p-1}\big(\delta+(p-1)(dp^*(a,d)-p(a+1))\big),$$
$$\mu_4:=\big(\delta-p(a+1)+dp^*(a,d)\big)^{p-1}\big(\delta+(p-1)(p(a+1)-dp^*(a,d))\big).$$

The main result of this paper is summarized in the following theorem, which is new when $0<a<(N-p)/p$ and $0<\mu<\bar{\mu}$. We can verify that the sets used in Theorem 1.1 for the parameters $\mu$ and $q$ are not empty.

THEOREM 1.1. *Assume that one of the following conditions holds:*
*(i)* $1<p<2,\ 0\leqslant\mu<\bar{\mu}$ *and* $q_1<q<p^*(a,d)$.
*(ii)* $2\leqslant p<N,\ 0\leqslant\mu<\bar{\mu}$ *and* $q_2<q<p^*(a,d)$.
*(iii)* $1<p<2,\ q=p,\ 0<\lambda<\Lambda_{\mu,a,d},\ N>N'$ *and* $0\leqslant\mu<\min\{\mu_1,\ \mu_2\}$.
*(iv)* $2\leqslant p<N,\ q=p,\ 0<\lambda<\Lambda_{\mu,a,d},\ N>N''$ *and* $0\leqslant\mu<\min\{\mu_3,\ \mu_4\}$.

*Then the problem* (1.1) *has one pair of sign-changing solutions* $\pm u(x)$, *satisfying*

$$\int_\Omega\Big(\frac{|u|^{p^*(a,b)-p}}{|x|^{bp^*(a,b)}}+\lambda\frac{|u|^{q-p}}{|x|^{dp^*(a,d)}}\Big)\,v(u)^{p-1}u=0,$$

*where* $v(u)$ *is the first eigenfunction of the weighted eigenvalue problem*

$$\begin{cases}-\mathrm{div}\Big(\dfrac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}}\Big)-\mu\dfrac{|v|^{p-2}v}{|x|^{p(a+1)}}=\gamma\Big(\dfrac{|u|^{p^*(a,b)-p}}{|x|^{bp^*(a,b)}}+\lambda\dfrac{|u|^{q-p}}{|x|^{dp^*(a,d)}}\Big)|v|^{p-2}v,\\[2ex] v\in W^{1,p}_{0,a}(\Omega).\end{cases}$$

REMARK 1.1. It is known that $S_{0,a,b}$ has the explicit minimizers ([9]):

$$V(x) = C\varepsilon^{a-\frac{N-p}{p}}\left(1+\left(\frac{|x|}{\varepsilon}\right)^{\frac{p(a+1)-bp^*(a,b)}{p-1}}\right)^{-\frac{N-p(a+1)}{p(a+1)-bp^*(a,b)}},$$

where $C>0$ is a particular constant and $\varepsilon>0$ is an arbitrary constant. On the other hand, when $0\leqslant a<\frac{N-p}{p}$ and $0\leqslant\mu<\bar{\mu}$, the extremals of $S_{\mu,a,b}$ and the existence and properties of positive solutions to (1.1) were investigated in [10] and [11].

This paper is organized as follows. In Section 2 some preliminary results are established. In Section 3 the asymptotic properties of the extremal functions related to $S_{\mu,a,b}$ are investigated. At last, we verify Theorem 1.1 in Section 4. In the following argument, $\eta=O(\varepsilon^\tau)\,(\tau>0)$ means that there exists positive constant $C$ such that $|\eta|\leqslant C\varepsilon^\tau$ for $\varepsilon>0$ small enough, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t\to 0$ as $\varepsilon\to 0$ and $o(1)$ stands for a generic infinitesimal value. We always denote the positive constants as $C$ and omit $\mathrm{d}x$ in integrals for convenience.

## 2. Preliminary results

We summarize some required results.

LEMMA 2.1. *Suppose* $a\leqslant d<a+1$, $p\leqslant q\leqslant p^*(a,d)$, $0\leqslant\mu<\bar{\mu}$. *Then:*
$(i)$ *there exists a constant* $C>0$ *such that*

$$\left(\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}\right)^{p/q}\leqslant C\int_\Omega\left(\frac{|\nabla u|^p}{|x|^{ap}}-\mu\frac{|u|^p}{|x|^{p(a+1)}}\right),\quad \forall\, u\in W^{1,p}_{0,a}(\Omega);$$

$(ii)$ *the embedding* $W^{1,p}_{0,a}(\Omega)\hookrightarrow L^q(\Omega,|x|^{-dp^*(a,d)})$ *is compact if* $p\leqslant q<p^*(a,d)$.

*Proof.* The statement $(i)$ can be proved by employing the Hölder inequality, (1.2) and the equivalent norm (1.4) of $W^{1,p}_{0,a}(\Omega)$. The proof of $(ii)$ can be found in [21].

LEMMA 2.2. ([10],[22]) *Let us suppose:*

$$0\leqslant a<\frac{N-p}{p},\ a\leqslant b<a+1,\ \text{and}\ 0\leqslant\mu<\bar{\mu}.$$

*Then the best constant* $S_{\mu,a,b}$ *is achieved in* $\mathbb{R}^N$ *by the radial functions*

$$V_\varepsilon(x):=\varepsilon^{-\delta}U_{p,\mu}(\varepsilon^{-1}x)=\varepsilon^{-\delta}U_{p,\mu}(\varepsilon^{-1}|x|),\ \forall\,\varepsilon>0,$$

*that satisfy*

$$\int_{\mathbb{R}^N}\left(\frac{|\nabla V_\varepsilon(x)|^p}{|x|^{ap}}-\mu\frac{|V_\varepsilon(x)|^p}{|x|^{p(a+1)}}\right)=\int_{\mathbb{R}^N}\frac{|V_\varepsilon(x)|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}=(S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}}.$$

*The function* $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ *is the unique radial solution of the limiting problem:*

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) - \mu \dfrac{u^{p-1}}{|x|^{p(a+1)}} = \dfrac{u^{p^*(a,b)-1}}{|x|^{bp^*(a,b)}} \quad in \ \ \mathbb{R}^N \backslash \{0\}, \\ u \in D_a^{1,p}(\mathbb{R}^N), \qquad u > 0 \ \ in \ \ \mathbb{R}^N \backslash \{0\}, \end{cases} \tag{2.1}$$

*satisfying*

$$U_{p,\mu}(1) = \left(p^*(a,b)\,(\bar{\mu} - \mu)/p\right)^{\frac{1}{p^*(a,b)-p}}.$$

*Furthermore,* $U_{p,\mu}$ *have the following properties:*

$$\lim_{r\to 0} r^{\alpha(\mu)}\, U_{p,\mu}(r) = C_1, \ \ \lim_{r\to 0} r^{\alpha(\mu)+1}\, |U'_{p,\mu}(r)| = C_1\, \alpha(\mu),$$
$$\lim_{r\to +\infty} r^{\beta(\mu)}\, U_{p,\mu}(r) = C_2, \ \ \lim_{r\to +\infty} r^{\beta(\mu)+1}\, |U'_{p,\mu}(r)| = C_2\, \beta(\mu),$$

*where* $C_i (i = 1,2)$ *are positive constants and* $\alpha(\mu)$ *and* $\beta(\mu)$ *are zeroes of the function*

$$f(t) = (p-1)t^p - \big(N - p(a+1)\big)t^{p-1} + \mu, \quad t \geqslant 0,$$

*that satisfy*

$$0 \leqslant \alpha(\mu) < \frac{N-p(a+1)}{p} < \beta(\mu) < \frac{N-p(a+1)}{p-1}. \tag{2.2}$$

*Furthermore, there exist positive constants* $C_3 = C_3(\mu,p,a,b)$ *and* $C_4 = C_4(\mu,p,a,b)$ *such that*

$$C_3 \leqslant U_{p,\mu}(x)\left(|x|^{\frac{\alpha(\mu)}{\delta}} + |x|^{\frac{\beta(\mu)}{\delta}}\right)^{\delta} \leqslant C_4.$$

We mention that the properties of positive solutions to (1.1) were investigated in a recent paper [11] and the following results are already known.

LEMMA 2.3. ([11]) *Suppose* $1 < p < N$ *and* $0 < \mu < \bar{\mu}$. *Assume that* $u \in W_{0,a}^{1,p}(\Omega)$ *is a positive solution to the problem* (1.1). *Then:*

$(i)$ *there exists some constants* $\rho > 0$ *small and* $C > 0$, *such that*

$$u(x) \geqslant C\,|x|^{-\alpha(\mu)}, \quad \forall x \in B_\rho(0) \setminus \{0\};$$

$(ii)$ $\ u \in L^r(\Omega, |x|^{-bp^*(a,b)}), \ \ \forall r \in \left(1, \frac{N\tau_0}{\alpha(\mu)}\right)$;

$(iii)$ $\ |x|^{-a}\,|\nabla u| \in L^r(\Omega, |x|^{-bp^*(a,b)}), \ \ r \in \left(1, \frac{N\tau_0}{\alpha(\mu)+a+1}\right)$.

## 3. Asymptotic property of the extremal function

Let $V_\varepsilon(x)$ be the functions in Lemma 2.2. Take $\rho > 0$ small enough such that $B_\rho(0) \subset \Omega$, $\varphi(x) \in C_0^\infty(\Omega)$, $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ for $|x| \leqslant \frac{\rho}{2}$ and $\varphi(x) = 0$ for $|x| \geqslant \rho$. Setting $u_\varepsilon(x) = \varphi(x)V_\varepsilon(x)$, we have the following estimates.

LEMMA 3.1. ([12]) *As $\varepsilon \to 0$ we have:*

$$\int_{\mathbb{R}^N}\Big(\frac{|\nabla u_\varepsilon|^p}{|x|^{ap}} - \mu\frac{u_\varepsilon^p}{|x|^{p(a+1)}}\Big) = (S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}} + O(\varepsilon^{\beta(\mu)p+p(a+1)-N}),$$

$$\int_\Omega \frac{u_\varepsilon^{p^*(a,b)}}{|x|^{bp^*(a,b)}} = (S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}} + O(\varepsilon^{(\beta(\mu)+b)p^*(a,b)-N}),$$

$$\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} \geqslant \begin{cases} C\varepsilon^{N-dp^*(a,d)-q\delta}, & \frac{N-dp^*(a,d)}{\beta(\mu)} < q < p^*(a,d), \\ C\varepsilon^{q(\beta(\mu)-\delta)}|\ln\varepsilon|, & q = \frac{N-dp^*(a,d)}{\beta(\mu)}, \\ C\varepsilon^{q(\beta(\mu)-\delta)}, & 1 \leqslant q < \frac{N-dp^*(a,d)}{\beta(\mu)}, \end{cases}$$

$$\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} \to 0, \quad 1 \leqslant q < p^*(a,d).$$

LEMMA 3.2. *Suppose $0 < \mu < \bar{\mu}$ and $0 < q \leqslant p^*(a,b) - 1$. Assume that $u \in W_{0,a}^{1,p}(\Omega)$ is a positive solution of the problem* (1.1). *Then as $\varepsilon \to 0$ we have*

$$\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} = \begin{cases} O(\varepsilon^{p-1+\delta-\alpha(\mu)}), & \alpha(\mu)+(p-1)\beta(\mu) > p\delta, \\ O(\varepsilon^{p-1+\delta-\alpha(\mu)}|\ln\varepsilon|), & \alpha(\mu)+(p-1)\beta(\mu) = p\delta, \\ O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}), & \alpha(\mu)+(p-1)\beta(\mu) < p\delta, \end{cases} \tag{3.1}$$

$$\int_\Omega \frac{|\nabla u|^{p-1}|\nabla u_\varepsilon|}{|x|^{ap}} = \begin{cases} O(\varepsilon^{1+(p-1)(\delta-\alpha(\mu))}), & (p-1)\alpha(\mu)+\beta(\mu) > p\delta, \\ O(\varepsilon^{1+\beta(\mu)-\delta}|\ln\varepsilon|), & (p-1)\alpha(\mu)+\beta(\mu) = p\delta, \\ O(\varepsilon^{1+\beta(\mu)-\delta}), & (p-1)\alpha(\mu)+\beta(\mu) < p\delta, \end{cases} \tag{3.2}$$

$$\int_\Omega \frac{uu_\varepsilon^q}{|x|^{bp^*(a,b)}} = \begin{cases} O(\varepsilon^{N\tau_0-q\delta-\alpha(\mu)}), & \alpha(\mu)+q\beta(\mu) > N\tau_0, \\ O(\varepsilon^{q(\beta(\mu)-\delta)}|\ln\varepsilon|), & \alpha(\mu)+q\beta(\mu) = N\tau_0, \\ O(\varepsilon^{q(\beta(\mu)-\delta)}), & \alpha(\mu)+q\beta(\mu) < N\tau_0, \end{cases} \tag{3.3}$$

$$\int_\Omega \frac{u^q u_\varepsilon}{|x|^{bp^*(a,b)}} = \begin{cases} O(\varepsilon^{N\tau_0-\delta-q\alpha(\mu)}), & q\alpha(\mu)+\beta(\mu) > N\tau_0, \\ O(\varepsilon^{\beta(\mu)-\delta}|\ln\varepsilon|), & q\alpha(\mu)+\beta(\mu) = N\tau_0, \\ O(\varepsilon^{\beta(\mu)-\delta}), & q\alpha(\mu)+\beta(\mu) < N\tau_0, \end{cases} \tag{3.4}$$

$$\int_\Omega \frac{u u_\varepsilon^{p-1}}{|x|^{p(a+1)}} = \begin{cases} O(\varepsilon^{\delta-\alpha(\mu)}), & \alpha(\mu)+(p-1)\beta(\mu) > p\delta, \\ O(\varepsilon^{\delta-\alpha(\mu)}|\ln\varepsilon|), & \alpha(\mu)+(p-1)\beta(\mu) = p\delta, \\ O(\varepsilon^{(p-1)(\beta(\mu)-\delta)}), & \alpha(\mu)+(p-1)\beta(\mu) < p\delta, \end{cases} \tag{3.5}$$

$$\int_\Omega \frac{u^{p-1} u_\varepsilon}{|x|^{p(a+1)}} = \begin{cases} O(\varepsilon^{(p-1)(\delta-\alpha(\mu))}), & (p-1)\alpha(\mu)+\beta(\mu) > p\delta, \\ O(\varepsilon^{\beta(\mu)-\delta}|\ln\varepsilon|), & (p-1)\alpha(\mu)+\beta(\mu) = p\delta, \\ O(\varepsilon^{\beta(\mu)-\delta}), & (p-1)\alpha(\mu)+\beta(\mu) < p\delta. \end{cases} \tag{3.6}$$

*Proof.* The statements (3.1)-(3.6) can be verified by the Hölder inequality, Lemma 2.3 and Lemma 3.1. For simplicity we only prove (3.1). Note that the following equality is useful:

$$N\tau_0 + bp^*(a,b) = N. \tag{3.7}$$

VERIFICATION OF (3.1). Assume that $\alpha(\mu)+(p-1)\beta(\mu) > p\delta$. By taking $\tau > 0$ small we deduce that

$$N + \frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1} - \tau - 1} < ((a+1)(p-1)+\beta(\mu)(p-1))\frac{\frac{N\tau_0}{\alpha(\mu)+a+1} - \tau}{\frac{N\tau_0}{\alpha(\mu)+a+1} - \tau - 1}.$$

Consequently,

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} \\
&\leqslant \left(\int_\Omega \frac{(|x|^{-a}|\nabla u|)^{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}{|x|^{bp^*(a,b)}}\right)^{\frac{1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\quad \times \left(\int_\Omega |x|^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}} \left(|x|^{-a(p-1)}|\nabla u_\varepsilon|^{p-1}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(\int_0^{\frac{\rho}{\varepsilon}} \varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N} r^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N-1}\right. \\
&\quad \left. \times \left(r^{-(a+1)(p-1)}\left(r^{\frac{\alpha(\mu)}{\delta}} + r^{\frac{\beta(\mu)}{\delta}}\right)^{-\delta(p-1)}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}} dr\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(\int_0^1 + \int_1^{\frac{\rho}{\varepsilon}}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(O\left(\varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N}\right) + O\left(\varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N}\right)\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}
\end{aligned}$$

$$\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)+N-(\alpha(\mu)+a+1)-\frac{(\alpha(\mu)+a+1)\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}$$

$$\leqslant C\varepsilon^{(p-1)+(\delta-\alpha(\mu))-\frac{(\alpha(\mu)+a+1)\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}.$$

Since $\tau$ is arbitrary, taking the limit as $\tau \to 0$ we have

$$\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} \leqslant C\varepsilon^{(p-1)+(\delta-\alpha(\mu))}.$$

Assume that $\alpha(\mu)+(p-1)\beta(\mu) < p\delta$. By taking $\tau > 0$ small we deduce that

$$N + \frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1} > ((a+1)(p-1)+\beta(\mu)(p-1))\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}.$$

By direct calculation we have

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} \\
&\quad\leqslant \left(\int_\Omega \frac{(|x|^{-a}|\nabla u|)^{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}{|x|^{bp^*(a,b)}}\right)^{\frac{1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\qquad\times \left(\int_\Omega |x|^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\left(|x|^{-a(p-1)}|\nabla u_\varepsilon|^{p-1}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\quad\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(\int_0^{\frac{\rho}{\varepsilon}} \varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N} r^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N-1}\right. \\
&\qquad\times \left.\left(r^{-(a+1)(p-1)}\left(r^{\frac{\alpha(\mu)}{\delta}}+r^{\frac{\beta(\mu)}{\delta}}\right)^{-\delta(p-1)}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}} dr\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\quad\leqslant\ C\varepsilon^{-(\frac{N-p}{p})(p-1)+(a+1)(p-1)+\beta(\mu)(p-1)} \\
&\quad= O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}).
\end{aligned}$$

If $\alpha(\mu)+(p-1)\beta(\mu) = p\delta$, by repeating the above argument we have

$$\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} = O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}|\ln\varepsilon|).$$

The proof of this lemma is thus completed.

## 4. Existence of sign-changing solutions

In this section, we investigate the sign-changing solutions to the problem (1.1). We have to overcome the singularity of the positive solutions to (1.1).

For $\nu \geqslant 0$ small and $u \in W_{0,a}^{1,p}(\Omega)$, we define

$$J_\nu(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*(a,b)-\nu}\int_\Omega \frac{|u|^{p^*(a,b)-\nu}}{|x|^{bp^*(a,b)}} - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}, \tag{4.1}$$

$$\Lambda_\nu = \{u \in W_{0,a}^{1,p}(\Omega)\,;\, \langle J_\nu'(u)\,,\, u\rangle = 0\,,\, u \not\equiv 0\}, \tag{4.2}$$

$$c_{1,\nu} = \inf_{u\in\Lambda_\nu} J_\nu(u)\,. \tag{4.3}$$

Then $J_\nu \in C^1(W_{0,a}^{1,p}(\Omega),\,\mathbb{R})$. Moreover, for $\nu' > 0$ small enough, there exists $\alpha_0 > 0$ such that the following lower bound holds:

$$c_{1,\nu} \geqslant \alpha_0, \quad \forall\, \nu \in [0\,,\,\nu'\,].$$

We recall the following existence result related to the positive solutions of (1.1).

LEMMA 4.1. ([10]) *Suppose* $N \geqslant 3,\ \lambda > 0,\ a \leqslant b, d < a+1,\ 0 \leqslant \mu < \bar{\mu}$. *Assume that one of the following conditions holds:*

$(i)$ $q = p\,,\ 0 < \lambda < \Lambda_{\mu,a,d},\ N \geqslant p^2(a+1) + (1-p)dp^*(a,d)$ *and*

$$0 \leqslant \mu < \widetilde{\mu} := \frac{N - (p^2(a+1) + (1-p)dp^*(a,d))}{p}\left(\frac{N - dp^*(a,d)}{p}\right)^{p-1}.$$

$(ii)$ $\lambda > 0,\ \widetilde{q} < q < p^*(a,d)$, *where*

$$\widetilde{q} := \max\left\{p\,,\ \frac{N - dp^*(a,d)}{\beta(\mu)}\,,\ \frac{p(2N - dp^*(a,d) - p(a+1+\beta(\mu)))}{N - p(a+1)}\right\}.$$

*Then the problem* $(1.1)$ *has a mountain-pass-type positive solution* $u_1 \in \Lambda_0$.

It should be mentioned that the solution $u_1$ has the following property [20]:

$$J_0(u_1) = \sup_{t\in\mathbb{R}} J_0(t\,u_1) = c_{1,0} := \inf_{u\in\Lambda_0} J_0(u).$$

LEMMA 4.2. ([10]) *For* $\varepsilon > 0$ *small enough, there exists a constant* $C > 0$ *such that*

$$\sup_{t\geqslant 0} J_0(tu_\varepsilon) \leqslant \left(\frac{1}{p} - \frac{1}{p^*(a,b)}\right)(S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}} - C\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} + O\big(\varepsilon^{p(\beta(\mu)-\delta)}\big).$$

To obtain the sign-changing solutions, we employ the min-max principle ([7]). To this end, let $B \subset W_{0,a}^{1,p}(\Omega)$ be a closed symmetric set. Then the Krasnoselski genus $i(B)$ is well defined. Fix $\rho > 0$ and define

$$S_\rho = \big\{u | u \in W_{0,a}^{1,p}(\Omega),\ \|u\| = \rho\,\big\},$$

$$\mathscr{H} = \big\{h | h : W_{0,a}^{1,p}(\Omega) \to W_{0,a}^{1,p}(\Omega) \text{ is an odd homeomorphism}\big\},$$

$$\mathscr{F}_2 = \big\{B | B \subset W_{0,a}^{1,p}(\Omega) \text{ is closed symmetric },\ i\big(h(B) \cap S_\rho\,\big) \geqslant 2\,, \forall\, h \in \mathscr{H}\,\big\}.$$

The following result in the sub-critical case can be obtained by the min-max principle ([7]) and the proof is omitted for simplicity.

LEMMA 4.3. *There is a* $\nu^* > 0$, *such that for every* $\nu \in (0, \nu^*)$, *the problem*

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big) - \mu \dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = \dfrac{|u|^{p^*(a,b)-2-\nu}u}{|x|^{bp^*(a,b)}} + \lambda \dfrac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \\ u \in W_{0,a}^{1,p}(\Omega), \end{cases} \tag{4.4}$$

*has a pair of sign-changing solutions* $\pm u_{2,\nu}$ *satisfying*

$$\int_\Omega \Big( \frac{|u_{2,\nu}|^{p^*(a,b)-p-\nu}}{|x|^{bp^*(a,b)}} + \lambda \frac{|u_{2,\nu}|^{q-p}}{|x|^{dp^*(a,d)}} \Big) v(u_{2,\nu})^{p-1} u_{2,\nu} = 0,$$

*where* $v(u_{2,\nu})$ *is the first eigenfunction of the weighted eigenvalue problem*

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}}\Big) - \mu \dfrac{|v|^{p-2}v}{|x|^{p(a+1)}} = \gamma\Big( \dfrac{|u_{2,\nu}|^{p^*(a,b)-p-\nu}}{|x|^{bp^*(a,b)}} + \lambda \dfrac{|u_{2,\nu}|^{q-p}}{|x|^{dp^*(a,d)}} \Big)|v|^{p-2}v, \\ v \in W_{0,a}^{1,p}(\Omega). \end{cases}$$

*Furthermore,*

$$c_{2,\nu} := \inf_{A \in \mathscr{F}_2} \sup_{w \in A} J_\nu(w) = J_\nu(u_{2,\nu}).$$

Note that the sets in Theorem 1.1 for the parameters $q$ and $\mu$ are smaller than those in Lemma 4.1 respectively. Thus under the assumptions of Theorem 1.1 we can get a positive solution $u_1$ to (1.1). Furthermore, we have the following estimate for the sub-critical problem (4.4).

LEMMA 4.4. *Under the assumptions of Theorem 1.1, there exist* $\sigma > 0$ *and* $\nu^{**} > 0$ *such that*

$$c_{2,\nu} \leqslant c_{1,\nu} + \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - \sigma, \ \ \forall \nu \in (0, \nu^{**}). \tag{4.5}$$

*Proof.* Arguing as in [8] and [18], we have that $c_{1,\nu} \to c_{1,0}$ and $c_{2,\nu} \to c_{2,0}$ as $\nu \to 0$.

The following elementary inequalities are well known: $\forall r \in [1, +\infty)$, there exists a constant $C = C(r) > 0$ such that

$$|A+B|^r \leqslant |A|^r + B|^r + C\big(|A|^{r-1}|B| + |A|B|^{r-1}\big), \quad \forall A, B \in \mathbb{R},$$
$$|A+B|^r \geqslant |A|^r + |B|^r - C\big(|A|^{r-1}|B| + |A||B|^{r-1}\big), \quad \forall A, B \in \mathbb{R}.$$

Set $\Gamma_\varepsilon = \operatorname{span}\{u_\varepsilon, u_1\}$, where $u_\varepsilon$ and $u_1$ are the functions defined as in Lemmas 3.1 and 4.1. Then $\Gamma_\varepsilon \in \mathscr{F}_2$ and

$$c_{2,\nu} \leqslant \sup_{w \in \Gamma_\varepsilon} J_\nu(w) = \sup_{A,B \in \mathbb{R}} J_\nu(Au_1 + Bu_\varepsilon).$$

Consequently,

$$
\begin{aligned}
&J_\nu(Au_1+Bu_\varepsilon)\\
&\quad= \frac{1}{p}\|Au_1+Bu_\varepsilon\|^p - \frac{\lambda}{q}\int_\Omega \frac{|Au_1+Bu_\varepsilon|^q}{|x|^{dp^*(a,d)}} - \frac{1}{p^*(a,b)-\nu}\int_\Omega \frac{|Au_1+Bu_\varepsilon|^{p^*(a,b)-\nu}}{|x|^{bp^*(a,b)}}\\
&\quad\leqslant J_\nu(Au_1)+J_\nu(Bu_\varepsilon)\\
&\qquad+\ C|A|^{p-1}|B|\int_\Omega |x|^{-ap}|\nabla u_1|^{p-1}|\nabla u_\varepsilon| + C|A||B|^{p-1}\int_\Omega |x|^{-ap}|\nabla u_1||\nabla u_\varepsilon|^{p-1}\\
&\qquad+C|A|^{p^*(a,b)-1-\nu}|B|\int_\Omega \frac{u_1^{p^*(a,b)-1-\nu}u_\varepsilon}{|x|^{bp^*(a,b)}} + C|A||B|^{p^*(a,b)-1-\nu}\int_\Omega \frac{u_1u_\varepsilon^{p^*(a,b)-1-\nu}}{|x|^{bp^*(a,b)}}\\
&\qquad+C|A|^{p-1}|B|\int_\Omega \frac{u_1^{p-1}u_\varepsilon}{|x|^{p(a+1)}} \ +\ C|A||B|^{p-1}\int_\Omega \frac{u_1u_\varepsilon^{p-1}}{|x|^{p(a+1)}}\\
&\qquad+C|A|^{q-1}|B|\int_\Omega \frac{u_1^{q-1}u_\varepsilon}{|x|^{dp^*(a,d)}} \ +\ C|A||B|^{q-1}\int_\Omega \frac{u_1u_\varepsilon^{q-1}}{|x|^{dp^*(a,d)}}\\
&\quad\leqslant J_\nu(Au_1)+J_\nu(Bu_\varepsilon)\\
&\qquad+C(|A|^p+|B|^p)\,(\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\alpha(\mu)-\delta}+\varepsilon^{(p-1)(\alpha(\mu)-\delta)})|\ln\varepsilon|\\
&\qquad+C\big(|A|^{p^*(a,b)-\nu}+|B|^{p^*(a,b)-\nu}\big)\,(\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{\delta-\alpha(\mu)})|\ln\varepsilon|\\
&\qquad+C(|A|^q+|B|^q)\,(\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{\delta-\alpha(\mu)})|\ln\varepsilon|\,.
\end{aligned}
$$

By the above estimates we get that

$$
\lim_{A,B\to\infty} J_\nu(Au_1+Bu_\varepsilon) = -\infty \ \text{ for small enough } \ \varepsilon>0.
$$

Therefore we may assume that $A$ and $B$ are in bounded sets. From Lemmas 3.1, 3.2 and 4.3 it follows that

$$
\begin{aligned}
&J_\nu(Au_1+Bu_\varepsilon)\\
&\quad\leqslant J_\nu(Au_1)+J_\nu(Bu_\varepsilon)\\
&\qquad\quad+C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|\\
&\quad\leqslant c_{1,\nu}+J_0(Bu_\varepsilon)\\
&\qquad\quad+C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|+I_1\\
&\quad\leqslant c_{1,\nu}+\Big(\frac{1}{p}-\frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - C\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}}\\
&\qquad\quad+\ C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|+I_1,
\end{aligned}
$$

where

$$
I_1 := \frac{|B|^{p^*(a,b)}}{p^*(a,b)}\int_\Omega \frac{u_\varepsilon^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \ -\ \frac{|B|^{p^*(a,b)-\nu}}{p^*(a,b)-\nu}\int_\Omega \frac{u_\varepsilon^{p^*(a,b)-\nu}}{|x|^{bp^*(a,b)}}\,.
$$

$(i)$. Assume that $1<p<2$ and $q>q_1$, where

$$q_1=\max\{p,\ \tfrac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(p-1)(\delta-\alpha(\mu)}{\delta}\}.$$

Then

$$N-dp^*(a,d)-q\delta<(p-1)(\beta(\mu)-\delta)<\beta(\mu)-\delta,$$
$$N-dp^*(a,d)-q\delta<(p-1)(\delta-\alpha(\mu))<\delta-\alpha(\mu).$$

Since
$$\frac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta}>\frac{N-dp^*(a,d)}{\beta(\mu)},$$

from Lemma 3.1 it follows that there exists a constant $\sigma>0$ such that

$$C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon| \\ -C\int_\Omega\frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}}\leqslant-2\sigma. \qquad (4.6)$$

Choose $\nu^{**}>0$ small enough such that $I_1<\sigma$ for $0<\nu<\nu^{**}$. Then

$$c_{2,\nu}\leqslant J_\nu(Au_1+Bu_\varepsilon)\leqslant c_{1,\nu}+\Big(\frac{1}{p}-\frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}}-\sigma,\ \forall\,\nu\in(0,\nu^{**}).$$

$(ii)$. Assume that $p\geqslant 2$ and $q>q_2$, where

$$q_2=\max\{p,\ \tfrac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(\delta-\alpha(\mu)}{\delta}\}.$$

Then

$$N-dp^*(a,d)-q\delta<\beta(\mu)-\delta<(p-1)(\beta(\mu)-\delta),$$
$$N-dp^*(a,d)-q\delta<\delta-\alpha(\mu)<(p-1)(\delta-\alpha(\mu)).$$

Since
$$\frac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta}>\frac{N-dp^*(a,d)}{\beta(\mu)},$$

from Lemma 3.1 it follows that (4.6) for some constant $\sigma>0$. Choose $\nu^{**}>0$ small enough such that $I_1<\sigma$ for $0<\nu<\nu^{**}$. Therefore (4.5) holds.

$(iii)$. $1<p<2$, $q=p$ and $0<\lambda<\Lambda_{\mu,a,d}$. Discussing as above we also have

$$J_\nu(Au_1+Bu_\varepsilon)\leqslant c_{1,\nu}+\Big(\frac{1}{p}-\frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}}-C\int_\Omega\frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} \\ +C\big(\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{(p-1)(\beta(\mu)-\delta)}\big)|\ln\varepsilon|+I_1.$$

If $(p-1)(\beta(\mu)-\delta)>N-dp^*(a,d)-p\delta$, then

$$\beta(\mu)>\delta+\frac{p(a+1)-dp^*(a,d)}{p-1}>\frac{N-dp^*(a,d)}{p}.$$

If $(p-1)(\delta-\alpha(\mu)) > N - dp^*(a,d) - p\delta$, then

$$\alpha(\mu) < \delta - \frac{p(a+1)-dp^*(a,d)}{p-1} < \delta.$$

On the other hand, it's easy to verify that the function

$$f(t) = (p-1)t^p - \big(N-p(a+1)\big)t^{p-1} + \mu, \quad t \geqslant 0,$$

has the unique minimal point $\delta$. Furthermore, $f(t)$ is decreasing on the interval $(0,\delta)$ and is increasing on the interval $(\delta,+\infty)$.

If

$$N > \widetilde{N} := p((p+1)(a+1) - dp^*(a,d)),$$

then

$$\delta + dp^*(a,d) - p(a+1) > 0.$$

Furthermore,

$$\delta + \frac{p(a+1)-dp^*(a,d)}{p-1} < \beta(\mu),$$

which is equivalent to

$$f\big(\delta + \frac{p(a+1)-dp^*(a,d)}{p-1}\big) < f\big(\beta(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_1,$$

where

$$\mu_1 = \big(\delta + \tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta + dp^*(a,d) - p(a+1)).$$

If

$$N > N' = p(a+1+\frac{p(a+1)-dp^*(a,d)}{p-1}),$$

then

$$0 < \delta - \frac{p(a+1)-dp^*(a,d)}{p-1} < \delta.$$

Hence

$$\delta - \frac{p(a+1)-dp^*(a,d)}{p-1} > \alpha(\mu),$$

which is equivalent to

$$f\big(\delta - \frac{p(a+1)-dp^*(a,d)}{p-1}\big) < f\big(\alpha(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_2,$$

where

$$\mu_2 = \left(\delta - \tfrac{p(a+1)-dp^*(a,d)}{p-1}\right)^{p-1}(\delta + p(a+1) - dp^*(a,d)).$$

Now we choose $N$ and $\mu$ such that

$$N > \max\{\widetilde{N}, N'\} = N' \quad \text{and} \quad 0 \leqslant \mu < \min\{\mu_1, \mu_2\}.$$

Then taking $\varepsilon$ small we have

$$C\left(\varepsilon^{(p-1)(\delta-\alpha(\mu))} + \varepsilon^{(p-1)(\beta(\mu)-\delta)}\right)|\ln \varepsilon| - C\int_{\Omega} \frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} \leqslant -2\sigma$$

for some constant $\sigma > 0$. Choose $\nu^{**} > 0$ small enough such that $I_1 < \sigma$ for $0 < \nu < \nu^{**}$. Thus (4.5) holds.

$(iv)$. $p \geqslant 2$, $q = p$ and $0 < \lambda < \Lambda_{\mu,a,d}$. Discussing as above we also have

$$J_\nu(Au_1 + Bu_\varepsilon) \leqslant c_{1,\nu} + \left(\frac{1}{p} - \frac{1}{p^*(a,b)}\right)\left(S_{\mu,a,b}\right)^{\frac{p^*(a,b)}{p^*(a,b)-p}}$$
$$+ C\left(\varepsilon^{\beta(\mu)-\delta} + \varepsilon^{\delta-\alpha(\mu)}\right)|\ln \varepsilon| - C\int_{\Omega} \frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} + I_1.$$

If $\beta(\mu) - \delta > N - dp^*(a,d) - p\delta$, then

$$\beta(\mu) > \delta + p(a+1) - dp^*(a,d) > \frac{N - dp^*(a,d)}{p}.$$

Assume that $N > N'' = p(a+1+(p-1)(p(a+1)-dp^*(a,d)))$. Then

$$\delta + (p-1)(dp^*(a,d) - p(a+1)) > 0.$$

Furthermore,

$$\delta + p(a+1) - dp^*(a,d) < \beta(\mu),$$

which is equivalent to

$$f\left(\delta + p(a+1) - dp^*(a,d)\right) < f\left(\beta(\mu)\right) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_3,$$

where

$$\mu_3 = \left(\delta + p(a+1) - dp^*(a,d)\right)^{p-1}(\delta + (p-1)(dp^*(a,d) - p(a+1))).$$

If $\delta - \alpha(\mu) > N - dp^*(a,d) - p\delta$, then

$$\alpha(\mu) < \delta - p(a+1) + dp^*(a,d) < \delta.$$

Assume $N > \bar{N} := p((p+1)(a+1) - dp^*(a,d))$. Then

$$0 < \delta - p(a+1) + dp^*(a,d) < \delta.$$

Consequently,

$$\delta - p(a+1) + dp^*(a,d) > \alpha(\mu),$$

which is equivalent to

$$f\big(\delta - p(a+1) + dp^*(a,d)\big) < f\big(\alpha(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_4,$$

where

$$\mu_4 = \Big(\delta - p(a+1) + dp^*(a,d)\Big)^{p-1} (\delta + (p-1)(p(a+1) - dp^*(a,d))).$$

Now we choose $N$ and $\mu$ such that

$$N > \max\big\{ N'', \bar{N}\big\} = N'' \quad \text{and} \quad 0 \leqslant \mu < \min\{\mu_3, \mu_4\}.$$

Taking $\varepsilon$ small we have

$$C\big(\varepsilon^{(p-1)(\delta-\alpha(\mu))} + \varepsilon^{(p-1)(\beta(\mu)-\delta)}\big)|\ln\varepsilon| - C\int_\Omega \frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} \leqslant -2\sigma$$

for some constant $\sigma > 0$. Choose $\nu^{**} > 0$ small enough such that $I_1 < \sigma$ for $0 < \nu < \nu^{**}$. Therefore (4.5) holds.

The proof of the lemma is completed.

PROOF OF THEOREM 1.1. Set $\nu_0 = \min\{\nu', \nu^*, \nu^{**}\}$. From the fact $c_{1,\nu} \to c_{1,0}$ as $\nu \to 0$ and by Lemma 4.4 it follows that $c_{2,\nu}$ is bounded uniformly in $\nu \in (0, \nu_0)$. Let $u_{2,\nu}$ be the solution obtained in Lemma 4.3. Then there exists a constant $C > 0$ such that

$$\|u_{2,\nu}\| \leqslant C, \qquad \forall\, \nu \in (0, \nu_0). \tag{4.7}$$

Define $u^\pm(x) = \max\{\pm u(x), 0\}$ respectively for any $u \in W_{0,a}^{1,p}(\Omega)$, then $u^\pm \in W_{0,a}^{1,p}(\Omega)$. By (4.6) we can find $\nu_n \to 0$ such that for some $u \in W_{0,a}^{1,p}(\Omega)$,

$$u_{2,\nu_n}^\pm \rightharpoonup u^\pm \text{ weakly in } W_{0,a}^{1,p}(\Omega).$$

For convenience, we denote $u_{2,\nu_n}, J_{\nu_n}, c_{1,\nu_n}, c_{2,\nu_n}$ and $\Lambda_{\nu_n}$ as $u_n, J_n, c_{1,n}, c_{2,n}$ and $\Lambda_n$ respectively. Since $u_n^\pm \in \Lambda_n$, $J_n(u_n^\pm) \geqslant c_{1,n}$. From Lemma 4.4 it follows that

$$J_n(u_n^+) + J_n(u_n^-) = J_n(u_n) = c_{2,n} \leqslant c_{1,n} + \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - \sigma$$

for $n$ large. Necessarily,

$$J_n(u_n^\pm) \leqslant \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - \sigma.$$

By (4.6) and the fact that $u_n^{\pm} \in \Lambda_n$ we derive

$$C_1 \leqslant \int_{\Omega} \frac{|u_n^{\pm}|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \leqslant C_2$$

for some positive constants $C_1$ and $C_2$.

Now we study the convergence of $\{u_n^{\pm}\}$. Note that $\{u_n^{\pm}\}$ is bounded in $W_{0,a}^{1,p}(\Omega)$. By Lemma 2.1 and the concentration compactness theorem ([13],[14]) and up to a subsequence, we have that $u_n^{\pm} \to u^{\pm}$ strongly in $W_{0,a}^{1,p}(\Omega)$ for some $u \in W_{0,a}^{1,p}(\Omega)$ and $u^{\pm} \not\equiv 0$. Therefore $u$ changes sign in $\Omega$ , $u_n \rightharpoonup u$ weakly in $W_{0,a}^{1,p}(\Omega)$ and $u$ is a solution of (1.1). Since $c_{2,n} \to c_{2,0}$ as $n \to \infty$, it is easy to verify that $\{u_n\}$ is actually a $PS$ sequence for $J_0$ at level $c_{2,0}$. From $\lim_{n\to\infty} c_{1,n} = c_{1,0}$ and by standard arguments we can show that a subsequence of $\{u_n\}$ converges strongly to $u$ in $W_{0,a}^{1,p}(\Omega)$. Therefore $c_{2,0} = J_0(u)$. The proof of Theorem 1.1 is completed. $\square$

*Acknowledgements.* The author acknowledges the anonymous referee for carefully reading this paper and making many useful comments.

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(Received March 15, 2010)

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