SOME RESULTS ON WEIGHTED CRITICAL QUASILINEAR PROBLEMS

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Abstract. In this paper, we are concerned with a kind of quasilinear elliptic problems, which involves the Caffarelli-Kohn-Nirenberg inequality and critical exponents. By employing variational methods and analytical techniques, the existence of sign-changing solutions to the problem is proved.

1. Introduction

In this paper, we investigate the following elliptic problem:

\[
\begin{cases}
- \text{div} \left( \frac{|\nabla u|^{p-2} \nabla u}{|x|^{ap}} \right) - \mu \frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \frac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} + \lambda \frac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \\
\quad u \in W^{1,p}_{0,a}(\Omega),
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary and

\[
0 \in \Omega, \quad N \geq 3, \quad \lambda > 0, \quad 1 < p < N, \\
0 \leq \mu < \mu^* := \left( \frac{N-p}{p} - a \right)^p, \\
0 \leq a < \frac{N-p}{p}, \quad a \leq b, \quad d < a + 1, \quad p \leq q < p^*(a,d),
\]

where

\[
p^*(a,b) := \frac{Np}{N-p(a+1-b)} \quad \text{and} \quad p^*(a,d) := \frac{Np}{N-p(a+1-d)}
\]

are the critical Hardy-Sobolev exponents and \( p^*(0,0) = p^* := Np/(N-p) \) is the critical Sobolev exponent. The space \( W^{1,p}_{0,a}(\Omega) \) is the completion of \( C_0^\infty(\Omega) \) with respect to the norm \( \left( \int_{\Omega} |x|^{-ap} |\nabla u|^p \, dx \right)^{1/p} \).

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By a solution to the problem (1.1), we mean a function \( u \in W^{1,p}_{0,a}(\Omega) \) such that the following equality holds for all \( v \in W^{1,p}_{0,a}(\Omega) \):

\[
\int_{\Omega} \left( \frac{|\nabla u|^{p-2} \nabla u \nabla v}{|x|^{ap}} - \mu \frac{|u|^{p-2} uv}{|x|^{ap(a+1)}} - \lambda \frac{|u|^{q-2} uv}{|x|^{dp^*(a,d)}} \right) \, dx = 0.
\]

By the standard elliptic regularity argument, \( u \in C^1(\Omega \setminus \{0\}) \).

Problem (1.1) is related to the Caffarelli-Kohn-Nirenberg inequality [3]:

\[
\left( \int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \, dx \right)^{\frac{p}{p^*(a,b)}} \leq C \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}} \, dx, \quad \forall u \in C^\infty_0(\mathbb{R}^N),
\]

which is also named as the (weighted or general) Hardy-Sobolev inequality. For the sharp constants and extremal functions, see [9]. If \( b = a + 1 \), then \( p^*(a,b) = p \) and the following Hardy inequality holds [1] [16]:

\[
\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ap(p+1)}} \, dx \leq \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}} \, dx, \quad \forall u \in C^\infty_0(\mathbb{R}^N),
\]

where \( \bar{\mu} = \left( \frac{N-p}{p} - a \right)^p \).

We employ the following norm in the space \( W^{1,p}_{0,a}(\Omega) \) for \( \mu < \bar{\mu} \):

\[
||u|| = ||u||_{W^{1,p}_{0,a}(\Omega)} := \left( \int_{\Omega} \left( \frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{ap(a+1)}} \right) \, dx \right)^{\frac{1}{p}}.
\]

By (1.3) it is equivalent to the norm \( (\int_{\Omega} |x|^{-ap} |\nabla u|^p \, dx)^{1/p} \) of the space \( W^{1,p}_{0,a}(\Omega) \). According to the Hardy inequality, the following best constant is well-defined:

\[
S_{\mu,a,b} := \inf_{u \in D^1_{a,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left( \frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{ap(a+1)}} \right) \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \, dx \right)^{\frac{p}{p^*(a,b)}}},
\]

where the space \( D^1_{a,p}(\mathbb{R}^N) \) is the completion of \( C^\infty_0(\mathbb{R}^N) \) with respect to the norm \( (\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx)^{1/p} \). By Lemma 2.1 of this paper, we can also define the following constant:

\[
\Lambda_{\mu,a,d} := \inf_{u \in W^{1,p}_{0,a}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \left( \frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \, dx}{\int_{\Omega} \frac{|u|^p}{|x|^{dp^*(a,d)}} \, dx}.
\]

The elliptic problems involving the Hardy and Hardy-Sobolev inequalities have been studied by many authors, either in bounded domain or in the whole space \( \mathbb{R}^N \), see [2]-[6], [8]-[10], [15]-[19] and the references therein. Many important results were
obtained in these publications and the results give us very good insight into these problems. In particular, when \( p = 2, \ a = 0 \) and \( \mu < \left( \frac{N+2}{2} \right)^2 \), the problem (1.1) was investigated extensively. On the other hand, we know less about (1.1) when \( p \neq 2, \ a \neq 0 \) and \( \mu < \bar{\mu} \). Thus it is meaningful for us to study (1.1) deeply.

The purpose of this paper is to investigate the existence of the sign-changing solutions to (1.1). In order to state clearly the conclusions of this paper, we need to explain some notations. \( \alpha(\mu) \) and \( \beta(\mu) \) (\( \alpha(\mu) < \beta(\mu) \)) are the zeroes of the function
\[
f(t) = (p-1)t^p - (N-p(a+1))t^{p-1} + \mu, \quad t \geq 0, \quad 0 \leq \mu < \bar{\mu}.
\]

The following constants are well defined and will be used in this paper:
\[
\begin{align*}
\delta &:= \frac{N-p}{p} - a, \\
\tau_0 &:= \frac{N-p(a+1)}{N-p(a+1-b)}, \\
N' &:= p(a+1 + \frac{p(a+1-b)dp^*(a,d)-dp^*(a,d)}{p-1}), \\
N'' &:= p(a+1 + (p-1)(p(a+1) - dp^*(a,d))), \\
q_1 &:= \max\{p, \frac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta}, \frac{N-dp^*(a,d)-(p-1)(\delta-\alpha(\mu))}{\delta}\}, \\
q_2 &:= \max\{p, \frac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta}, \frac{N-dp^*(a,d)-(\delta-\alpha(\mu))}{\delta}\}, \\
\mu_1 &:= (\delta + \frac{p(a+1)-dp^*(a,d)}{p-1})^{p-1}(\delta + dp^*(a,d) - p(a+1)), \\
\mu_2 &:= (\delta - \frac{p(a+1)-dp^*(a,d)}{p-1})^{p-1}(\delta + p(a+1) - dp^*(a,d)), \\
\mu_3 &:= (\delta + p(a+1) - dp^*(a,d))^{p-1}(\delta + (p-1)(dp^*(a,d) - p(a+1))), \\
\mu_4 &:= (\delta - p(a+1) + dp^*(a,d))^{p-1}(\delta + (p-1)(p(a+1) - dp^*(a,d))).
\end{align*}
\]

The main result of this paper is summarized in the following theorem, which is new when \( 0 < a < (N-p)/p \) and \( 0 < \mu < \bar{\mu} \). We can verify that the sets used in Theorem 1.1 for the parameters \( \mu \) and \( q \) are not empty.

**THEOREM 1.1.** Assume that one of the following conditions holds:

(i) \( 1 < p < 2, \ 0 \leq \mu < \bar{\mu} \) and \( q_1 < q < p^*(a,d) \).

(ii) \( 2 \leq p < N, \ 0 \leq \mu < \bar{\mu} \) and \( q_2 < q < p^*(a,d) \).

(iii) \( 1 < p < 2, \ q = p, \ 0 < \lambda < \Lambda_{\mu,a,d}, \ N > N' \) and \( 0 \leq \mu < \min\{\mu_1, \mu_2\} \).

(iv) \( 2 \leq p < N, \ q = p, \ 0 < \lambda < \Lambda_{\mu,a,d}, \ N > N'' \) and \( 0 \leq \mu < \min\{\mu_3, \mu_4\} \).

Then the problem (1.1) has one pair of sign-changing solutions \( \pm u(x) \), satisfying
\[
\int_{\Omega} \left( \frac{|u|^{p^*(a,b)-p}}{|x|^{bp^*(a,b)}} + \frac{|u|^{q-p}}{|x|^{dp^*(a,d)}} \right) v(u)^{p-1} u = 0,
\]
where \( v(u) \) is the first eigenfunction of the weighted eigenvalue problem
\[
\begin{align*}
- \text{div} \left( \frac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}} \right) - \mu \frac{|v|^{p-2}v}{|x|^{p(a+1)}} &= \gamma \left( \frac{|u|^{p^*(a,b)-p}}{|x|^{bp^*(a,b)}} + \lambda \frac{|u|^{q-p}}{|x|^{dp^*(a,d)}} \right) |v|^{p-2}v, \\
v &\in W^{1,p}_{0,a}(\Omega).
\end{align*}
\]
REMARK 1.1. It is known that $S_{0,a,b}$ has the explicit minimizers ([9]):

$$ V(x) = C \varepsilon^{a - \frac{N-p}{p}} \left( 1 + \frac{|x|}{\varepsilon} \right)^{\frac{p(a+1)-bp^*(a,b)}{p-1}} - \frac{N-p(a+1)}{p(a+1)-bp^*(a,b)}, $$

where $C > 0$ is a particular constant and $\varepsilon > 0$ is an arbitrary constant. On the other hand, when $0 < a < \frac{N-p}{p}$ and $0 \leq \mu < \bar{\mu}$, the extremals of $S_{\mu,a,b}$ and the existence and properties of positive solutions to (1.1) were investigated in [10] and [11].

This paper is organized as follows. In Section 2 some preliminary results are established. In Section 3 the asymptotic properties of the extremal functions related to $S_{\mu,a,b}$ are investigated. At last, we verify Theorem 1.1 in Section 4. In the following argument, $\eta = O(\varepsilon^\tau) (\tau > 0)$ means that there exists positive constant $C$ such that $|\eta| \leq C \varepsilon^\tau$ for $\varepsilon > 0$ small enough, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ stands for a generic infinitesimal value. We always denote the positive constants as $C$ and omit $dx$ in integrals for convenience.

## 2. Preliminary results

We summarize some required results.

**LEMMA 2.1.** Suppose $a \leq a < a + 1$, $p \leq q \leq p^*(a,d)$, $0 \leq \mu < \bar{\mu}$. Then:

(i) there exists a constant $C > 0$ such that

$$ \left( \frac{\int_{\Omega} |u|^{q} |x|^{-ap^*(a,d)}}{|x|^{ap^*(a,d)}} \right)^{p/q} \leq C \int_{\Omega} \left( \frac{|\nabla u|^p |x|^{-ap}}{|x|^{ap+1}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right), \quad \forall u \in W^{1,p}_{0,a} (\Omega); $$

(ii) the embedding $W^{1,p}_{0,a} (\Omega) \hookrightarrow L^q(\Omega, |x|^{-dp^*(a,d)})$ is compact if $p \leq q < p^*(a,d)$.

**Proof.** The statement (i) can be proved by employing the H"older inequality, (1.2) and the equivalent norm (1.4) of $W^{1,p}_{0,a} (\Omega)$. The proof of (ii) can be found in [21].

**LEMMA 2.2.** ([10],[22]) Let us suppose:

$$ 0 \leq a < \frac{N-p}{p}, \ a \leq b < a + 1, \ \text{and} \ 0 \leq \mu < \bar{\mu}. $$

Then the best constant $S_{\mu,a,b}$ is achieved in $\mathbb{R}^N$ by the radial functions

$$ V_\varepsilon(x) := \varepsilon^{-\delta} U_{p,\mu}(\varepsilon^{-1}x) = \varepsilon^{-\delta} U_{p,\mu}(\varepsilon^{-1}|x|), \ \forall \varepsilon > 0, $$

that satisfy

$$ \int_{\mathbb{R}^N} \left( \frac{|\nabla V_\varepsilon(x)|^p}{|x|^{ap}} - \mu \frac{|V_\varepsilon(x)|^p}{|x|^{p(a+1)}} \right) = \int_{\mathbb{R}^N} \frac{|V_\varepsilon(x)|^p}{|x|^{bp^*(a,b)}} = (S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}}. $$
The function $U_{p, \mu}(x) = U_{p, \mu}(|x|)$ is the unique radial solution of the limiting problem:

$$
\begin{aligned}
- \text{div} \left( \frac{|\nabla u|^p - 2 \nabla u}{|x|^p} \right) - \mu \frac{u^{p-1}}{|x|^{p(a+1)}} &= \frac{u^{p^*(a,b)-1}}{|x|^{bp^*(a,b)}} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
u \in D^{1,p}_a(\mathbb{R}^N), \quad u > 0 & \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\end{aligned}
$$

(2.1)

satisfying

$$
U_{p, \mu}(1) = \left( p^*(a,b) (\bar{\mu} - \mu) / p \right)^{\frac{1}{p^*(a,b)-p}}.
$$

Furthermore, $U_{p, \mu}$ have the following properties:

$$
\lim_{r \to 0} r^{\alpha(\mu)} U_{p, \mu}(r) = C_1, \quad \lim_{r \to 0} r^{\alpha(\mu)+1} |U_{p, \mu}'(r)| = C_1 \alpha(\mu),
$$

$$
\lim_{r \to +\infty} r^{\beta(\mu)} U_{p, \mu}(r) = C_2, \quad \lim_{r \to +\infty} r^{\beta(\mu)+1} |U_{p, \mu}'(r)| = C_2 \beta(\mu),
$$

where $C_i (i = 1, 2)$ are positive constants and $\alpha(\mu)$ and $\beta(\mu)$ are zeroes of the function

$$
f(t) = (p-1)t^p - (N - p(a+1))t^{p-1} + \mu, \quad t \geq 0,
$$

that satisfy

$$
0 \leq \alpha(\mu) < \frac{N - p(a+1)}{p} < \beta(\mu) < \frac{N - p(a+1)}{p-1}.
$$

(2.2)

Furthermore, there exist positive constants $C_3 = C_3(\mu, p, a, b)$ and $C_4 = C_4(\mu, p, a, b)$ such that

$$
C_3 \leq U_{p, \mu}(x) \left( |x|^{\frac{\alpha(\mu)}{\delta}} + |x|^{\frac{\beta(\mu)}{\delta}} \right) \delta \leq C_4.
$$

We mention that the properties of positive solutions to (1.1) were investigated in a recent paper [11] and the following results are already known.

**Lemma 2.3.** ([11]) Suppose $1 < p < N$ and $0 < \mu < \bar{\mu}$. Assume that $u \in W^{1,p}_{0,a}(\Omega)$ is a positive solution to the problem (1.1). Then:

(i) there exists some constants $\rho > 0$ small and $C > 0$, such that

$$
u(x) \geq C |x|^{-\alpha(\mu)} , \quad \forall \in B_\rho(0) \setminus \{0\};
$$

(ii) $u \in L^r(\Omega, |x|^{-bp^*(a,b)})$, $\forall r \in \left(1, \frac{N \tau_0}{\alpha(\mu)} \right)$;

(iii) $|x|^{-a} |\nabla u| \in L^r(\Omega, |x|^{-bp^*(a,b)})$, $r \in \left(1, \frac{N \tau_0}{\alpha(\mu)+a+1} \right)$.
3. Asymptotic property of the extremal function

Let \( V_\varepsilon(x) \) be the functions in Lemma 2.2. Take \( \rho > 0 \) small enough such that \( B_\rho(0) \subset \Omega \), \( \phi(x) \in C^\infty_0(\Omega) \), \( 0 \leq \phi(x) \leq 1 \), \( \phi(x) = 1 \) for \( |x| \leq \frac{\rho}{2} \) and \( \phi(x) = 0 \) for \( |x| \geq \rho \). Setting \( u_\varepsilon(x) = \phi(x) V_\varepsilon(x) \), we have the following estimates.

**Lemma 3.1.** ([12]) As \( \varepsilon \to 0 \) we have:

\[
\int_{\mathbb{R}^N} \left( \frac{|\nabla u_\varepsilon|^p}{|x|^{ap}} - \mu \frac{u_\varepsilon^p}{|x|^{p(a+1)}} \right) = (S_{\mu,a,b})^{p^*(a,b) - p} + O(\varepsilon^\beta(\mu)p + p(a+1) - N),
\]

\[
\int_\Omega \frac{u_\varepsilon^{p^*(a,b)}}{|x|^{b p^*(a,b)}} = (S_{\mu,a,b})^{p^*(a,b) - p} + O(\varepsilon^{(\beta(\mu)+b)p^*(a,b) - N}),
\]

\[
\int_\Omega \frac{u_\varepsilon^q}{|x|^{d p^*(a,d)}} \leq \begin{cases} C \varepsilon^{N-d p^*(a,d) - q \delta}, & \frac{N-d p^*(a,d)}{\beta(\mu)} < q < p^*(a,d), \\
C \varepsilon^{q(\beta(\mu) - \delta)} |\ln \varepsilon|, & q = \frac{N-d p^*(a,d)}{\beta(\mu)}, \\
C \varepsilon^{q(\beta(\mu) - \delta)}, & 1 \leq q < \frac{N-d p^*(a,d)}{\beta(\mu)}, \end{cases}
\]

\[
\int_\Omega \frac{u_\varepsilon^q}{|x|^{d p^*(a,d)}} \to 0, \quad 1 \leq q < p^*(a,d).
\]

**Lemma 3.2.** Suppose \( 0 < \mu < \bar{\mu} \) and \( 0 < q \leq p^*(a,b) - 1 \). Assume that \( u \in W^{1,p}_0(\Omega) \) is a positive solution of the problem (1.1). Then as \( \varepsilon \to 0 \) we have

\[
\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} = \begin{cases} O(\varepsilon^{p-1+\delta - \alpha(\mu)}), & \alpha(\mu) + (p-1)\beta(\mu) > p\delta, \\
O(\varepsilon^{p-1+\delta - \alpha(\mu)}|\ln \varepsilon|), & \alpha(\mu) + (p-1)\beta(\mu) = p\delta, \\
O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}), & \alpha(\mu) + (p-1)\beta(\mu) < p\delta, \end{cases}
\]

\[
(3.1)
\]

\[
\int_\Omega \frac{|\nabla u|^{p-1} |\nabla u_\varepsilon|}{|x|^{ap}} = \begin{cases} O(\varepsilon^{1+\beta(\mu)-\delta}|\ln \varepsilon|), & (p-1)\alpha(\mu) + \beta(\mu) = p\delta, \\
O(\varepsilon^{1+\beta(\mu)-\delta}), & (p-1)\alpha(\mu) + \beta(\mu) < p\delta, \end{cases}
\]

\[
(3.2)
\]

\[
\int_\Omega \frac{uu_\varepsilon^q}{|x|^{b p^*(a,b)}} = \begin{cases} O(\varepsilon^{N\tau_0 - q\delta - \alpha(\mu)}), & \alpha(\mu) + q\beta(\mu) > N\tau_0, \\
O(\varepsilon^{q(\beta(\mu) - \delta)}|\ln \varepsilon|), & \alpha(\mu) + q\beta(\mu) = N\tau_0, \\
O(\varepsilon^{q(\beta(\mu) - \delta)}), & \alpha(\mu) + q\beta(\mu) < N\tau_0, \\
O(\varepsilon^{N\tau_0 - q\alpha(\mu)}), & q\alpha(\mu) + \beta(\mu) > N\tau_0, \end{cases}
\]

\[
(3.3)
\]

\[
\int_\Omega \frac{u^q u_\varepsilon}{|x|^{b p^*(a,b)}} = \begin{cases} O(\varepsilon^{\beta(\mu) - \delta}|\ln \varepsilon|), & q\alpha(\mu) + \beta(\mu) = N\tau_0, \\
O(\varepsilon^{\beta(\mu) - \delta}), & q\alpha(\mu) + \beta(\mu) < N\tau_0, \end{cases}
\]

\[
(3.4)
\]
\[
\int \frac{uv_e^{p-1}}{|x|^{p(a+1)}} = \begin{cases}
O(e^{\delta - \alpha(\mu)}), & \alpha(\mu) + (p-1)\beta(\mu) > p\delta, \\
O(e^{\delta - \alpha(\mu)} |\ln e|), & \alpha(\mu) + (p-1)\beta(\mu) = p\delta, \\
O(e^{(p-1)(\beta(\mu) - \delta)}), & \alpha(\mu) + (p-1)\beta(\mu) < p\delta,
\end{cases}
\tag{3.5}
\]

\[
\int \frac{u^{p-1} v_e}{|x|^{p(a+1)}} = \begin{cases}
O(e^{\beta(\mu) - \delta} |\ln e|), & (p-1)\alpha(\mu) + \beta(\mu) = p\delta, \\
O(e^{\beta(\mu) - \delta}), & (p-1)\alpha(\mu) + \beta(\mu) < p\delta.
\end{cases}
\tag{3.6}
\]

**Proof.** The statements (3.1)-(3.6) can be verified by the Hölder inequality, Lemma 2.3 and Lemma 3.1. For simplicity we only prove (3.1). Note that the following equality is useful:

\[
N_{\tau_0} + bp^*(a, b) = N.
\tag{3.7}
\]

**Verification of (3.1).** Assume that \(\alpha(\mu) + (p-1)\beta(\mu) > p\delta\). By taking \(\tau > 0\) small we deduce that

\[
N + \frac{bp^*(a, b)}{\alpha(\mu) + a + 1} - \tau - 1 < ((a+1)(p-1) + \beta(\mu)(p-1)) \frac{N_{\tau_0}}{\alpha(\mu) + a + 1} - \tau
\]

Consequently,

\[
\int \frac{|\nabla u| |\nabla v_e|^{p-1}}{|x|^{ap}} \leq \left( \int \frac{\left( |x|^{-a} |\nabla u| \right)^{\frac{N_{\tau_0}}{\alpha(\mu) + a + 1} - \tau}}{|x|^{bp^*(a, b)}} \right)^{\frac{1}{\alpha(\mu) + a + 1 - \tau}} \times \left( \int \frac{\left( |x|^{-a(p-1)} |\nabla v_e|^{p-1} \right)^{\frac{N_{\tau_0}}{\alpha(\mu) + a + 1} - \tau}}{|x|^{\frac{bp^*(a, b)}{\alpha(\mu) + a + 1} - \tau}} \right)^{\frac{1}{\alpha(\mu) + a + 1 - \tau}}
\]

\[
\leq C e^{-\frac{N-\tau}{p}(p-1)} \left( \int_0^\frac{N_{\tau_0}}{\alpha(\mu) + a + 1} - \tau \right) r^{\frac{bp^*(a, b)}{\alpha(\mu) + a + 1} - \tau} + N \frac{bp^*(a, b)}{\alpha(\mu) + a + 1 - \tau} + N - 1
\]

\[
\times \left( r^{-a(1)(p-1)} \left( \frac{\alpha(\mu)}{\delta} + \frac{\beta(\mu)}{\delta} - \delta(p-1) \right) \frac{N_{\tau_0}}{\alpha(\mu) + a + 1} - \tau \right) dr
\]

\[
\leq C e^{-\frac{N-\tau}{p}(p-1)} \left( \int_0^1 + \int_1^\frac{N_{\tau_0}}{\alpha(\mu) + a + 1} \right) \frac{N_{\tau_0}}{\alpha(\mu) + a + 1} - \tau
\]

\[
\leq C e^{-\frac{N-\tau}{p}(p-1)} \left( O \left( \frac{bp^*(a, b)}{\alpha(\mu) + a + 1} - \tau \right) + O \left( \frac{bp^*(a, b)}{\alpha(\mu) + a + 1} - \tau \right) + N \frac{bp^*(a, b)}{\alpha(\mu) + a + 1} - \tau \right)
\]
\[-\frac{(N-p)(p-1)+N-(\alpha(\mu)+a+1)-\frac{(\alpha(\mu)+a+1)}{N_0}}{\alpha(\mu)+a+1}\tau \leq C\varepsilon\]

\[\leq C\varepsilon\]

Since \(\tau\) is arbitrary, taking the limit as \(\tau \to 0\) we have
\[\int_{\Omega} \frac{\nabla u}{|x|^{a\varepsilon}} \leq C\varepsilon^{(p-1)+(\delta -\alpha(\mu))}.\]

Assume that \(\alpha(\mu) + (p-1)\beta(\mu) < p\delta\). By taking \(\tau > 0\) small we deduce that
\[N + \frac{b \alpha(p-1)}{N_0} - \tau - 1 > ((a+1)(p-1) + \beta(\mu)(p-1)) \frac{N_0}{\alpha(\mu)+a+1} - \tau.\]

By direct calculation we have
\[\int_{\Omega} \frac{\nabla u}{|x|^{a\varepsilon}} \leq C\varepsilon^{(p-1)+(\delta -\alpha(\mu))}.\]

\[
\leq C\varepsilon^{(p-1)} \left(\int_{E} \frac{b \alpha(p-1)}{N_0} - \tau - 1 \right) + N \frac{b \alpha(p-1)}{N_0} - \tau - 1
+ \frac{N_0}{\alpha(\mu)+a+1} - \tau - 1
\]

\[\leq O\left(\varepsilon^{(p-1)}(\beta(\mu) - \delta + 1)\right).\]

If \(\alpha(\mu) + (p-1)\beta(\mu) = p\delta\), by repeating the above argument we have
\[\int_{\Omega} \frac{\nabla u}{|x|^{a\varepsilon}} = O\left(\varepsilon^{(p-1)}(\beta(\mu) - \delta + 1)\ln|\varepsilon|\right).\]

The proof of this lemma is thus completed.

### 4. Existence of sign-changing solutions

In this section, we investigate the sign-changing solutions to the problem (1.1). We have to overcome the singularity of the positive solutions to (1.1).
Then \( J_\nu \in C^1(W_{0,a}^{1,p}(\Omega), \mathbb{R}) \). Moreover, for \( \nu' > 0 \) small enough, there exists \( \alpha_0 > 0 \) such that the following lower bound holds:

\[
J_{\nu'} < 0, \quad \forall \nu' \in [0, \nu'].
\]

We recall the following existence result related to the positive solutions of (1.1).

**Lemma 4.1.** ([10]) Suppose \( N \geq 3, \lambda > 0, a \leq b, d < a + 1, 0 \leq \mu < \tilde{\mu} \). Assume that one of the following conditions holds:

(i) \( q = p, 0 < \lambda < \Lambda_{\mu,a,d}, N \geq (p^2(a+1) + (1-p)d)p^*(a,d) \) and

\[
0 \leq \mu < \tilde{\mu} = \frac{N - (p^2(a+1) + (1-p)d)p^*(a,d))}{p} \left( \frac{N - dp^*(a,d)}{p} \right)^{p-1}.
\]

(ii) \( \lambda > 0, \tilde{q} < q < p^*(a,d), \) where

\[
\tilde{q} := \max \left\{ p, \frac{N - dp^*(a,d)}{\beta(\mu)} + \frac{p(2N - dp^*(a,d) - p(a+1+\beta(\mu)))}{N - p(a+1)} \right\}.
\]

Then the problem (1.1) has a mountain-pass-type positive solution \( u_1 \in \Lambda_0 \).

It should be mentioned that the solution \( u_1 \) has the following property [20]:

\[
J_0(u_1) = \sup_{t \in \mathbb{R}} J_0(tu_1) = c_{1,0} := \inf_{u \in \Lambda_0} J_0(u).
\]

**Lemma 4.2.** ([10]) For \( \varepsilon > 0 \) small enough, there exists a constant \( C > 0 \) such that

\[
\sup_{t \geq 0} J_0(tu_\varepsilon) \leq \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right) (S_{\mu,a,b})^{-\frac{p^*(a,b)}{p^*(a,b)-p}} - C \int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} + O(\varepsilon p^{(\beta(\mu) - \delta)}).
\]

To obtain the sign-changing solutions, we employ the min-max principle ([7]). To this end, let \( B \subset W_{0,a}^{1,p}(\Omega) \) be a closed symmetric set. Then the Krasnoselski genus \( i(B) \) is well defined. Fix \( \rho > 0 \) and define

\[
S_\rho = \{ u | u \in W_{0,a}^{1,p}(\Omega), \| u \| = \rho \},
\]

\[
S = \{ h | h : W_{0,a}^{1,p}(\Omega) \to W_{0,a}^{1,p}(\Omega) \text{ is an odd homeomorphism} \},
\]

\[
\mathcal{F}_2 = \{ B | B \subset W_{0,a}^{1,p}(\Omega) \text{ is closed symmetric} \}
\]

The following result in the sub-critical case can be obtained by the min-max principle ([7]) and the proof is omitted for simplicity.
Lemma 4.3. There is a $v^* > 0$, such that for every $v \in (0, v^*)$, the problem
\[
-\text{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = |u|^{p^*(a,b)-2-v}u + \lambda|u|^{q-2}u,
\]
\[u \in W_{0,a}^{1,p}(\Omega),\]
has a pair of sign-changing solutions $\pm u_{2,v}$ satisfying
\[
\int_{\Omega}\left(\frac{|u_{2,v}|^{p^*(a,b)-p-v}}{|x|^{bp^*(a,b)}} + \lambda \frac{|u_{2,v}|^{q-p}}{|x|^{dp^*(a,d)}}\right)v(u_{2,v})^{p-1}u_{2,v} = 0,
\]
where $v(u_{2,v})$ is the first eigenfunction of the weighted eigenvalue problem
\[
-\text{div}\left(\frac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}}\right) - \mu\frac{|v|^{p-2}v}{|x|^{p(a+1)}} = \gamma\left(\frac{|u_{2,v}|^{p^*(a,b)-p-v}}{|x|^{bp^*(a,b)}} + \lambda \frac{|u_{2,v}|^{q-p}}{|x|^{dp^*(a,d)}}\right)|v|^{p-2}v,
\]
$v \in W_{0,a}^{1,p}(\Omega)$.

Furthermore,
\[
c_{2,v} := \inf_{A \in \mathcal{F}_2}\sup_{w \in A} J_v(w) = J_v(u_{2,v}).
\]

Note that the sets in Theorem 1.1 for the parameters $q$ and $\mu$ are smaller than those in Lemma 4.1 respectively. Thus under the assumptions of Theorem 1.1 we can get a positive solution $u_1$ to (1.1). Furthermore, we have the following estimate for the sub-critical problem (4.4).

Lemma 4.4. Under the assumptions of Theorem 1.1, there exist $\sigma > 0$ and $v^{**} > 0$ such that
\[
c_{2,v} \leq c_{1,v} + \left(\frac{1}{p} - \frac{1}{p^*(a,b)}\right) \left(S_{\mu,a,b}\right)\frac{p^*(a,b)}{p^*(a,b)-p} - \sigma, \quad \forall v \in (0, v^{**}).
\]

Proof. Arguing as in [8] and [18], we have that $c_{1,v} \to c_{1,0}$ and $c_{2,v} \to c_{2,0}$ as $v \to 0$.

The following elementary inequalities are well known: $\forall r \in [1, +\infty)$, there exists a constant $C = C(r) > 0$ such that
\[
|A + B|^r \leq |A|^r + |B|^r + C(|A|^{r-1}|B| + |A||B|^{r-1}), \quad \forall A, B \in \mathbb{R},
\]
\[
|A + B|^r \geq |A|^r + |B|^r - C(|A|^{r-1}|B| + |A||B|^{r-1}), \quad \forall A, B \in \mathbb{R}.
\]

Set $\Gamma_\epsilon = \text{span}\{u_\epsilon, u_1\}$, where $u_\epsilon$ and $u_1$ are the functions defined as in Lemmas 3.1 and 4.1. Then $\Gamma_\epsilon \in \mathcal{F}_2$ and
\[
c_{2,v} \leq \sup_{w \in \Gamma_\epsilon} J_v(w) = \sup_{A,B \in \mathbb{R}} J_v(Au_1 + Bu_\epsilon).
\]
Consequently,

\[ J_{\nu}(Au_1 + Bu_\varepsilon) \]

\[ = \frac{1}{p} ||Au_1 + Bu_\varepsilon||^p - \frac{\lambda}{d} \int_{\Omega} |Au_1 + Bu_\varepsilon|^q - \frac{1}{p^*(a, b) - \nu} \int_{\Omega} |Au_1 + Bu_\varepsilon|^{p^*(a, b) - \nu} \]

\[ \leq J_{\nu}(Au_1) + J_{\nu}(Bu_\varepsilon) \]

\[ + C|A|^{p-1}|B| \int_{\Omega} |x|^{-ap} |\nabla u_1|^{p-1} |\nabla u_\varepsilon| + C|A||B|^{p-1} \int_{\Omega} |x|^{-ap} |\nabla u_1| |\nabla u_\varepsilon|^{p-1} \]

\[ + C|A|^{p^*(a, b) - 1 - \nu} |B| \int_{\Omega} \frac{|u_1|^{p^*(a, b) - 1 - \nu} u_\varepsilon}{|x|^{p^*(a, b)}} + C|A||B|^{p^*(a, b) - 1 - \nu} \int_{\Omega} \frac{u_1 u_\varepsilon^{p^*(a, b) - 1 - \nu}}{|x|^{p^*(a, b)}} \]

\[ + C|A|^{q-1}|B| \int_{\Omega} \frac{|u_1|^{q-1} u_\varepsilon}{|x|^{dp^*(a,d)}} + C|A||B|^{q-1} \int_{\Omega} \frac{u_1 u_\varepsilon^{q-1}}{|x|^{dp^*(a,d)}} \]

\[ \leq J_{\nu}(Au_1) + J_{\nu}(Bu_\varepsilon) \]

\[ + C(|A|^p + |B|^p) (\varepsilon^{\beta(\mu) - \delta} + \varepsilon^{(p-1)\beta(\mu) - \delta} + \varepsilon^{\alpha(\mu) - \delta} + \varepsilon^{(p-1)\alpha(\mu) - \delta}) |\ln \varepsilon| \]

\[ + C(|A|^{p^*(a, b) - \nu} + |B|^{p^*(a, b) - \nu}) (\varepsilon^{\beta(\mu) - \delta} + \varepsilon^{\delta - \alpha(\mu)}) |\ln \varepsilon| \]

\[ + C(|A|^q + |B|^q) (\varepsilon^{\beta(\mu) - \delta} + \varepsilon^{\delta - \alpha(\mu)}) |\ln \varepsilon| . \]

By the above estimates we get that

\[ \lim_{A, B \to \infty} J_{\nu}(Au_1 + Bu_\varepsilon) = -\infty \text{ for small enough } \varepsilon > 0. \]

Therefore we may assume that \( A \) and \( B \) are in bounded sets. From Lemmas 3.1, 3.2 and 4.3 it follows that

\[ J_{\nu}(Au_1 + Bu_\varepsilon) \]

\[ \leq J_{\nu}(Au_1) + J_{\nu}(Bu_\varepsilon) \]

\[ + C(\varepsilon^{(p-1)\beta(\mu) - \delta} + \varepsilon^{\alpha(\mu) - \delta} + \varepsilon^{(p-1)\delta - \alpha(\mu)}) |\ln \varepsilon| \]

\[ \leq c_{1, \nu} + J_0(Bu_\varepsilon) \]

\[ + C(\varepsilon^{(p-1)\beta(\mu) - \delta} + \varepsilon^{\alpha(\mu) - \delta} + \varepsilon^{(p-1)\delta - \alpha(\mu)}) |\ln \varepsilon| + I_1 \]

\[ \leq c_{1, \nu} + \left( \frac{1}{p} - \frac{1}{p^*(a, b)} \right) (S_{\mu, a, b}) \frac{p^*(a, b)}{p^*(a, b) - p} \int_{\Omega} \frac{|u_\varepsilon|^{p^*(a, b)}}{|x|^{dp^*(a,d)}} \]

\[ + C(\varepsilon^{(p-1)\beta(\mu) - \delta} + \varepsilon^{\alpha(\mu) - \delta} + \varepsilon^{(p-1)\delta - \alpha(\mu)}) |\ln \varepsilon| + I_1, \]

where

\[ I_1 := \frac{|B|^{p^*(a, b)}}{p^*(a, b)} \int_{\Omega} \frac{|u_\varepsilon|^{p^*(a, b)}}{|x|^{bp^*(a,b)}} - \frac{|B|^{p^*(a, b) - \nu}}{p^*(a, b) - \nu} \int_{\Omega} \frac{|u_\varepsilon|^{p^*(a, b) - \nu}}{|x|^{bp^*(a,b)}} . \]
(i). Assume that $1 < p < 2$ and $q > q_1$, where
\[ q_1 = \max\{p, \frac{N - dp^*(a,d) - (p-1)(\beta(\mu) - \delta)}{\delta}, \frac{N - dp^*(a,d) - (p-1)(\delta - \alpha(\mu))}{\delta}\}. \]

Then
\[ \frac{N - dp^*(a,d) - q\delta}{\delta} < (p-1)(\beta(\mu) - \delta) < \beta(\mu) - \delta, \]
\[ \frac{N - dp^*(a,d) - q\delta}{\delta} < (p-1)(\delta - \alpha(\mu)) < \delta - \alpha(\mu). \]

Since
\[ \frac{N - dp^*(a,d) - (p-1)(\beta(\mu) - \delta)}{\delta} > \frac{N - dp^*(a,d)}{\beta(\mu)}, \]
from Lemma 3.1 it follows that there exists a constant $\sigma > 0$ such that
\[ C(\varepsilon^{(p-1)(\beta(\mu) - \delta)} + \varepsilon^{\beta(\mu) - \varepsilon} + \varepsilon^{(p-1)(\delta - \alpha(\mu))} + \varepsilon^{\delta - \alpha(\mu)})|\ln \varepsilon| \]
\[ - C \int_{|x| < \varepsilon} \frac{u^p_{\varepsilon}}{d^{p^*(a,d)}} \leq -2\sigma. \quad (4.6) \]

Choose $\nu^{**} > 0$ small enough such that $I_1 < \sigma$ for $0 < \nu < \nu^{**}$. Then
\[ c_{2,\nu} \leq J_{\nu}(Au_1 + Bu_\varepsilon) \leq c_{1,\nu} + \left(\frac{1}{p} - \frac{1}{p^{*}(a,b)}\right)(S_{\mu,a,b})^{p^*(a,b)} - p - \sigma, \forall \nu \in (0, \nu^{**}). \]

(ii). Assume that $p \geq 2$ and $q > q_2$, where
\[ q_2 = \max\{p, \frac{N - dp^*(a,d) - (p-1)(\beta(\mu) - \delta)}{\delta}, \frac{N - dp^*(a,d) - (p-1)(\delta - \alpha(\mu))}{\delta}\}. \]

Then
\[ \frac{N - dp^*(a,d) - q\delta}{\delta} < \beta(\mu) - \delta < (p-1)(\beta(\mu) - \delta), \]
\[ \frac{N - dp^*(a,d) - q\delta}{\delta} < \delta - \alpha(\mu) < (p-1)(\delta - \alpha(\mu)). \]

Since
\[ \frac{N - dp^*(a,d) - (p-1)(\beta(\mu) - \delta)}{\delta} > \frac{N - dp^*(a,d)}{\beta(\mu)}, \]
from Lemma 3.1 it follows that (4.6) for some constant $\sigma > 0$. Choose $\nu^{**} > 0$ small enough such that $I_1 < \sigma$ for $0 < \nu < \nu^{**}$. Therefore (4.5) holds.

(iii). $1 < p < 2$, $q = p$ and $0 < \lambda < \Lambda_{\mu,a,d}$. Discussing as above we also have
\[ J_{\nu}(Au_1 + Bu_\varepsilon) \leq c_{1,\nu} + \left(\frac{1}{p} - \frac{1}{p^{*}(a,b)}\right)(S_{\mu,a,b})^{p^*(a,b)} - p - \sigma + C \int_{|x| < \varepsilon} \frac{u^p_{\varepsilon}}{d^{p^*(a,d)}} \]
\[ + C(\varepsilon^{(p-1)(\delta - \alpha(\mu))} + \varepsilon^{(p-1)(\beta(\mu) - \delta)})|\ln \varepsilon| + I_1. \]

If $(p-1)(\beta(\mu) - \delta) > N - dp^*(a,d) - p\delta$, then
\[ \beta(\mu) > \delta + \frac{p(a+1) - dp^*(a,d)}{p-1} > \frac{N - dp^*(a,d)}{p}. \]
If \((p - 1)(\delta - \alpha(\mu)) > N - dp^*(a, d) - p\delta\), then

\[
\alpha(\mu) < \delta - \frac{p(a + 1) - dp^*(a, d)}{p - 1} < \delta.
\]

On the other hand, it’s easy to verify that the function

\[
f(t) = (p - 1)t^p - (N - p(a + 1))t^{p-1} + \mu, \quad t \geq 0,
\]

has the unique minimal point \(\delta\). Furthermore, \(f(t)\) is decreasing on the interval \((0, \delta)\) and is increasing on the interval \((\delta, +\infty)\).

If

\[
N > \tilde{N} := p((p + 1)(a + 1) - dp^*(a, d)),
\]

then

\[
\delta + dp^*(a, d) - p(a + 1) > 0.
\]

Furthermore,

\[
\delta + \frac{p(a + 1) - dp^*(a, d)}{p - 1} < \beta(\mu),
\]

which is equivalent to

\[
f(\delta + \frac{p(a + 1) - dp^*(a, d)}{p - 1}) < f(\beta(\mu)) = 0,
\]

and it is equivalent to

\[
0 \leq \mu < \mu_1,
\]

where

\[
\mu_1 = (\delta + \frac{p(a + 1) - dp^*(a, d)}{p - 1})^{p-1}(\delta + dp^*(a, d) - p(a + 1)).
\]

If

\[
N > N' = p(a + 1 + \frac{p(a + 1) - dp^*(a, d)}{p - 1}),
\]

then

\[
0 < \delta - \frac{p(a + 1) - dp^*(a, d)}{p - 1} < \delta.
\]

Hence

\[
\delta - \frac{p(a + 1) - dp^*(a, d)}{p - 1} > \alpha(\mu),
\]

which is equivalent to

\[
f(\delta - \frac{p(a + 1) - dp^*(a, d)}{p - 1}) < f(\alpha(\mu)) = 0,
\]

and it is equivalent to

\[
0 \leq \mu < \mu_2,
\]
Furthermore, 
\[ \nu < \sigma \] for some constant \( \sigma > 0 \). Choose \( \nu^* > 0 \) small enough such that \( I_1 < \sigma \) for \( 0 < \nu < \nu^* \). Thus (4.5) holds.

(iv). \( p \geq 2, \ q = p \) and \( 0 < \lambda < \Lambda_{\mu,a,d} \). Discussing as above we also have

\[
J_v(Au_1 + Bu_{\varepsilon}) \leq c_{1,v} + \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right) (S_{\mu,a,b})^{-1} p^*(a,b)^{-p} + C(e^{\beta(\mu)-\delta} + e^{\delta-\alpha(\mu)})|\ln \varepsilon| - C\int_{\Omega} \frac{u_{\varepsilon}^p}{|x|^{dp^*(a,d)}} \leq -2\sigma
\]

for some constant \( \sigma > 0 \). Choose \( \nu^* > 0 \) small enough such that \( I_1 < \sigma \) for \( 0 < \nu < \nu^* \). Thus (4.5) holds.

Assume that \( N > N'' = p(a+1+(p-1)(p(a+1) - dp^*(a,d))) \). Then

\[
\delta + (p-1)(dp^*(a,d) - p(a+1)) > 0.
\]

Furthermore,

\[
\delta + p(a+1) - dp^*(a,d) < \beta(\mu),
\]

which is equivalent to

\[
f(\delta + p(a+1) - dp^*(a,d)) < f(\beta(\mu)) = 0,
\]

and it is equivalent to

\[
0 < \mu < \mu_3,
\]

where

\[
\mu_3 = (\delta + p(a+1) - dp^*(a,d))^{p-1} (\delta + (p-1)(dp^*(a,d) - p(a+1))).
\]

If \( \delta - \alpha(\mu) > N - dp^*(a,d) - p\delta \), then

\[
\alpha(\mu) < \delta - p(a+1) + dp^*(a,d) < \delta.
\]

Assume \( N > \tilde{N} := p((p+1)(a+1) - dp^*(a,d)) \). Then

\[
0 < \delta - p(a+1) + dp^*(a,d) < \delta.
\]
Consequently,
\[ \delta - p(a + 1) + dp^*(a, d) > \alpha(\mu), \]
which is equivalent to
\[ f(\delta - p(a + 1) + dp^*(a, d)) < f(\alpha(\mu)) = 0, \]
and it is equivalent to
\[ 0 \leq \mu < \mu_4, \]
where
\[ \mu_4 = \left( \delta - p(a + 1) + dp^*(a, d) \right)^{p-1} \left( \delta + (p - 1)(p(a + 1) - dp^*(a, d)) \right). \]

Now we choose \( N \) and \( \mu \) such that
\[ N > \max \{ N''', \overline{N} \} = N'' \text{ and } 0 < \mu < \min \{ \mu_3, \mu_4 \}. \]

Taking \( \varepsilon \) small we have
\[ C(e^{(p-1)(\delta-\alpha(\mu))} + e^{(p-1)(\beta(\mu) - \delta)}) |\ln \varepsilon| - C \int_{\Omega} \frac{u^p_+}{|x|^{p^*(a,d)}} \leq -2\sigma \]
for some constant \( \sigma > 0 \). Choose \( v^{**} > 0 \) small enough such that \( I_1 < \sigma \) for \( 0 < v < v^{**} \). Therefore (4.5) holds.

The proof of the lemma is completed.

**PROOF OF THEOREM 1.1.** Set \( v_0 = \min \{ v', v^*, v^{**} \} \). From the fact \( c_{1,v} \to c_{1,0} \)
as \( v \to 0 \) and by Lemma 4.4 it follows that \( c_{2,v} \) is bounded uniformly in \( v \in (0, v_0) \).

Let \( u_{2,v} \) be the solution obtained in Lemma 4.3. Then there exists a constant \( C > 0 \) such that
\[ \| u_{2,v} \| \leq C, \quad \forall v \in (0, v_0). \quad (4.7) \]

Define \( u^{\pm}(x) = \max \{ \pm u(x), 0 \} \) respectively for any \( u \in W^{1,p}_{0,a}(\Omega) \), then \( u^{\pm} \in W^{1,p}_{0,a}(\Omega) \).

By (4.6) we can find \( v_n \to 0 \) such that for some \( u \in W^{1,p}_{0,a}(\Omega) \),
\[ u^{\pm}_{2,v_n} \rightharpoonup u^\pm \text{ weakly in } W^{1,p}_{0,a}(\Omega). \]

For convenience, we denote \( u^{\pm}_{2,v_n}, J_{v_n}, c_{1,v_n}, c_{2,v_n} \) and \( \Lambda_{v_n} \) as \( u_n, J_n, c_{1,n}, c_{2,n} \) and \( \Lambda_n \) respectively. Since \( u^{\pm}_n \in \Lambda_n, J_n(u^{\pm}_n) \geq c_{1,n} \).

From Lemma 4.4 it follows that
\[ J_n(u^{+}_n) + J_n(u^{-}_n) = J_n(u_n) = c_{2,n} \leq c_{1,n} + \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right)(S_{\mu,a,b})^{p^*(a,b)-p} - \sigma \]
for \( n \) large. Necessarily,
\[ J_n(u^{\pm}_n) \leq \left( \frac{1}{p} - \frac{1}{p^*(a,b)} \right)(S_{\mu,a,b})^{p^*(a,b)-p} - \sigma. \]
By (4.6) and the fact that $u_\pm^\in \Lambda_n$ we derive

$$C_1 \leq \int_\Omega \frac{|u_\pm^n|^p(a,b)}{|x|^{bp^*(a,b)}} \leq C_2$$

for some positive constants $C_1$ and $C_2$.

Now we study the convergence of $\{u_\pm^\in\}$. Note that $\{u_\pm^\in\}$ is bounded in $W^{1,p}_0(\Omega)$. By Lemma 2.1 and the concentration compactness theorem ([13],[14]) and up to a subsequence, we have that $u_\pm^n \rightharpoonup u_\pm$ strongly in $W^{1,p}_0(\Omega)$ for some $u \in W^{1,p}_0(\Omega)$ and $u_\pm \neq 0$. Therefore $u$ changes sign in $\Omega$, $u_n \rightharpoonup u$ weakly in $W^{1,p}_{0,a}(\Omega)$ and $u$ is a solution of (1.1). Since $c_{2,n} \to c_{2,0}$ as $n \to \infty$, it is easy to verify that $\{u_n\}$ is actually a $PS$ sequence for $J_0$ at level $c_{2,0}$ and by standard arguments we can show that a subsequence of $\{u_n\}$ converges strongly to $u$ in $W^{1,p}_{0,a}(\Omega)$. Therefore $c_{2,0} = J_0(u)$. The proof of Theorem 1.1 is completed. □

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REFERENCES


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