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A BREZIS–NIRENBERG TYPE THEOREM ON LOCAL MINIMIZERS FOR $p(x)$ –KIRCHHOFF DIRICHLET PROBLEMS AND APPLICATIONS

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Abstract. This paper deals with a class of $p(x)$-Kirchhoff Dirichlet problems possessing a variational structure which corresponds to the variational functional E defined on $W_0^{1,p(x)}(\Omega)$. We prove a Brezis-Nirenberg type theorem which asserts that every local minimizer of E in the $C^1(\overline{\Omega})$ topology is also a local minimizer of E in the $W_0^{1,p(x)}(\Omega)$ topology. Some applications of this theorem are given.

1. Introduction

The Kirchhoff type equations, characterized by involving the nonlocal term, and the differential equations with variable exponent are two important research fields having wide-ranging application backgrounds. We refer to [2], [3], [6], [10], [11], [15], [23], [24], [26], [27], [31], [32] and references therein for the former and to [1], [4], [5], [9], [14], [16]-[19], [22], [28], [29], [30], [33]-[36] and references therein for the latter. The study of the Kirchhoff type equations with variable exponent is a new and interesting topic (see [12], [13]). In this paper we consider the following $p(x)$-Kirchhoff Dirichlet problem:

$$\begin{cases} -a\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\Delta_{p(x)}u = b\left(\int_\Omega F(x,u)dx\right)f(x,u(x)) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$,

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right),$$

$$F(x,t) = \int_0^t f(x,s)ds, \quad \text{for } x\in\Omega \text{ and } t\in\mathbb{R},$$

and p, a, b and f satisfy the following conditions:

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(p_0) $p \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and

$$1 < p_- := \inf_{x\in\overline{\Omega}} p(x) \leqslant p_+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty;$$

(a_0) $a : [0,+\infty) \to (0,+\infty)$ is continuous;

(b_0) $b : \mathbb{R} \to \mathbb{R}$ is continuous;

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and there exists $q \in C^0(\overline{\Omega})$ such that $1 < q(x) < p^*(x)$ for $x \in \overline{\Omega}$ and

$$|f(x,t)| \leqslant c_1 + c_2 |t|^{q(x)-1} \text{ for } x \in \Omega \text{ and } t \in \mathbb{R},$$

where c_1 and c_2 are positive constants, and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

Define

$$\widehat{a}(t) = \int_0^t a(s)ds, \forall t \geqslant 0, \ \ \widehat{b}(t) = \int_0^t b(s)ds, \forall t \in \mathbb{R},$$

$$E(u) = \widehat{a}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) - \widehat{b}\left(\int_\Omega F(x,u)dx\right), \ \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Then $E \in C^1\left(W_0^{1,p(\cdot)}(\Omega), \mathbb{R}\right)$. E is the variational functional corresponding to problem (1.1). Every critical point of E is just a weak solution of (1.1).

For the usual Laplacian case, that is, when $a(t) \equiv 1$, $b(t) \equiv 1$ and $p(x) \equiv 2$ in (1.1), Brezis and Nirenberg [8] proved a famous theorem which asserts that every local minimizer of E in the $C^1(\overline{\Omega})$ topology is also a local minimizer of E in the $W_0^{1,2}(\Omega)$ topology. This theorem has been extended to the p-Laplacian case (see [7,21]) and to the $p(x)$-Laplacian case (see [17]). A main result of the present paper is the following theorem which extends the Brezis-Nirenberg's theorem to the $p(x)$-Kirchhoff Dirichlet problem (1.1).

THEOREM 1.1. *Let (p_0), (a_0), (b_0) and (f_0) hold, and let $u_0 \in W_0^{1,p(\cdot)}(\Omega)$ be a local minimizer (resp. a strictly local minimizer) of E in the $C^1(\overline{\Omega})$ topology. Then u_0 is a local minimizer (resp. a strictly local minimizer) of E in the $W_0^{1,p(\cdot)}(\Omega)$ topology.*

Theorem 1.1 is also a new result even for the case that $p(x) \equiv 2$ in (1.1).

It is well known that the fact that u_0 is a local minimizer of E in the $W_0^{1,p(\cdot)}(\Omega)$ topology is more useful than that u_0 is a local minimizer of E in the $C^1(\overline{\Omega})$ topology.

In Section 2 we give the proof of Theorem 1.1. In Section 3 we give some applications of Theorem 1.1 to the existence and multiplicity for problem (1.1).

2. Proof of Theorem 1.1

Before proving Theorem 1.1, let us give some preliminaries.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u|^{p(x)}\, dx < \infty \right\}$$

with the norm

$$|u|_{L^{p(\cdot)}(\Omega)} = |u|_{p(\cdot)} = \inf\left\{ \sigma > 0 \mid \int_\Omega \left| \frac{u}{\sigma} \right|^{p(x)} dx \leqslant 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \mid \ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}.$$

Denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. $|\nabla u|_{p(\cdot)}$ is an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$. In this paper we denote $\|u\| = |\nabla u|_{p(\cdot)}$ for $u \in W_0^{1,p(\cdot)}(\Omega)$. We refer to [5, 14, 20, 25, 34] for the elementary properties of the space $W^{1,p(x)}(\Omega)$.

In what follows, for brevity, we shall write X instead of $W_0^{1,p(\cdot)}(\Omega)$.

The function $u \in X$ is called a (weak) solution of (1.1) if for all $v \in X$,

$$a\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx = b\left(\int_\Omega F(x,u) dx \right) \int_\Omega f(x,u) v dx.$$

Define

$$I_1(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx, \ I_2(u) = \int_\Omega F(x,u) dx, \forall u \in X,$$

$$J(u) = \widehat{a}(I_1(u)) = \widehat{a}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \right), \forall u \in X,$$

$$\Phi(u) = \widehat{b}(I_2(u)) = \widehat{b}\left(\int_\Omega F(x,u) dx \right), \forall u \in X,$$

$$E(u) = J(u) - \Phi(u), \forall u \in X.$$

PROPOSITION 2.1. *Let (p_0), (a_0), (b_0) and (f_0) hold. Then the following statements hold:*

1) $\widehat{a} \in C^1([0,\infty))$, $\widehat{a}(0) = 0$, $\widehat{a}'(t) = a(t) > 0$ *for any* $t \geqslant 0$, $\widehat{a}$ *is strictly increasing on* $[0,\infty)$*;* $\widehat{b} \in C^1(\mathbb{R})$, $\widehat{b}(0) = 0$.

2) $J, \Phi, E \in C^1(X)$, $J(0) = \Phi(0) = E(0) = 0$. *For every* $u, v \in X$, *there holds*

$$E'(u)v = a\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx \\ - b\left(\int_\Omega F(x,u)dx\right)\int_\Omega f(x,u)v dx.$$

Thus, $u \in X$ *is a (weak) solution of* (1.1) *if and only if* u *is a critical point of* E.

3) *The functional* $J : X \to \mathbb{R}$ *is sequentially weakly lower semi-continuous,* $\Phi : X \to \mathbb{R}$ *is sequentially weakly continuous, and thus* E *is sequentially weakly lower semi-continuous.*

Proof. The proof of statements 1) and 2) is immediate. Since the function $\widehat{a}(t)$ is increasing and the convex functional I_1 is sequentially weakly lower semi-continuous, we can see that the functional $J : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous. Noting that the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is compact, we can see that Φ is sequentially weakly continuous. So $E : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous. □

Now let us give the proof of Theorem 1.1 which is similar to the proof of Theorem 3.1 in [17] and is based on the regularity results established in [1,16,19] for the weak solutions of the variable exponent elliptic equations in divergence form.

PROOF OF THEOREM 1.1. We only consider the case that $u_0 \in X$ is a local minimizer of E in the C^1 topology because the proof in the case that u_0 is a strictly local minimizer of E in the C^1 topology is very similar. Now let $u_0 \in X$ be a local minimizer of E in the C^1 topology. Then $E'(u_0)v = 0$ for every $v \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in X, $E'(u_0)v = 0$ for every $v \in X$. Thus, u_0 is a weak solution of (1.1). By the regularity results of [1,16,19], $u_0 \in C^{1,\beta_1}(\overline{\Omega})$ with some $\beta_1 \in (0,1)$. Define

$$G(u) = \int_\Omega \frac{|\nabla u - \nabla u_0|^{p(x)}}{p(x)}dx, \ \forall u \in X.$$

For $\varepsilon \in (0,1)$, put $D_\varepsilon = \{u \in X : G(u) \leqslant \varepsilon\}$. Then D_ε is a bounded, closed and convex subset of X, and it is a neighborhood of u_0 in $W_0^{1,p(x)}(\Omega)$. Since $E : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous and D_ε is a sequentially weakly compact subset of X, $\inf_{D_\varepsilon} E$ is achieved at some $u_\varepsilon \in D_\varepsilon$. By the Lagrange multiplier rule, there exists $\mu_\varepsilon \leqslant 0$ such that $E'(u_\varepsilon) = \mu_\varepsilon G'(u_\varepsilon)$, that is,

$$-a\left(\int_\Omega \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}dx\right)\operatorname{div}(|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon)) - b\left(\int_\Omega F(x,u_\varepsilon)dx\right)f(x,u_\varepsilon) \\ = -\mu_\varepsilon \operatorname{div}(|\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0)). \tag{2.1}$$

Arguing by contradiction, assume that u_0 is not a local minimizer of E in the $W_0^{1,p(x)}(\Omega)$ topology. Then for each $\varepsilon \in (0,1)$, $u_\varepsilon \neq u_0$ and $E(u_\varepsilon) < E(u_0)$. Note that $u_\varepsilon \to u_0$ in $W_0^{1,p(x)}(\Omega)$ as $\varepsilon \to 0$. Below we shall prove that $u_\varepsilon \to u_0$ in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$, which contradicts with the fact that u_0 is a local minimizer of E in the C^1 topology. Writing:

$$a\left(\int_\Omega \frac{|\nabla u_0|^{p(x)}}{p(x)}dx\right) = a_0 \text{ and } a\left(\int_\Omega \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}dx\right) = a_\varepsilon,$$
$$b\left(\int_\Omega F(x,u_0)dx\right) = b_0 \text{ and } b\left(\int_\Omega F(x,u_\varepsilon)dx\right) = b_\varepsilon,$$

then (2.1) becomes

$$-\mathrm{div}\left[a_\varepsilon|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon - \mu_\varepsilon|\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0)\right] = b_\varepsilon f(x,u_\varepsilon). \tag{2.2}$$

Note that $a_\varepsilon \to a_0 > 0$ and $b_\varepsilon \to b_0$ as $\varepsilon \to 0$. Without loss of generality, we can assume $a_\varepsilon \geqslant \frac{1}{2}a_0$ and $|b_\varepsilon| \leqslant |b_0| + 1$.

Dividing both sides of (2.2) by $a_\varepsilon - \mu_\varepsilon$, yields

$$\begin{aligned}-\mathrm{div}&\left\{\frac{1}{a_\varepsilon - \mu_\varepsilon}\left[a_\varepsilon|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon - \mu_\varepsilon|\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0)\right]\right\}\\ &= \frac{b_\varepsilon}{a_\varepsilon - \mu_\varepsilon}f(x,u_\varepsilon).\end{aligned} \tag{2.3}$$

Define $A_\varepsilon : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $B_\varepsilon : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$A_\varepsilon(x,\eta) = \frac{1}{a_\varepsilon - \mu_\varepsilon}\left[a_\varepsilon|\eta|^{p(x)-2}\eta - \mu_\varepsilon|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0)\right],$$

$$B_\varepsilon(x,t) = \frac{b_\varepsilon}{a_\varepsilon - \mu_\varepsilon}f(x,t).$$

Then u_ε is a weak solution of the following problem:

$$\begin{cases} -\mathrm{div}A_\varepsilon(x,\nabla u) = B_\varepsilon(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

We can verify that A_ε and B_ε satisfy the following conditions:

$$A_\varepsilon(x,\eta)\eta \geqslant c_3|\eta|^{p(x)} - c_4,\ \forall x \in \overline{\Omega},\ \eta \in \mathbb{R}^N, \tag{2.5}$$

$$|A_\varepsilon(x,\eta)| \leqslant c_5|\eta|^{p(x)-1} + c_6,\ \forall x \in \overline{\Omega},\ \eta \in \mathbb{R}^N, \tag{2.6}$$

$$|B_\varepsilon(x,t)| \leqslant c_7 + c_8|t|^{q(x)-1},\ \forall x \in \Omega,\ t \in \mathbb{R}, \tag{2.7}$$

where c_i is a positive constant independent of $\varepsilon \in (0,1)$.

The verification of (2.6) and (2.7) is simple, here we only give the proof of (2.5). By the definition of $A_\varepsilon(x,\eta)$,

$$\begin{aligned}A_\varepsilon(x,\eta)\eta &= \frac{1}{a_\varepsilon-\mu_\varepsilon}\Big[(a_\varepsilon|\eta|^{p(x)-2}\eta-\mu_\varepsilon|\eta|^{p(x)-2}\eta)\\ &\qquad -\mu_\varepsilon(|\eta-\nabla u_0|^{p(x)-2}(\eta-\nabla u_0)-|\eta|^{p(x)-2}\eta)\Big]\eta\\ &=: \frac{1}{a_\varepsilon-\mu_\varepsilon}[(a_\varepsilon-\mu_\varepsilon)|\eta|^{p(x)}-\mu_\varepsilon\Gamma],\end{aligned}$$

where $\Gamma=(|\eta-\nabla u_0|^{p(x)-2}(\eta-\nabla u_0)-|\eta|^{p(x)-2}\eta)\eta$. In the proof of Theorem 3.1 in [17], it was proved that

$$|\Gamma|\leqslant\frac{1}{2}|\eta|^{p(x)}+c,$$

where c is a generic positive constant independent of ε. Thus we have

$$\begin{aligned}A_\varepsilon(x,\eta)\eta &\geqslant \frac{1}{a_\varepsilon-\mu_\varepsilon}[(a_\varepsilon-\mu_\varepsilon)|\eta|^{p(x)}-|\mu_\varepsilon|(\frac{1}{2}|\eta|^{p(x)}+c)]\\ &\geqslant \frac{1}{a_\varepsilon+|\mu_\varepsilon|}[(a_\varepsilon+\frac{1}{2}|\mu_\varepsilon|)|\eta|^{p(x)}-c|\mu_\varepsilon|]\\ &\geqslant \frac{1}{2}|\eta|^{p(x)}-c,\end{aligned}$$

and so (2.5) is proved.

It follows from Theorem 4.1 in [19] that $u_\varepsilon\in L^\infty(\Omega)$ and $|u_\varepsilon|_{L^\infty(\Omega)}\leqslant c$ because $\|u_\varepsilon\|_{W_0^{1,p(x)}(\Omega)}$ is bounded uniformly for $\varepsilon\in(0,1)$, where c is a positive constant independent of ε. Furthermore, by Theorem 4.4 in [19], $u_\varepsilon\in C^{0,\alpha_1}(\overline{\Omega})$ and $\|u_\varepsilon\|_{C^{0,\alpha_1}(\overline{\Omega})}\leqslant c$, where $\alpha_1\in(0,1)$ and c are positive constants independent of ε. In addition, by Lemma 4.1 in [16], there exists $\delta_0>0$ such that $u_\varepsilon\in W^{1,p(x)(1+\delta_0)}(\Omega)$.

Below we shall use Theorem 1.2 from [16] to prove that there exist $\alpha\in(0,1)$ and a positive constant c independent of ε such that $\|u_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})}\leqslant c$ for sufficiently small $\varepsilon>0$ for the following two cases, respectively.

Case i): $\mu_\varepsilon\in[-1,0]$.

Noting that u_0 satisfies the equation

$$-\mathrm{div}(|\nabla u_0|^{p(x)-2}\nabla u_0)=\frac{b_0}{a_0}f(x,u_0), \tag{2.8}$$

then (2.2) is equivalent to the following

$$\begin{aligned}&-\mathrm{div}\left\{|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon-\frac{\mu_\varepsilon}{a_\varepsilon}|\nabla u_\varepsilon-\nabla u_0|^{p(x)-2}(\nabla u_\varepsilon-\nabla u_0)-\frac{\mu_\varepsilon}{a_\varepsilon}|\nabla u_0|^{p(x)-2}\nabla u_0\right\}\\ &\qquad=\frac{b_\varepsilon}{a_\varepsilon}f(x,u_\varepsilon)-\frac{\mu_\varepsilon b_0}{a_\varepsilon a_0}f(x,u_0).\end{aligned}$$

Denote $\overline{A}_\varepsilon : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ and $\overline{B}_\varepsilon : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ by

$$\overline{A}_\varepsilon(x,\eta) = |\eta|^{p(x)-2}\eta - \frac{\mu_\varepsilon}{a_\varepsilon}|\eta-\nabla u_0|^{p(x)-2}(\eta-\nabla u_0) - \frac{\mu_\varepsilon}{a_\varepsilon}|\nabla u_0|^{p(x)-2}\nabla u_0,$$
$$\overline{B}_\varepsilon(x,t) = \frac{b_\varepsilon}{a_\varepsilon} f(x,t) - \frac{\mu_\varepsilon b_0}{a_\varepsilon a_0} f(x,u_0).$$

Then u_ε is a weak solution of the following problem:

$$\begin{cases} -\mathrm{div}\overline{A}_\varepsilon(x,\nabla u) = \overline{B}_\varepsilon(x,u) \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

We can prove that, for $x,y\in\overline{\Omega},\ \eta\in\mathbb{R}^N\setminus\{0\},\ \xi\in\mathbb{R}^N,\ t\in\mathbb{R}$, the following statements are true:

$$\overline{A}_\varepsilon(x,0) = 0, \tag{2.10}$$
$$\sum_{i,j=1}^N \frac{\partial(\overline{A}_\varepsilon)_j}{\partial\eta_i}(x,\eta)\xi_i\xi_j \geqslant c_1|\eta|^{p(x)-2}|\xi|^2, \tag{2.11}$$
$$\sum_{i,j=1}^N \left|\frac{\partial(\overline{A}_\varepsilon)_j}{\partial\eta_i}(x,\eta)\right| |\eta| \leqslant c_2(1+|\eta|^{p(x)-1}), \tag{2.12}$$
$$\left|\overline{B}_\varepsilon(x,t)\right| \leqslant c_3 + c_4|t|^{q(x)-1}, \tag{2.13}$$

where c_i is a positive constant independent of ε, and for sufficiently small $\delta>0$, there exists a positive constant C_δ, depending on $p_+,\ p_-$ and δ, but independent of $\mu_\varepsilon\in[-1,0]$, such that

$$\left|\overline{A}_\varepsilon(x,\eta)-\overline{A}_\varepsilon(y,\eta)\right| \leqslant C_\delta|x-y|^\beta(1+|\eta|^{p_{xy}-1+\delta}), \tag{2.14}$$

where $p_{xy} = \max\{p(x),p(y)\}$.

Here we omit the proof of (2.10)-(2.14). The proof of (2.10)-(2.13) is immediate (see [7]), and the proof of (2.14) is similar to the proof of (3.15) in [17].

By Theorem 1.2 in [16], under the conditions (2.10)-(2.14), there exist $\alpha\in(0,1)$ and a positive constant c independent of ε such that $u_\varepsilon\in C^{1,\alpha}(\overline{\Omega})$ and $\|u_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})}\leqslant c$. From this and $u_\varepsilon\to u_0$ in $W_0^{1,p(x)}(\Omega)$ it follows that $u_\varepsilon\to u_0$ in $C^1(\overline{\Omega})$ as $\varepsilon\to 0$.

Case ii): $\mu<-1$.

Set $v_\varepsilon = u_\varepsilon - u_0$. Then from (2.2) and (2.8) we know that v_ε satisfies the equation

$$\begin{aligned} -\mathrm{div}[\frac{1}{|\mu_\varepsilon|}&|\nabla v_\varepsilon+\nabla u_0|^{p(x)-2}(\nabla v_\varepsilon+\nabla u_0) \\ &+\frac{1}{a_\varepsilon}|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon - \frac{1}{|\mu_\varepsilon|}|\nabla u_0|^{p(x)-2}\nabla u_0] \\ = \frac{b_\varepsilon}{a_\varepsilon|\mu_\varepsilon|}&f(x,v_\varepsilon+u_0) - \frac{b_0}{|\mu_\varepsilon|a_0}f(x,u_0). \end{aligned}$$

Define

$$\widetilde{A}_{\varepsilon}(x,\eta)=\frac{1}{|\mu_{\varepsilon}|}|\eta+\nabla u_0|^{p(x)-2}(\eta+\nabla u_0)+\frac{1}{a_{\varepsilon}}|\eta|^{p(x)-2}\eta-\frac{1}{|\mu_{\varepsilon}|}|\nabla u_0|^{p(x)-2}\nabla u_0,$$

$$\widetilde{B}_{\varepsilon}(x,t)=\frac{b_{\varepsilon}}{a_{\varepsilon}\,|\mu_{\varepsilon}|}f(x,t+u_0)-\frac{b_0}{|\mu_{\varepsilon}|\,a_0}f(x,u_0).$$

Then v_{ε} is a solution of the following problem:

$$\begin{cases}-\mathrm{div}\widetilde{A}_{\varepsilon}(x,\nabla v)=\widetilde{B}_{\varepsilon}(x,v) \text{ in } \Omega,\\ v=0 \text{ on } \partial\Omega.\end{cases}$$

Analogously to the case i), we can prove that $\widetilde{A}_{\varepsilon}$ and $\widetilde{B}_{\varepsilon}$ satisfy the corresponding conditions (2.10)-(2.14). So by Theorem 1.2 in [16], $v_{\varepsilon}\in C^{1,\alpha}(\overline{\Omega})$ and $\|v_{\varepsilon}\|_{C^{1,\alpha}(\overline{\Omega})}\leqslant c$, furthermore $v_{\varepsilon}\to 0$ in $C^1(\overline{\Omega})$, that is $u_{\varepsilon}\to u_0$ in $C^1(\overline{\Omega})$ as $\varepsilon\to 0$. The proof of Theorem 1.1 is complete. $\square$

REMARK 2.1. From the proof of Theorem 1.1 we see that, the fact that a_{ε}, as $\varepsilon\to 0^+$, is bounded from below by a positive constant, which is guaranteed by condition (a_0), plays an important role. Sometimes, it is possible to encounter the case that condition (a_0) is not satisfied at $t=0$. It is obvious that, if we replace condition (a_0) by the following condition

$(a_0)'$ $a:(0,+\infty)\to(0,+\infty)$ is continuous and $a\in L^1(0,t)$ for any $t>0$,

then the conclusion of Theorem 1.1 is valid for $u_0\in X\backslash\{0\}$. If we replace condition (a_0) by the following condition

$(a_0)'_+$ a satisfies $(a_0)'$ and $\liminf_{t\to 0+}a(t)>0$,

then the conclusion of Theorem 1.1 is also valid for $u_0=0$, namely, Theorem 1.1 is valid if condition (a_0) in Theorem 1.1 is replaced by the weaker condition $(a_0)'_+$. Note that condition $(a_0)'_+$ implies that $\widehat{a}\in C^0([0,+\infty))$, $\widehat{a}\in C^1((0,+\infty))$, $\widehat{a}(0)=0$, $\widehat{a}'(t)=a(t)>0$ for any $t>0$, and $\widehat{a}$ is strictly increasing on $[0,+\infty)$. A typical example of the function a satisfying condition $(a_0)'_+$ is the function $a(t)=t^{\alpha}$ for $t>0$, where $\alpha\in(-1,0)$.

REMARK 2.2. Problem (1.1), which is considered in Theorem 1.1, it possesses Dirichlet boundary value condition. It is easy to see that the same assertion as in Theorem 1.1 is also true for the corresponding Neumann boundary value problems.

3. Applications of Theorem 1.1

Let us continue to use the notations as in Sections 1 and 2.

For two real numbers $s<t$, define $D_{[s,t]}:=\{u\in X: s\leqslant u(x)\leqslant t \text{ for a.e. } x\in\Omega\}$.

THEOREM 3.1. *Let (p_0), $(a_0)'_+$, (b_0) and (f_0) hold. Suppose that the following conditions are satisfied:*

(f_1) *$f(x,t)=0$ for $x\in\Omega$ and $t\leqslant 0$;*

(a_1) *there exist positive constants* α, M *and* C *such that* $\widehat{a}(t) \geqslant Ct^{\alpha}$ *for* $t \geqslant M$*;*

(b_1) $b(t) \geqslant 0$ *for* $t \in \mathbb{R}$*;*

(f_2) *there exist* $0 < \xi < \eta$ *such that* $F(x,\xi) \geqslant F(x,t)$ *for* $x \in \Omega$ *and* $t \in [\xi,\eta]$*.*
Then problem (1.1) *has a nonnegative solution* u_0 *such that* $u_0(x) \in [0,\xi]$ *for* $x \in \Omega$*,* u_0 *is a global minimizer of the restriction of* E *on* $D_{[0,\eta]}$ *and a local minimizer of* E (*in the* $W_0^{1,p(\cdot)}(\Omega)$ *topology*).

Proof. Let $r_0 > 0$ be fixed. Then $D_{[-r_0,\eta]}$ is a convex and sequentially weakly closed subset of X. It is obvious that there exists a positive constant C_1 such that $\widehat{b}(\int_\Omega F(x,u)dx) \leqslant C_1$ for $u \in D_{[-r_0,\eta]}$. It follows from (a_1) that for $\|u\|$ large enough,

$$\widehat{a}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) \geqslant \widehat{a}\left(\frac{1}{p_+}\|u\|^{p_-}\right) \geqslant C\left(\frac{1}{p_+}\right)^{\alpha}\|u\|^{\alpha p_-}.$$

Thus we have

$$E(u) \to +\infty \text{ as } u \in D_{[-r_0,\eta]} \text{ and } \|u\| \to \infty,$$

that is, E is coercive on $D_{[-r_0,\eta]}$. Since E is sequentially weakly lower semi-continuous, there exists $u_0 \in D_{[-r_0,\eta]}$ such that $E(u_0) = \inf_{u \in D_{[-r_0,\eta]}} E(u)$.

We claim that

$$0 \leqslant u_0(x) \leqslant \xi \text{ for } x \in \Omega. \tag{3.1}$$

To see this, define

$$u_1(x) = \begin{cases} u_0(x) & \text{if } u_0(x) \geqslant 0, \\ 0 & \text{if } u_0(x) < 0, \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} u_0(x) & \text{if } u_0(x) \leqslant \xi, \\ \xi & \text{if } u_0(x) > \xi. \end{cases}$$

It is sufficient to show that $u_0 = u_1$ and $u_0 = u_2$. Here we only give the proof of $u_0 = u_2$ because the proof of $u_0 = u_1$ is similar and simpler. Obviously, $u_2 \in D_{[-r_0,\eta]}$, and thus, $E(u_0) \leqslant E(u_2)$, that is,

$$\widehat{a}(I_1(u_0)) - \widehat{a}(I_1(u_2)) \leqslant \widehat{b}(I_2(u_0)) - \widehat{b}(I_2(u_2)).$$

It follows from (f_2) that $I_2(u_0) \leqslant I_2(u_2)$. By ($b_1$), $\widehat{b}$ is nondecreasing, so $\widehat{b}(I_2(u_0)) \leqslant \widehat{b}(I_2(u_2))$ and consequently, $\widehat{a}(I_1(u_0)) \leqslant \widehat{a}(I_1(u_2))$. Setting

$$\Omega_1 := \{x \in \Omega : u_0(x) \leqslant \xi\} \text{ and } \Omega_2 := \{x \in \Omega : \xi < u_0(x) \leqslant \eta\},$$

then $\nabla u_0(x) = \nabla u_2(x)$ for $x \in \Omega_1$, and $\nabla u_2(x) = 0$ for $x \in \Omega_2$. It is obvious that $I_1(u_0) \geqslant I_1(u_2)$ and $\widehat{a}(I_1(u_0)) \geqslant \widehat{a}(I_1(u_2))$. Thus we have that $\widehat{a}(I_1(u_0)) = \widehat{a}(I_1(u_2))$, and consequently, $I_1(u_0) = I_1(u_2)$, that is

$$\int_\Omega \frac{|\nabla u_0|^{p(x)}}{p(x)}dx = \int_\Omega \frac{|\nabla u_2|^{p(x)}}{p(x)}dx.$$

Since

$$\int_{\Omega_1} \frac{|\nabla u_0|^{p(x)}}{p(x)}dx = \int_{\Omega_1} \frac{|\nabla u_2|^{p(x)}}{p(x)}dx,$$

we have that

$$\int_{\Omega_2} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx = \int_{\Omega_2} \frac{|\nabla u_2|^{p(x)}}{p(x)} dx = 0.$$

From this it follows that

$$\begin{aligned}\int_{\Omega} \frac{|\nabla u_0 - \nabla u_2|^{p(x)}}{p(x)} dx &= \int_{\Omega_1} \frac{|\nabla u_0 - \nabla u_2|^{p(x)}}{p(x)} dx + \int_{\Omega_2} \frac{|\nabla u_0 - \nabla u_2|^{p(x)}}{p(x)} dx \\ &= 0 + \int_{\Omega_2} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx = 0,\end{aligned}$$

and thus $u_0 = u_2$. The claim (3.1) is proved.

The statement (3.1) implies that u_0 is a global minimizer of the restriction of E on $D_{[0,\eta]}$ and a local minimizer u_0 of E in the C^1 topology. By Theorem 1.1 and Remark 2.1, u_0 is a local minimizer of E in the $W_0^{1,p(\cdot)}(\Omega)$ topology. □

THEOREM 3.2. *Let (p_0), $(a_0)'$, (a_1), (b_0), (b_1), (f_0) and (f_1) hold. Suppose that the following conditions are satisfied:*

(a_2) *there exist positive constants δ_1, e_1 and α_1 such that $\widehat{a}(t) \leqslant e_1 t^{\alpha_1}$ for $t \in [0,\delta_1]$;*

(b_2) *there exist positive constants δ_2, e_2 and β_1 such that $\widehat{b}(t) \geqslant e_2 t^{\beta_1}$ for $t \in [0,\delta_2]$;*

(f_3) *for every $n \in \mathbb{N}$, there exist $\xi_n, \eta_n \in \mathbb{R}$ with $0 < \xi_n < \eta_n$ and $\lim_{n\to\infty} \eta_n = 0$ such that*

$$F(x,\xi_n) \geqslant F(x,t) \ \textit{ for } x \in \Omega \textit{ and } t \in [\xi_n, \eta_n];$$

(f_4) *there exist a non-empty open set $U \subset \Omega$, a positive constant L and a sequence $\{\tau_n\}$ with $\tau_n > 0$ and $\lim_{n\to\infty} \tau_n = 0$, such that*

$$\lim_{n\to\infty} \frac{\operatorname{ess\,inf}_{x\in U} F(x,\tau_n)}{\tau_n^r} = +\infty$$

and

$$\operatorname{ess\,inf}_{x\in U} \left(\inf_{t\in[0,\tau_n]} F(x,t) \right) \geqslant -L \cdot \operatorname{ess\,inf}_{x\in U} F(x,\tau_n),$$

where $r = \frac{\alpha_1 p_-}{\beta_1}$.

Then problem (1.1) has a sequence $\{u_n\}$ of non-trivial nonnegative solutions such that u_n is a local minimizer of E, $E(u_n) < 0$, $\lim_{n\to\infty} E(u_n) = 0$, $\lim_{n\to\infty} |u_n|_{L^\infty(\Omega)} = 0$ and $\lim_{n\to\infty} \|u_n\| = 0$.

Proof. From Theorem 3.1 we know that, for every $n \in \mathbb{N}$, there exists $u_n \in D_{[0,\xi_n]}$ such that $E(u_n) = \inf_{u\in D_{[0,\eta_n]}} E(u)$. We shall prove that $E(u_n) < 0$. Now let $n \in \mathbb{N}$ be fixed. Choose a compact set $K \subset U$ such that $0 < |K| = (L+1)|U \backslash K|$. Define a function $v \in X$ such that $v(x) = 1$ if $x \in K$; $v(x) = 0$ if $x \in \Omega \backslash U$; $v(x) \in [0,1]$ if $x \in U \backslash K$.

Define

$$h_k = \operatorname{ess\,inf}_{x\in U} F(x,\tau_k) \text{ for each } k \in \mathbb{N},$$

$$d := \frac{(L+1)}{|K|}\left(\frac{2e_1}{e_2}\right)^{\frac{1}{\beta_1}}\left(\int_\Omega \frac{|\nabla v|^{p(x)}}{p(x)}dx\right)^{\frac{\alpha_1}{\beta_1}}. \tag{3.2}$$

By (f_4), there exists $k \in \mathbb{N}$ large enough such that:

$$\begin{gathered}\tau_k \leqslant \min\{\eta_n, 1\},\ h_k > d(\tau_k)^r,\\ \int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx \leqslant \delta_1 \quad \text{and} \quad \left|\int_\Omega F(x,\tau_k v)dx\right| \leqslant \delta_2.\end{gathered} \tag{3.3}$$

Then by (a_2),

$$\begin{aligned}\widehat{a}\left(\int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx\right) &\leqslant e_1\left(\int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx\right)^{\alpha_1}\\ &\leqslant e_1(\tau_k)^{\alpha_1 p_-}\left(\int_\Omega \frac{|\nabla v|^{p(x)}}{p(x)}dx\right)^{\alpha_1},\end{aligned} \tag{3.4}$$

by (f_4),

$$\begin{aligned}\int_\Omega F(x,\tau_k v)dx &= \int_K F(x,\tau_k v)dx + \int_{U\setminus K} F(x,\tau_k v)dx\\ &\geqslant h_k|K| - Lh_k|U\setminus K| = \frac{h_k}{L+1}|K|,\end{aligned} \tag{3.5}$$

and by (b_2),

$$\begin{aligned}\widehat{b}\left(\int_\Omega F(x,\tau_k v)dx\right) &\geqslant e_2\left(\int_\Omega F(x,\tau_k v)dx\right)^{\beta_1} \geqslant e_2\left(\frac{h_k}{L+1}|K|\right)^{\beta_1}\\ &> e_2\left(\frac{d(\tau_k)^r}{L+1}|K|\right)^{\beta_1}.\end{aligned} \tag{3.6}$$

From (3.2) and $r = \frac{\alpha_1 p_-}{\beta_1}$, we obtain

$$\widehat{b}\left(\int_\Omega F(x,\tau_k v)dx\right) > 2\widehat{a}\left(\int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx\right) \tag{3.7}$$

and consequently,

$$\begin{aligned}E(\tau_k v) &= \widehat{a}\left(\int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx\right) - \widehat{b}\left(\int_\Omega F(x,\tau_k v)dx\right)\\ &< -\frac{1}{2}\widehat{b}\left(\int_\Omega F(x,\tau_k v)dx\right) < 0.\end{aligned}$$

Since $0 \leqslant \tau_k v(x) \leqslant \tau_k \leqslant \eta_n$, we have $E(u_n) \leqslant E(\tau_k v) < 0$. This implies that $u_n \neq 0$ because $E(0) = 0$. By Remark 2.1, Theorem 1.1 is valid for $u_n \neq 0$ when condition (a_0) is replaced by $(a_0)'$, and thus u_n is a local minimizer of E and is a solution of problem (1.1).

Since $|u_n|_{L^\infty(\Omega)} \leqslant \xi_n$, we have $|u_n|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$, and consequently,

$$\widehat{b}\left(\int_\Omega F(x,u_n)dx\right) \to 0 \text{ as } n \to \infty.$$

Noting that $E(u_n) < 0$ and $\widehat{a}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) > 0$, we have

$$\widehat{a}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) \to 0 \text{ as } n \to \infty.$$

From this we obtain that $\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx \to 0$, $\|u_n\| \to 0$ and $E(u_n) \to 0$ as $n \to \infty$. $\square$

REMARK 3.1. Theorem 3.2 is a generalization of the main results established in [23] and [13]. In [23] and [13] the Kirchhoff Dirichlet problem:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

and the $p(x)$-Kirchhoff Dirichlet problem:

$$\begin{cases} -\left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

were considered respectively, where a and b are positive constants. In [23] an example of the function f satisfying the corresponding conditions (f_3) and (f_4) was given.

In Theorem 3.2, the function $f(x,t)$ is oscillatory near $t = 0$. Similarly, we can consider the case that the function $f(x,t)$ is oscillatory at infinity, and obtain the following theorem.

THEOREM 3.3. *Let (p_0) $(a_0)'$, (a_1), (b_0), (b_1), (f_0) and (f_1) hold. Suppose that the following conditions are satisfied:*

(a_3) *there exist positive constants M_1, e_1 and α_1 such that $\widehat{a}(t) \leqslant e_1 t^{\alpha_1}$ for $t \geqslant M_1$;*

(b_3) *there exist positive constants M_2, e_2 and β_1 such that $\widehat{b}(t) \geqslant e_2 t^{\beta_1}$ for $t \geqslant M_2$;*

(f_5) *for every $n \in \mathbb{N}$, there exist $\xi_n, \eta_n \in \mathbb{R}$ with $0 < \xi_n < \eta_n$ and $lim_{n\to\infty}\eta_n = +\infty$ such that*

$$F(x,\xi_n) \geqslant F(x,t) \text{ for } x \in \Omega \text{ and } t \in [\xi_n, \eta_n];$$

(f_6) *there exist a non-empty open set $U \subset \Omega$, a positive constant L and a sequence $\{\tau_n\}$ with $\tau_n > 0$ and $\lim_{n\to\infty}\tau_n = +\infty$, such that*

$$\lim_{n\to\infty} \frac{ess\ inf_{x\in U} F(x,\tau_n)}{\tau_n^r} = +\infty$$

and

$$\text{ess } \inf_{x\in U}\left(\inf_{t\in[0,\tau_n]}F(x,t)\right) \geqslant -L\cdot \text{ess } \inf_{x\in U}F(x,\tau_n),$$

where $r = \frac{\alpha_1 p_-}{\beta_1}$. *Then problem* (1.1) *has a sequence* $\{u_n\}$ *of non-trivial nonnegative solutions such that* u_n *is a local minimizer of* E, $E(u_n) < 0$, $\lim_{n\to\infty} E(u_n) = -\infty$, $\lim_{n\to\infty} |u_n|_{L^\infty(\Omega)} = \infty$ *and* $\lim_{n\to\infty} \|u_n\| = \infty$.

Proof. Let K, v, h_k and d be as in the proof of Theorem 3.2. Then, similar to the proof of Theorem 3.2, (3.5) holds, and consequently, for sufficiently large k, $\int_\Omega F(x,\tau_k v)dx \geqslant M_2$. By (f_6), there exists $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$,

$$h_k > d(\tau_k)^r,\ \int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx \geqslant M_1 \ \text{ and } \ \int_\Omega F(x,\tau_k v)dx \geqslant M_2. \tag{3.8}$$

For every $k \geqslant k_0$, similar to the proof of Theorem 3.2, we can obtain (3.4), (3.6) and (3.7), and consequently,

$$\begin{aligned} E(\tau_k v) &= \widehat{a}\left(\int_\Omega \frac{|\nabla(\tau_k v)|^{p(x)}}{p(x)}dx\right) - \widehat{b}\left(\int_\Omega F(x,\tau_k v)dx\right) \\ &< -\frac{1}{2}\widehat{b}\left(\int_\Omega F(x,\tau_k v)dx\right) \leqslant -\frac{1}{2}e_2\left(\frac{d(\tau_k)^r}{L+1}|K|\right)^{\beta_1}. \end{aligned} \tag{3.9}$$

The statement (3.9) shows that $E(\tau_k v) < 0$ for every $k \geqslant k_0$ and $E(\tau_k v) \to -\infty$ as $k \to \infty$. Since $\lim_{n\to\infty}\eta_n = +\infty$, for every $k \geqslant k_0$, there exists $n_k \in \mathbb{N}$ such that $\tau_k < \eta_{n_k}$. By Theorem 3.1, there exists $u_{n_k} \in D_{[0,\xi_{n_k}]}$ such that $E(u_{n_k}) = \inf_{u\in D_{[0,\eta_{n_k}]}} E(u)$. Thus $E(u_{n_k}) \leqslant E(\tau_k v) < 0$ and $E(u_{n_k}) \to -\infty$ as $k \to \infty$. This shows that $u_{n_k} \neq 0$, u_{n_k} is a local minimizer of E, $\|u_{n_k}\| \to \infty$ and $|u_{n_k}|_{L^\infty(\Omega)} \to \infty$ as $k \to \infty$. □

The following theorem provides a simple example of applying Theorem 1.1 and the mountain pass theorem in combination.

THEOREM 3.4. *Let* (p_0), $(a_0)'_+$, (b_0) *and* (f_0) *hold. Suppose that the following conditions are satisfied:*

(f_7) *there exists* $\delta > 0$ *such that* $F(x,t) \leqslant 0$ *for* $x \in \Omega$ *and* $t \in [-\delta,\delta]$;

(b_4) $\widehat{b}(t) \leqslant 0$ *for* $t \leqslant 0$;

(E_1) *there exists* $u_* \in W_0^{1,p(\cdot)}(\Omega)$ *such that* $E(u_*) < 0$;

(E_2) E *satisfies the* $(P.S)_c$ *condition for every* $c > 0$.

Then problem (1.1) *has a mountain pass type solution with positive energy.*

Proof. We know $E(0) = 0$. For any $u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}$ with $|u|_{L^\infty(\Omega)} \leqslant \delta$, we obtain from (a_0) that $\widehat{a}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) > 0$ and from (f_7) and (b_4) that

$$\widehat{b}\left(\int_\Omega F(x,u)dx\right) \leqslant 0,$$

and consequently, $E(u) > 0$. This shows that 0 is a strictly local minimizer of E in the $C^0(\overline{\Omega})$ topology, and hence 0 is a strictly local minimizer of E in the $C^1(\overline{\Omega})$ topology. By Theorem 1.1 and Remark 2.1, 0 is a strictly local minimizer of E in the $W_0^{1,p(\cdot)}(\Omega)$ topology. Thus there exists $r > 0$ such that $E(u) > 0$ for every $u \in X\backslash\{0\}$ with $\|u\| \leqslant r$.

We claim that $\inf_{\|u\|=r} E(u) > 0$. To prove this claim, arguing by contradiction, assume that there exists a sequence $\{u_n\} \subset X$ with $\|u_n\| = r$ such that $E(u_n) \to 0$ as $n \to \infty$. We may assume that $u_n \rightharpoonup u_0$ weakly in X. Since E is sequentially weakly lower semi-continuous, we have that $E(u_0) = 0$ and hence $u_0 = 0$. Since Φ is sequentially weakly continuous, we have that $\Phi(u_n) \to \Phi(0) = 0$, and hence $J(u_n) = E(u_n) + \Phi(u_n) \to 0$. It follows from this that $u_n \to 0$ in X which contradicts with $\|u_n\| = r$.

In virtue of conditions (E_1) and (E_2), we can apply the mountain pass theorem and complete the proof of Theorem 3.4. □

REMARK 3.2. It is easy to give some sufficient conditions in order that conditions (E_1) and (E_2) hold.

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