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ON AN EIGHTH ORDER OVERDETERMINED ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. We consider the overdetermined boundary value problem for the 4-harmonic operator, $\Delta^4 = \Delta(\Delta^3)$, and show that if the solution of the problem exists, then the domain must be an open N-ball $(N \geqslant 2)$. As a consequence of overdetermined problems mean value results are obtained for harmonic, biharmonic, triharmonic and 4-harmonic functions.

1. Introduction

In 1971, J. Serrin [20] used the moving plane method also called "reflection method" and proved that if D is a bounded domain in $\mathbb{R}^N$ and u satisfies the overdetermined problem:

$$\begin{cases} \Delta u = -1 & \text{in } D, \\ u = 0,\ \frac{\partial u}{\partial n} = C \text{ (constant)} & \text{on } \partial D, \end{cases} \tag{1}$$

where Δ is the N-dimensional Laplace operator, $\frac{\partial}{\partial n}$ is the outer normal derivative operator on the boundary and C is a negative constant, then D is a ball of radius $N|C|$ and that the solution of (1) is given by

$$u = \frac{(NC)^2 - r^2}{2N}.$$

A simpler proof of this result based on Hopf maximum principle and Green identities was presented by Weinberger [22]. Weinberger's method has been extended by several authors [5], [9] and [19]. Bennett [5] modified Weinberger's argument and proved that if the problem

$$\begin{cases} \Delta^2 u = -1 & \text{in } D, \\ u = \frac{\partial u}{\partial n} = 0, \Delta u = C & \text{on } \partial D, \end{cases}$$

has a solution then D is a N-ball of radius $[|C|N(N+2)]^{1/2}$ and that

$$u(x) = -\frac{1}{2N}\left[\frac{1}{4}(N+2)(NC)^2 + \frac{NC}{2}r^2 + \frac{r^4}{4(N+2)}\right].$$

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Later, Dalmasso [9] used Serrin's method of moving plane and gave alternating proof of Bennett's result. Payne and Schaefer [17] considered a number of fourth order problems and showed, in particular, that if the problem

$$\begin{cases} \Delta^2 u = 1 & \text{in } D, \\ u = \Delta u = 0, \frac{\partial u}{\partial n} = C & \text{on } \partial D, \end{cases}$$

has a solution for $N = 2$ and D is star-shaped then D is an open disk. It was conjectured that this result holds for $N > 2$ and for more general domains. The validity of this conjecture was proved in [18] by the Serrin reflection method. A new proof of the conjecture by means of suitably defined auxiliary functions was also presented in [12]. In [15] the authors considered second, fourth and $2N$-th order overdetermined problems and introduced a new method for determining the geometry of the domain. First an integral identity equivalent to the problem was obtained and then this integral dual was used to conclude that the domain must be a N-ball. Authors also conjectured that if the problem

$$\begin{cases} \Delta^3 u = -1 & \text{in } D, \\ u = \frac{\partial u}{\partial n} = \Delta u = 0, \frac{\partial \Delta u}{\partial n} = -C & \text{on } \partial D, \end{cases}$$

where D is a bounded domain in $\mathbb{R}^N (N > 2)$ with $C^{6+\varepsilon}$ boundary ∂D, has a sufficiently smooth solution in $C^6(\bar{D})$, then D is a N-ball. The proof of this problem was completed in [11].

All these methods have been extensively used to investigate a variety of overdetermiend problems. In particular see [1, 2, 15, 16, 19, 23] and references therein.

Here, in this note we consider the eighth order overdetermined elliptic boundary value problem and use the method of Weinberger [22] and Bennett [5] to show that if the solution of the problem exists, then the domain is a N-ball $(N \geqslant 2)$. An alternative proof, based on the method of [15], is also obtained. As a consequence of overdetermined problems mean value type results are obtained for harmonic, biharmonic, triharmonic and 4-harmonic functions. We shall use the comma notation for partial differentiation and the summation convention, i.e., a repeated index in a term signifies summation over the index from 1 to N.

2. Main result

We prove the following theorem:

THEOREM 1. *Let D be a bounded domain in $\mathbb{R}^N$ with $C^{8+\varepsilon}$ boundary ∂D and suppose that the eighth order overdetermined problem*

$$\Delta^4 u = -1 \text{ in } D, \tag{2}$$

$$u = \frac{\partial u}{\partial n} = \Delta u = \frac{\partial \Delta u}{\partial n} = 0 \text{ on } \partial D, \tag{3}$$

$$\Delta^2 u = -C\ (Const)\ \ on\ \partial D, \tag{4}$$

has a solution $u \in C^8(\bar{D})$. Then D is an open N-ball $(N \geqslant 2)$ of radius

$$R = (|C|N(N+2)(N+4)(N+6))^{1/4} \tag{5}$$

and

$$\begin{aligned} u(x) = &-\frac{r^8}{384N(N+2)(N+4)(N+6)} + \frac{Cr^6}{96(|C|N(N+2)(N+4)(N+6))^{1/2}} \\ &- \frac{Cr^4}{64} + \frac{C(|C|N(N+2)(N+4)(N+6))^{1/2}}{96} r^2 \\ &- \frac{C^2(N(N+2)(N+4)(N+6))}{384}, \end{aligned} \tag{6}$$

where r is the distance of x from the center of D.

Proof. First we prove the following lemmas.

LEMMA 1. *If $u \in C^8(\bar{D})$ is the solution of the problem (2), (3) and (4), then*

$$\int_D u dx = -\frac{NVC^2}{N+8}, \tag{7}$$

where V denotes the volume of D.

Proof. We note that if u satisfies (2) and r denotes the distance from x to the fixed origin of D, then

$$\Delta^4 (r\frac{\partial u}{\partial r}) = r\frac{\partial}{\partial r}\Delta^4 u + 8\Delta^4 u = -8. \tag{8}$$

With the help of (8), we get

$$\begin{aligned} \int_D \big(-r\frac{\partial u}{\partial r} + 8u\big)dx &= \int_D \Big[r\frac{\partial u}{\partial r}\Delta^4 u - u\Delta^4\big(r\frac{\partial u}{\partial r}\big)\Big]dx \\ &= \int_{\partial D}\Big[r\frac{\partial u}{\partial r}\frac{\partial \Delta^3 u}{\partial n} - \Delta^3 u\frac{\partial}{\partial n}\big(r\frac{\partial u}{\partial r}\big) \\ &\qquad + \big(r\frac{\partial \Delta u}{\partial r} + 2\Delta u\big)\frac{\partial \Delta^2 u}{\partial n} - \Delta^2 u\frac{\partial}{\partial n}\big(r\frac{\partial \Delta u}{\partial r} + 2\Delta u\big)\Big]ds \\ &= \int_{\partial D}\Big[r\frac{\partial r}{\partial n}\frac{\partial u}{\partial n}\frac{\partial \Delta^3 u}{\partial n} - \Delta^3 u\frac{\partial}{\partial n}\big(r\frac{\partial r}{\partial n}\frac{\partial u}{\partial n}\big) \\ &\qquad + \big(r\frac{\partial r}{\partial n}\frac{\partial \Delta u}{\partial n} + 2\Delta u\big)\frac{\partial \Delta^2 u}{\partial n} \\ &\qquad - \Delta^2 u\frac{\partial}{\partial n}\big(r\frac{\partial r}{\partial n}\frac{\partial \Delta u}{\partial n} + 2\Delta u\big)\Big]ds \end{aligned}$$

$$= \int_{\partial D} \Big[-\Delta^3 u \Big(\frac{\partial u}{\partial n} \frac{\partial}{\partial n} \Big(r \frac{\partial r}{\partial n} \Big) + r \frac{\partial r}{\partial n} \frac{\partial^2 u}{\partial n^2} \Big)$$

$$- \Delta^2 u \Big(r \frac{\partial r}{\partial n} \frac{\partial^2 \Delta u}{\partial n^2} + \frac{\partial \Delta u}{\partial n} \frac{\partial}{\partial n} \Big(r \frac{\partial r}{\partial n} \Big) + 2 \frac{\partial \Delta u}{\partial n} \Big) \Big] ds, \quad (9)$$

where in the second equality we used the Green identity for the 4-Laplacian and in the last equality we used the fact that $\frac{\partial u}{\partial n} = \Delta u = \frac{\partial \Delta u}{\partial n} = 0$ on the boundary. Now since $u = \frac{\partial u}{\partial n} = 0$ on ∂D, we have $\frac{\partial^2 u}{\partial n^2} = \Delta u$. Also, in view of, $\Delta u = \frac{\partial \Delta u}{\partial n} = 0$ on the boundary, we observe that $\frac{\partial^2 \Delta u}{\partial n^2} = \Delta^2 u$. Consequently (9) reduces to

$$\int_D \left(-r \frac{\partial u}{\partial r} + 8u \right) dx = -C^2 \int_{\partial D} r \frac{\partial r}{\partial n} ds = -C^2 NV, \quad (10)$$

by the second Green identity, where V is the volume of D and N is the number of dimensions. Furthermore,

$$\int_D r \frac{\partial u}{\partial r} dx = \int_D \text{grad}\, \Big(\frac{r^2}{2}\Big)\, \text{grad}\, u\, dx$$

$$= -\int_D u \Delta \Big(\frac{r^2}{2}\Big) dx = -N \int_D u dx, \quad (11)$$

where we used the Green's first identity and the fact that $u = 0$ on ∂D. Consequently, by (10) and (11), we get (7), which completes the proof of Lemma 1. □

LEMMA 2. *The function A defined by*

$$A = (\Delta u)_{,ij} (\Delta u)_{,ij} - (\Delta u)_{,i} (\Delta^2 u)_{,i} - \frac{1}{4} (\Delta^2 u)^2 + \frac{1}{2} \Delta u \Delta^3 u + \frac{u}{2} + \Delta \alpha$$

attains its maximum value on ∂D provided the function $\alpha(x)$ satisfies:

$$\Delta^2 \alpha = \frac{3}{2} \frac{N-2}{N+2} (\Delta^2 u)_{,i} (\Delta^2 u)_{,i} \quad \text{in } D, \quad (12)$$

$$\Delta \alpha = -\frac{N+6}{N+8} C^2 \quad \text{on } \partial D, \quad (13)$$

$$\frac{\partial \alpha}{\partial n} = -\frac{2NVC^2}{S(N+8)} \quad \text{on } \partial D, \quad (14)$$

where S is the surface area of D.

Proof. First, we show that the problem (12), (13) and (14) has a solution. Clearly, if α is a solution, $\alpha + constant$ is also a solution. We assert that for fixed $\Delta^2 \alpha$ and $\Delta \alpha$ there is a unique $\frac{\partial \alpha}{\partial n}$ to ensure the existence of α. To show this, we let

$$\beta(x) = \Delta \alpha + \frac{N+6}{N+8} C^2.$$

Then β satisfies the Dirichlet problem

$$\begin{cases} \Delta\beta = \frac{3}{2}\frac{N-2}{N+2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} & \text{in } D, \\ \beta = 0 & \text{on } \partial D. \end{cases}$$

Thus, a unique β is guaranteed and by the maximum principle $\beta < 0$ in D. To determine α, we have

$$\begin{cases} \Delta\alpha = \beta(x) - \frac{N+6}{N+8}C^2 & \text{in } D, \\ \frac{\partial\alpha}{\partial n} = -\frac{2NVC^2}{S(N+8)} & \text{on } \partial D. \end{cases}$$

Integrating the equation over D and using the second Green identity, we have

$$\int_{\partial D} \frac{\partial\alpha}{\partial n} ds = \int_D \beta dx - \frac{N+6}{N+8}C^2 V, \tag{15}$$

or,

$$\left(\frac{-2NVC^2}{S(N+8)}\right) S = \int_D \beta dx - \frac{N+6}{N+8}C^2 V. \tag{16}$$

Remember that $\int_D \beta dx$ is uniquely determined by $\frac{3}{2}\frac{N-2}{N+2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i}$, so for fixed $\Delta^2\alpha$ and $\Delta\alpha$ there is only one $\left(-\frac{2NVC^2}{S(N+8)}\right)$, given by the relation (15) above, to ensure the existence of α. Now we compute

$$\begin{aligned} \Delta A &= 2(\Delta u)_{,ijk}(\Delta u)_{,ijk} + 2(\Delta u)_{,ij}(\Delta^2 u)_{,ij} - 2(\Delta u)_{,ij}(\Delta^2 u)_{,ij} - (\Delta u)_{,i}(\Delta^3 u)_{,i} \\ &\quad - (\Delta^2 u)_{,i}(\Delta^2 u)_{,i} - \frac{1}{2}\Delta^2 u \Delta^3 u - \frac{1}{2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} \\ &\quad - \frac{1}{2}\Delta u + \frac{1}{2}\Delta^2 u \Delta^3 u + (\Delta u)_{,i}(\Delta^3 u)_{,i} + \frac{\Delta u}{2} + \Delta^2\alpha \\ &= 2(\Delta u)_{,ijk}(\Delta u)_{,ijk} - \frac{3}{2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} + \frac{3}{2}\frac{N-2}{N+2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} \\ &= 2\big((\Delta u)_{,ijk}(\Delta u)_{,ijk} - \frac{3}{N+2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i}\big). \end{aligned} \tag{17}$$

To show that the righthand side of (17) is nonnegative we observe, as in [5], that for an arbitrary real number γ,

$$\sum_{i,j,k} \left[(\Delta u)_{,ijk} - \gamma\{(\Delta^2 u)_{,i}\delta_{jk} + (\Delta^2 u)_{,j}\delta_{ik} + (\Delta^2 u)_{,k}\delta_{ij}\}\right]^2 \geqslant 0. \tag{18}$$

This inequality reduces to

$$(\Delta u)_{,ijk}(\Delta u)_{,ijk} - 6\gamma(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} + 3\gamma^2(N+2)(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} \geqslant 0, \tag{19}$$

and the discriminant condition for the quadratic expression in γ is

$$\Delta A = (\Delta u)_{,ijk}(\Delta u)_{,ijk} - \frac{3}{N+2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} \geqslant 0.$$

Hence A is subharmonic in D and consequently attains its maximum value on ∂D. This proves Lemma 2.

Next, we show that A is constant in D. We note by the boundary conditions (3), (4) and (13) that

$$A = C^2 - \frac{C^2}{4} - \frac{N+6}{N+8}C^2 = -\frac{NC^2}{4(N+8)} \text{ on } \partial D. \tag{20}$$

Now integrating A on D

$$\int_D A dx = -\frac{7}{4}\int_D u dx + \int_{\partial D} \frac{\partial \alpha}{\partial n} ds = -\frac{NC^2}{4(N+8)}V, \tag{21}$$

where we have used Lemma 1, Green identities and the boundary condition (14). Hence, by (20) and (21),

$$A \equiv -\frac{NC^2}{4(N+8)} \quad \text{in } \bar{D}.$$

This implies that ΔA vanishes identically in $\bar{D}$ and therefore

$$(\Delta u)_{,ijk}(\Delta u)_{,ijk} - \frac{3}{N+2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} \equiv 0 \quad \text{in } \bar{D}. \tag{22}$$

Hence, each term of the sum in (18) vanishes when $\gamma = \frac{1}{N+2}$. By differentiating each term by x_k and adding, we get

$$(\Delta u)_{,ijkk} = (\Delta^2 u)_{,ij} = \frac{1}{N+2}\left[2(\Delta^2 u)_{,ij} + \Delta^3 u \delta_{ij}\right], \tag{23}$$

or,

$$(\Delta^2 u)_{,ij} = \frac{\Delta^3 u}{N}\delta_{ij}. \tag{24}$$

Taking Laplacian of both the sides,

$$(\Delta^3 u)_{,ij} = -\frac{1}{N}\delta_{ij}.$$

Integrating, for a suitable choice of origin, we find that

$$\Delta^3 u = \frac{1}{2N}(A_1 - r^2), \tag{25}$$

where A_1 is an arbitrary constant. Now (24) and (25) yield

$$(\Delta^2 u)_{,ij} = \frac{1}{2N^2}(A_1 - r^2)\delta_{ij}. \tag{26}$$

Again integrating and using the fact that $\Delta^2 u = -C$ on ∂D, we see that

$$r^4 - 6A_1 r^2 - 24N^2 A_2 r - 24N^2(A_3 + C) = 0, \tag{27}$$

where A_2 and A_3 are constants. (27) implies that $r =$ constant on ∂D. Thus, D is a ball with its center at the origin.

When D is a N-ball, to determine the solution of the problem we use the particular solution

$$u = -\frac{r^8}{384N(N+2)(N+4)(N+6)} + B_1 r^6 + B_2 r^4 + B_3 r^2 + B_4, \tag{28}$$

of (2) where B_1, B_2, B_3 and B_4 are constants. Using the boundary conditions (3) and (4) it is easily checked that the radius R of the ball and the solution of the problem are given respectively by (5) and (6). This completes the proof of Theorem 1. $\square$

As a consequence of Theorem 1 we derive the following corollary.

COROLLARY 1. *Let D be a bounded domain in $\mathbb{R}^N (N \geqslant 2)$ with $C^{8+\varepsilon}$ boundary ∂D of positive Gaussian curvature and suppose there is a constant M such that*

$$\int_D B(1+Pu) = -M \int_{\partial D} \frac{\partial \Delta B}{\partial n} ds, \tag{29}$$

for every function B satisfying

$$\begin{cases} \Delta^4 B = PB & \text{in } D, \\ B = \frac{\partial B}{\partial n} = \Delta B = 0 & \text{on } \partial D, \end{cases} \tag{30}$$

where the function $P \geqslant 0$ and $u \in C^8(\bar{D})$ is the solution of the boundary value problem

$$\begin{cases} \Delta^4 u = -1 & \text{in } D, \\ u = \frac{\partial u}{\partial n} = \Delta u = \frac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial D, \end{cases} \tag{31}$$

then D is a N-ball.

Proof. From the Green identity for the 4-harmonic operator, (30) and (31) it follows that

$$\int_D B(1+Pu)dx = \int_{\partial D} \Delta^2 u \frac{\partial \Delta B}{\partial n} ds. \tag{32}$$

We see from (29) that

$$\int_{\partial D} (\Delta^2 u + M) \frac{\partial \Delta B}{\partial n} ds = 0. \tag{33}$$

Now we choose $B \in C^8(\bar{D})$ to be the solution of

$$\begin{cases} \Delta^4 B = PB & \text{in } D, \\ B = \frac{\partial B}{\partial n} = \Delta B = 0, \frac{\partial \Delta B}{\partial n} = \Delta^2 u + M & \text{on } \partial D. \end{cases}$$

It is immediate from (33) that

$$\Delta^2 u = -M \text{ on } \partial D.$$

Hence, Theorem 1 implies that D is an open N-ball $(N \geqslant 2)$. This completes the proof of Corollary 1.

3. Alternative proof

An alternative proof of Theorem 1 can be given by reformulating the problem in an equivalent integral form. An integral dual for (2), (3) and (4) is

$$-\int_D F\,dx = C\int_{\partial D}\frac{\partial \Delta F}{\partial n}ds, \tag{34}$$

where F is any function satisfying

$$\begin{cases}\Delta^4 F = 0 & \text{in } D,\\ F = \frac{\partial F}{\partial n} = \Delta F = 0 & \text{on } \partial D.\end{cases} \tag{35}$$

The equality (34) is a consequence of

$$\int_{\partial D}\frac{\partial \Delta F}{\partial n}(\Delta^2 u + C)ds = \int_D F\,dx + C\int_{\partial D}\frac{\partial \Delta F}{\partial n}ds, \tag{36}$$

which is obtained from the Green identity for 4-Laplacian. Now letting $F = x_i u_{,i} - 8u$ in (34), it is immediate that (7) holds true.

We define the function λ such that

$$\begin{cases}\Delta\lambda = -1 & \text{in } D,\\ \lambda = 0 & \text{on } \partial D,\end{cases} \tag{37}$$

and let A be as in Lemma 2. Then $\Delta A \geqslant 0$ and from the second Green identity, (20) and (21) we deduce that

$$\int_D \lambda \Delta A\,dx = 0. \tag{38}$$

Since $\lambda > 0$, we conclude that

$$\Delta A = \Delta u_{,ijk}\,(\Delta u)_{,ijk} - \frac{3}{N+2}(\Delta^2 u)_{,i}\,(\Delta^2 u)_{,i} = 0, \tag{39}$$

in D. Consequently, as in the proof of Theorem 1, it follows that the set D is a N-ball $(N \geqslant 2)$. □

4. Mean value type results

As a consequence of overdetermined boundary value problems we derive Mean value type results for harmonic, biharmonic, triharmonic and 4-harmonic functions.

In [15] it was shown that the second order over-determined boundary value problem

$$\Delta u = -1 \qquad \text{in } D, \tag{40}$$

$$u = 0, \frac{\partial u}{\partial n} = -C \text{ (C is a const.) on } \partial D, \tag{41}$$

is equivalent to

$$\int_D h dx = C \int_{\partial D} h ds, \tag{42}$$

for all functions h harmonic in D and that D is a N-ball of radius $R = NC$.

By circumferential mean value theorem of harmonic functions

$$h_0 = \frac{1}{W_N R^{N-1}} \int_{\partial D} h ds, \tag{43}$$

where h_0 is the value of h at the center of D and W_N is the surface area of the unit sphere in N-dimensions. Since $C = \frac{R}{N}$, from (42) and (43)

$$h_0 = \frac{N}{W_N R^N} \int_D h dx, \tag{44}$$

which is the well-known areal mean value result for harmonic functions. $\square$

It was shown in [20, 22] that the solution u of (40) and (41) when D is an open N-ball of radius $R = NC$ is given by

$$u(x) = \frac{R^2 - r^2}{2N},$$

where r is the distance of x from the center of the ball and that

$$\int_D u dx = \frac{C^2 NV}{N+2}.$$

One immediately concludes that u also satisfies the mean value property

$$\frac{1}{V} \int_D u dx = \frac{2u(0)}{N+2}. \quad \square \tag{45}$$

Next, the fourth order overdetermined problem

$$\Delta^2 u = 1 \text{ in } D, \tag{46}$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial D, \tag{47}$$

$$\Delta u = C \text{ on } \partial D, \tag{48}$$

was proved in [15] to be equivalent to

$$\int_D B dx = -C \int_{\partial D} \frac{\partial B}{\partial n} ds, \tag{49}$$

for all biharmonic functions B such that

$$\Delta^2 B = 0 \text{ in } D, \tag{50}$$

$$B = 0 \text{ on } \partial D, \tag{51}$$

and that D is an N-ball of radius R, where $C = \frac{R^2}{N(N+2)}$. From (49)

$$\int_D B dx = -C \int_{\partial D} \frac{\partial B}{\partial n} ds = -C \int_D \Delta B dx = -C \frac{W_N R^N}{N} \Delta B_0, \tag{52}$$

where ΔB_0 is the value of the harmonic function ΔB at the center of D.

Also by [8], we have, for any biharmonic function B,

$$\frac{1}{W_N R^{N-1}} \int_{\partial D} B ds = B_0 + \frac{R^2}{2N} \Delta B_0. \tag{53}$$

Since $B = 0$ on ∂D, we conclude from (52) and (53)

$$\frac{N}{W_N R^N} \int_D B dx = \frac{2}{N+2} B_0, \tag{54}$$

which is the areal mean value result for biharmonic functions satisfying the equalities (50) and (51). $\square$

REMARK 1. For $N = 2$ Nicolesco [13] obtained the mean value result for biharmonic functions in the form

$$B_0 = \frac{2}{\pi R^2} \int_D B dx - \frac{1}{2\pi R} \int_{\partial D} B ds. \tag{55}$$

If $B = 0$ on ∂D (54) agrees with (55) in case $N = 2$. However, if $B \neq 0$ on ∂D then (55) implies (54).

To obtain a mean value result for triharmonic functions we observe from [15] that the problem

$$\Delta^3 u = -1 \text{ in } D, \tag{56}$$

$$u = \frac{\partial u}{\partial n} = \Delta u = 0 \text{ on } \partial D, \tag{57}$$

$$\frac{\partial \Delta u}{\partial n} = -C \text{ on } \partial D, \tag{58}$$

is equivalent to

$$\int_D t dx = C \int_{\partial D} \Delta t ds, \tag{59}$$

for all triharmonic functions t such that

$$\Delta^3 t = 0 \text{ in } D, \tag{60}$$

$$t = \frac{\partial t}{\partial n} = 0 \text{ on } \partial D, \tag{61}$$

and that D is a N-ball of radius $R = (CN(N+2)(N+4))^{1/3}$.

Since Δt is biharmonic, we get by [8]

$$\frac{1}{R^{N-1}W_N}\int_{\partial D}\Delta t\,ds=\Delta t_0+\frac{R^2}{2N}\Delta^2 t_0. \tag{62}$$

Multiplying (62) by R^{N-1} and integrating with respect to R from 0 to R,

$$\int_D \Delta t\,dx=W_N\frac{R^N}{N}(\Delta t_0+\frac{R^2}{2(N+2)}\Delta^2 t_0). \tag{63}$$

By Green identity and the fact that $\frac{\partial t}{\partial n}=0$ on ∂D, (63) reduces to

$$\Delta t_0=-\frac{R^2}{2(N+2)}\Delta^2 t_0. \tag{64}$$

Also from [8] for a triharmonic function t, we have

$$\frac{1}{W_N R^{N-1}}\int_{\partial D} t\,ds=t_0+\frac{R^2}{2N}\Delta t_0+\frac{R^4}{8N(N+2)}\Delta^2 t_0. \tag{65}$$

Since $t=0$ on ∂D, (65) with the help of (64) yields

$$\Delta t_0=-\frac{4N}{R^2}t_0. \tag{66}$$

Now (59), with the help of (62), (64) and (66) reduces to

$$\frac{N}{W_N R^N}\int_D t\,dx=\frac{8}{(N+2)(N+4)}t_0, \tag{67}$$

which is the areal mean value result for triharmonic functions satisfying (60) and (61) subject to (59). $\square$

In [11] it was shown that for the overdetermined problem (56), (57) and (58),

$$\frac{N}{W_N R^N}\int_D u\,dx=\frac{NC^2}{N+6}, \tag{68}$$

and that

$$u(x)=-\frac{r^6}{48N(N+2)(N+4)}+\left(\frac{C^2}{N(N+2)(N+4)}\right)^{1/3}\frac{r^4}{16} \\ -\left(C^4N(N+2)(N+4)\right)^{1/3}\frac{r^2}{16}+\frac{C^2N(N+2)(N+4)}{48}, \tag{69}$$

where D is an open N-ball of radius R given by

$$R=\{CN(N+2)(N+4)\}^{1/3}, \tag{70}$$

and r is the distance of x from the center of D.

From (68), (69) and (70) it is easily seen that the solutions of (56), (57) and (58) also satisfy the areal mean value property

$$\frac{N}{W_N R^N}\int_D u\,dx = \frac{48u(0)}{(N+2)(N+4)(N+6)}. \qquad \square \tag{71}$$

If $N=2$, a different mean value result can also be derived from (55) for triharmonic functions t.

For biharmonic Δt, (55) can be written as

$$\Delta t_0 = \frac{2}{\pi R^2}\int_D \Delta t\,dx - \frac{1}{2\pi R}\int_{\partial D}\Delta t\,ds, \tag{72}$$

where R is the radius of the disc D. Since $\frac{\partial t}{\partial n}$ is zero on ∂D, (72) reduces to

$$C\int_{\partial D}\Delta t\,ds = -2\pi RC\Delta t_0. \tag{73}$$

Also, for t equal to zero on ∂D, we get from (65),

$$\frac{R^2}{4}\Delta t_0 = -t_0 - \frac{R^4}{64}\Delta^2 t_0. \tag{74}$$

Hence, by virtue of (73), (74) and $C=\frac{R^3}{48}$, (59) yields

$$\frac{6}{\pi R^2}\int_D t\,dx = t_0 + \frac{R^4}{64}\Delta^2 t_0, \tag{75}$$

which is the mean value result for $N=2$. $\square$

Lastly, we consider the 4-harmonic problem (2), (3) and (4) which is equivalent to (34) where F is 4-harmonic function that satisfies (35) and D is a N-ball of radius $R=[CN(N+2)(N+4)(N+6)]^{1/4}$.

For 4-harmonic function F we get from [8],

$$\frac{1}{W_N R^{N-1}}\int_{\partial D}F\,ds = F_0 + \frac{R^2}{2N}\Delta F_0 + \frac{R^4}{8N(N+2)}\Delta^2 F_0 + \frac{R^6}{48N(N+2)(N+4)}\Delta^3 F_0. \tag{76}$$

Since $F=0$ on ∂D, hence

$$-F_0 - \frac{R^2}{2N}\Delta F_0 = \frac{R^4}{8N(N+2)}\Delta^2 F_0 + \frac{R^6}{48N(N+2)(N+4)}\Delta^3 F_0. \tag{77}$$

$\Delta^2 F$ is biharmonic, therefore, by [8],

$$\frac{1}{W_N R^{N-1}}\int_{\partial D}\Delta^2 F\,ds = \Delta^2 F_0 + \frac{R^2}{2N}\Delta^3 F_0. \tag{78}$$

Multiplying by R^{N-1} and integrating with respect to R from 0 to R,

$$\frac{1}{W_N}\int_D \Delta^2 F\,dx = \frac{R^N\Delta^2 F_0}{N} + \frac{R^{N+2}}{2N(N+2)}\Delta^3 F_0. \tag{79}$$

By Green identity and (79), (34) reduces to

$$-\frac{N}{W_N R^N}\int_D F\,dx = C[\Delta^2 F_0 + \frac{R^2\Delta^3 F_0}{2(N+2)}]. \tag{80}$$

From [8], since ΔF is triharmonic

$$\frac{1}{W_N R^{N-1}}\int_{\partial D}\Delta F\,ds = \Delta F_0 + \frac{R^2}{2N}\Delta^2 F_0 + \frac{R^4}{8N(N+2)}\Delta^3 F_0. \tag{81}$$

For $\Delta F = 0$ on ∂D, we get

$$\Delta F_0 + \frac{R^2}{2N}\Delta^2 F_0 + \frac{R^4}{8N(N+2)}\Delta^3 F_0 = 0. \tag{82}$$

Multiplying (81) by R^{N-1} and integrating with respect to R from 0 to R,

$$\frac{1}{W_N}\int_D \Delta F\,dx = \frac{R^N}{N}\Delta F_0 + \frac{R^{N+2}}{2N(N+2)}\Delta^2 F_0 + \frac{R^{N+4}}{8N(N+2)(N+4)}\Delta^3 F_0. \tag{83}$$

Green identity and $\frac{\partial F}{\partial n} = 0$ on ∂D, reduce it to

$$\Delta F_0 + \frac{R^2}{2(N+2)}\Delta^2 F_0 + \frac{R^4\Delta^3 F_0}{8(N+2)(N+4)} = 0. \tag{84}$$

By (82) and (84)

$$\Delta^2 F_0 = -\frac{4(N+2)}{R^2}\Delta F_0. \tag{85}$$

From (82) with the help (85)

$$\Delta^3 F_0 = \frac{8(N+2)(N+4)}{R^4}\Delta F_0. \tag{86}$$

Using (85) and (86) in (80), we get

$$-\frac{N}{W_N R^N}\int_D F\,dx = C\frac{8}{R^2}\Delta F_0. \tag{87}$$

From (77) by (85) and (86)

$$-F_0 = \frac{R^2}{6N}\Delta F_0. \tag{88}$$

Finally, from (87) and (88)

$$\frac{N}{W_N R^N}\int_D F\,dx = \frac{48F_0}{(N+2)(N+4)(N+6)}, \tag{89}$$

where we have used

$$C = \frac{R^4}{N(N+2)(N+4)(N+6)}.$$

The equality (89) is the areal mean value result for 4-harmonic functions F which satisfy (35) subject to (34).

Likewise, it is an immediate conclusion from (6) and (7) that the solutions u of the problem (2), (3) and (4) are subject to the areal mean value property

$$\frac{N}{W_N R^N}\int_D u\,dx = \frac{384u(0)}{(N+2)(N+4)(N+6)(N+8)}. \qquad \square \tag{90}$$

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