Differential Equations & Applications
Volume 2, Number 4 (2010), 569–574

EXISTENCE OF POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATION ON RIEMANNIAN MANIFOLDS

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(*Communicated by D. Žubrinić*)

Abstract. Let $(\mathscr{M},g)$ be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. We study the existence of positive weak solutions for the following quasilinear elliptic equation

$$-(\Delta_p)_g u + u^{p-1} = f(x,u,\nabla_g u) \quad \text{in } \mathscr{M},$$

where $(\Delta_p)_g u = div_g(|\nabla u|_g^{p-2}\nabla u)$ is the p-Laplacian operator on Riemannian manifold $(\mathscr{M},g)$ with $1 < p < n$.

1. Introduction

Let $(\mathscr{M},g)$ be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. We consider the existence of positive weak solutions for the following quasilinear elliptic equation

$$-(\Delta_p)_g u + u^{p-1} = f(x,u,\nabla_g u) \quad \text{in } \mathscr{M}, \tag{1.1}$$

where $(\Delta_p)_g u = div_g(|\nabla u|_g^{p-2}\nabla u)$ is the p-Laplacian operator on the Riemannian manifold $(\mathscr{M},g)$, $1 < p < n$. $f : \mathscr{M} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and nonnegative.

Problem (1.1) does not have variational structure in general. In bounded domains of the whole space $\mathbb{R}^n$, such a problem has been extensively studied for the case $p = 2$, see for instance [3], [5], [6] etc. A main ingredient in establishing existence results in [6] is to obtain a priori bound for solutions. This was done by a blow up argument, where a Liouville theorem was used. The existence of solutions is then obtained by topological method. For the case $p \neq 2$, the Liouville theorem in the half space is unknown up to now, so the blow up argument can not be applied in general. Results for $p \neq 2$ were obtained in convex domains in [1], [11]. The idea is to show maximal points of solutions remain inside the domain by using moving plane method. Hence, it needs only to apply Liouville type theorem in the whole space.

Mathematics subject classification (2010): 35J60, 53C21, 58J60.
Keywords and phrases: Riemannian manifold, quasilinear elliptic equation, a priori bound, existence of solution.
This work was supported by grants N10961016 and 10631030 from the NNSF of China and 2009GZS0011 from NSF of Jiangxi.

Paper DEA-02-34

Recently, nonlinear elliptic equations on compact Riemannian manifold have been brought much attention. Multiple solutions were obtained in [4] for the problem

$$-\varepsilon^2\Delta_g u + u = |u|^{p-2}u \quad \text{in } \mathscr{M}, \tag{1.2}$$

where $(\mathscr{M}, g)$ is a compact, connected, orientable, Riemannian manifold of class C^∞ with Riemannian metric g, $\dim \mathscr{M} = n \geqslant 3$, $2 < p < 2^* = \frac{2n}{n-2}$ and Δ_g is the Laplace-Beltrami operator. While for zero mass case, similar result was obtained in [14].

In this paper, we are interested in the existence of positive weak solutions for problem (1.1), that is a $C^{1,\alpha}-$solution of problem (1.1). Suppose that $f : \mathscr{M} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a nonnegative continuous function and satisfies the following condition:

(A) there exist $L > 0$ and $c_0 \geqslant 1$ such that

$$u^s - L|\eta|^q \leqslant f(x,u,\eta) \leqslant c_0 u^s + L|\eta|^q$$

for $(x,u,\eta) \in \mathscr{M} \times \mathbb{R} \times \mathbb{R}^n$, where $s \in (p-1, p_\star - 1)$, $q \in (p-1, ps/(s+1))$, $p_\star = p(n-1)/(n-p)$.

Our main result is the following.

THEOREM 1.1. *If we assume condition* (A) *holds, then problem* (1.1) *has at least one positive weak solution of* $\mathscr{C}^{1,\alpha}$ *regularity.*

In section 2, a priori bound for positive weak solutions of (1.1) is obtained; By topological methods, we obtain a positive weak solution in section 3 for problem (1.1).

2. A priori estimates

In this section, we establish a priori bound for positive weak solutions of problem (1.1) by using blow up arguments. Such an argument is based on the Liouville-type theorem in [12], which is described in Lemma 2.1 as follows, see [12, Corollary II(iii)]. Related works on differential inequalities on Riemannian manifolds can be found in [9] and [10].

LEMMA 2.1. *Assume* $1 < p < n$. *Then the differential inequality* $\Delta_p v + v^s \leqslant 0, v \geqslant 0$ *has a positive solution in* $\mathbb{R}^n$ *if and only if* $s > p_\star - 1$, *where* $p_\star = p(n-1)/(n-p)$.

Our main result in this section is the following.

PROPOSITION 2.1. *Suppose condition* (A) *holds. Let* $u \in \mathscr{C}^{1,\alpha}(\mathscr{M}), 0 < \alpha < 1$ *be a positive solution of problem* (1.1). *Then there exists a positive constant* C *independent of* u, *such that* $\|u\|_{L^\infty(\mathscr{M})} \leqslant C$.

Proof. We argue by contradiction. Suppose that there exists a sequence of positive solutions $\{u_k\}$ of problem (1.1) such that $\|u_k\|_{L^\infty(\mathscr{M})} \to +\infty$ as $k \to \infty$. Since the manifold $\mathscr{M}$ is compact and $u_k \in C^{1,\alpha}(\mathscr{M})$, we assume that there exist $\{x_k\} \subset \mathscr{M}$ such that $m_k := u_k(x_k) = \max_{x\in\mathscr{M}} u_k \to +\infty$ as well as $x_k \to x_0$ for some point $x_0 \in \mathscr{M}$.

Let i_g be the injectivity radius of $(\mathscr{M}, g)$ at the point x_0, that is, i_g is the largest number $r > 0$ for which any geodesic starting from x_0 and of length less than r is minimizing. The fact $\mathscr{M}$ being compact implies $i_g > 0$. Let (U, z^i) be normal coordinate system at x_0, taking $0 < \delta_1 < i_g$, for any $x \in B_g(x_k, \delta_1) \subset \mathscr{M}$ and k large, by the exponential map at x_0, we have $z = \exp_{x_0}^{-1}(x) \in B(z_k, \delta_2) \subset U$ for some $\delta_2 > 0$, $B_g(x_k, \delta_1)$ denotes the ball at the center x_k with the radius δ_1 on Riemannian manifold $(\mathscr{M}, g)$, and $B(z_k, \delta_2)$ denotes the ball at the center z_k with the radius δ_2 in $\mathbb{R}^n$.

In normal coordinate system (U, z^i), we have

$$g_{ij}(z) = \delta_{ij} + O(|z|^2), \quad \det g_{ij}(z) = 1 + O(|z|^2).$$

Set

$$\tilde{u}_k(z) = u_k(\exp_{x_0}(z))$$

for $z \in B(z_k, \delta_2)$. Since $\{u_k\}$ is a sequence of positive solutions of problem (1.1), $\tilde{u}_k(z)$ satisfies the following equation

$$-\frac{1}{\sqrt{\det g(z)}}\frac{\partial}{\partial z^i}\left(\sqrt{\det g(z)}g^{ij}(z)\frac{\partial \tilde{u}_k(z)}{\partial z^j}\left(g^{lh}(z)\frac{\partial \tilde{u}_k(z)}{\partial z^l}\frac{\partial \tilde{u}_k(z)}{\partial z^h}\right)^{\frac{p-2}{2}}\right) + \tilde{u}_k^{p-1}(z) = f(\exp_{x_0}(z), \tilde{u}_k(z), \nabla_g \tilde{u}_k(z)). \quad (2.1)$$

By properties of exponential map, $z_k = \exp_{x_0}^{-1}(x_k) \to 0$ as $k \to \infty$. Define

$$v_k(z) = m_k^{-1}\tilde{u}_k(\lambda_k z + z_k),$$

where $\lambda_k = m_k^{(p-1-s)/p} > 0$, $\lambda_k \to 0$ as $k \to \infty$ since $s > p-1$, and $m_k \to \infty$ as $k \to \infty$. Then the function v_k is well defined in $B(0, \lambda_k^{-1}\delta_2)$, and $v_k(0) = 1$. We also have

$$\nabla_g v_k(z) = m_k^{-1}\lambda_k \nabla_g \tilde{u}_k(\lambda_k z + z_k); \;\; (\Delta_p)_g v_k(z) = m_k^{1-p}\lambda_k^p (\Delta_p)_g \tilde{u}_k(\lambda_k z + z_k)$$

and v_k satisfies

$$\begin{aligned} -\frac{1}{b_k(z)}\frac{\partial}{\partial z^i}\left(b_k(z)a_k^{ij}\frac{\partial v_k(z)}{\partial z^j}\left(a_k^{lh}(z)\frac{\partial v_k(z)}{\partial z^l}\frac{\partial v_k(z)}{\partial z^h}\right)^{\frac{p-2}{2}}\right) + \lambda_k^p v_k^{p-1}(z) \\ = m_k^{1-p}\lambda_k^p f(\exp_{x_0}(\lambda_k z + z_k), m_k v_k(z), m_k \lambda_k^{-1}\nabla_{\tilde{g}_k} v_k(z))) \\ := \phi_k(z, v_k(z), \nabla_{\tilde{g}_k} v_k(z)), \end{aligned} \quad (2.2)$$

for $z \in B(0, \lambda_k^{-1}\delta_2)$, where $\tilde{g}_k = g(\lambda_k z + z_k)$ is a metric on $\mathbb{R}^n$, $\tilde{g}_k \to \xi$ in $C^1(B(0,R))$ for any $R > 0$ as $k \to \infty$, ξ is the standard metric on $\mathbb{R}^n$, and

$$a_k^{ij}(z) = \tilde{g}_k^{ij} = g^{ij}(\lambda_k z + z_k) \to \delta_{ij} \quad \text{as } k \to \infty, \quad (2.3)$$

$$b_k(z)=\sqrt{\det \tilde{g}_k}=\sqrt{\det g(\lambda_k z+z_k)}\to 1 \quad \text{as} \quad k\to\infty \tag{2.4}$$

uniformly in $\mathscr{C}^1(B(0,R))$ for any $R>0$. By assumption (A),

$$\begin{aligned}|\phi_k(z,v_k(z),\nabla_{\tilde{g}_k}v_k(z))| &\leqslant c_0 m_k^{1-p}\lambda_k^p(m_k|v_k(z)|)^s+Lm_k^{1-p}\lambda_k^p(m_k\lambda_k^{-1})^q|\nabla_{\tilde{g}_k}v_k|^q\\ &= c_0 m_k^{1-p+s}\lambda_k^p|v_k|^s+Lm_k^{1-p+q}\lambda_k^{p-q}|\nabla_{\tilde{g}_k}v_k|^q\end{aligned}$$

for $z\in B(0,R)$ for any $R>0$. Since $q<ps/(s+1)$, it yields

$$\begin{aligned}|\phi_k(z,v_k(z),\nabla_{\tilde{g}_k}v_k(z))| &\leqslant c_0|v_k|^s+Lm_k^{-s+\frac{q(s+1)}{p}}|\nabla_{\tilde{g}_k}v_k|^q\\ &\leqslant c_0|v_k|^s+|\nabla_{\tilde{g}_k}v_k|^q\end{aligned}$$

for $z\in B(0,R)$ and k large enough. Applying $C^{1,\alpha}$ regularity result up to boundary due to Lieberman [8], we obtain that there exists a positive constant C independent of k such that $|\nabla_{\tilde{g}_k}v_k|_{L^\infty(B(0,R))}\leqslant C$. Hence, for any $R>0$, $B(0,R)\subset B(0,\lambda_k^{-1}\delta_2)$ for k large, we conclude $\|v_k\|_{\mathscr{C}^{1,\alpha}(B(0,R))}\leqslant C$ for certain C independent of k. Therefore, we may assume $v_k\to v$ in $\mathscr{C}^{1,\alpha}(B(0,R))$ as $k\to\infty$. By condition (A),

$$\phi_k(z,v_k(z),\nabla_{\tilde{g}_k}v_k(z))-|v_k|^s\geqslant -Lm_k^{-s+\frac{q(s+1)}{p}}|\nabla_{\tilde{g}_k}v_k|^q\to 0 \quad \text{as} \quad k\to\infty. \tag{2.5}$$

Thus, taking limit in (2.2), by (2.3)-(2.5), we obtain

$$\Delta_p v+v^s\leqslant 0,\quad v\geqslant 0 \quad \text{in} \quad B(0,R),$$

where Δ_p is the $p-$Laplacian in $\mathbb{R}^n$.

Choosing a sequence $\{R_j\}$ with $R_j\to+\infty$ as $j\to\infty$, by a diagonal procedure, we see that v satisfies

$$\Delta_p v+v^s\leqslant 0,\quad v\geqslant 0 \quad \text{in} \quad \mathbb{R}^n,\quad v(0)=\max\, v=1,$$

where $s\in(p-1,p_\star-1)$, which contradicts to Lemma 2.1. The assertion follows.

REMARK 2.1. By a result in [2] and [13], we know that the solution u in Proposition 2.1 is actually a $C^{1,\alpha}-$solution.

3. Proof of Theorem 1.1

In order to prove the existence of positive weak solutions for problem (1.1), we use a version of a theorem of Krasnoselskii [7] about the existence of fixed points on compact operators defined in a cone.

LEMMA 3.1. (*Krasnoselskii*) *Let* $\mathscr{C}$ *be a cone in a Banach space and* $T:\mathscr{C}\to\mathscr{C}$ *a compact operator such that* $T(0)=0$. *Assume that there exists* $0<r<R$, *such that:*
(a) $u\neq tTu$ *for* $0\leqslant t\leqslant 1$, $u\in\mathscr{C}$, $\|u\|=r$.
(b) There exists a compact map $H:[0,1]\times\bar{B}_R\to\mathscr{C}$, *such that:*

(b_1) *$H(0,u) = Tu$ for $\|u\| = R$,*
(b_2) *$H(t,u) \neq u$ for $\|u\| = R$ and $t \geqslant 0$,*
(b_3) *there exists $t_0 > 0$, such that for $t \geqslant t_0, H(t,u) = u$ has no solution in $\bar{B}(0,R)$.*
Then

$$i_X(T,B_r) = 1, \;\; i_X(T,B_R) = 1, \;\; i_X(T,D) = -1,$$

where $D = \{u \in X : r < \|u\| < R\}$. So T has a fixed point in D.

PROOF OF THEOREM 1.1: Let $C^{1,\alpha}(\mathscr{M})$ be a Banach space equipped with the usual norm $\|\cdot\|_{C^{1,\alpha}}$.

For each function $v \in C^1(\mathscr{M})$, we know that the problem:

$$-(\Delta_p)_g K(v) + K(v)^{p-1} = v \text{ in } \mathscr{M}$$

has a unique weak solution $K(v)$. A bootstrap argument implies that $K(v) \in L^\infty(\mathscr{M})$, whence by results in [2] and [13], we find $K(v) \in C^{1,\alpha}(\mathscr{M})$. By Lemma 2.4 in [15], $K : C^1(\mathscr{M}) \to C^{1,\alpha}(\mathscr{M})$ is continuous.

Define $N : C^{1,\alpha}(\mathscr{M}) \to C^1(\mathscr{M})$ by $N(u) = f(x,u,\nabla_g u)$. The continuity of f and the compactness of the inclusion $C^{1,\alpha}(\mathscr{M}) \to C^1(\mathscr{M})$ imply that N is compact. Let $T = K \circ N$. Then, T is also compact.

Let $X := \{u \in E : u \geqslant 0\}$ be a cone in $E = \mathscr{C}^{1,\alpha}(\mathscr{M})$. We will show T has a nontrivial fixed point in X. For this purpose, we will use Lemma 3.1. Now, we verify conditions of Lemma 3.1.

By condition (A), $N(0) = 0$. Hence, $T(0) = K \circ N(0) = 0$. By the maximum principle, $T(u) > 0$ for $u \in X \setminus \{0\}$. The fact $T(u) \in C^{1,\alpha}(\mathscr{M})$ implies $T(X) \subset X$.

Now, we verify (a). Let $u \in X \setminus \{0\}$ be a solution of $u = tTu$ for some $t \in [0,1]$, namely,

$$-(\Delta_p)_g u + u^{p-1} = t^{p-1} f(x,u,\nabla_g u) \quad \text{in } \mathscr{M}. \tag{3.1}$$

Multiplying by u and integrating by part, we obtain

$$\int_{\mathscr{M}} (|\nabla_g u|^p + u^p)\, d\mu_g \tag{3.2}$$
$$= t^{p-1} \int_{\mathscr{M}} f(x,u,\nabla_g u)\, d\mu_g \leqslant \int_{\mathscr{M}} f(x,u,\nabla_g u)\, d\mu_g$$
$$\leqslant c_0 \int_{\mathscr{M}} |u|^{s+1}\, d\mu_g + L \int_{\mathscr{M}} |\nabla u|^q u\, d\mu_g$$
$$\leqslant C\Big(\int_{\mathscr{M}} |u|^p\, d\mu_g\Big)^{\frac{s+1}{p}} + C\Big(\int_{\mathscr{M}} |\nabla u|^p\, d\mu_g\Big)^{\frac{q}{p}} \Big(\int_{\mathscr{M}} |u|^{\frac{p}{p-q}}\, d\mu_g\Big)^{\frac{p-q}{p}}$$
$$\leqslant \|u\|^{s+1}_{W^{1,p}(\mathscr{M})} + \|u\|^{q+1}_{W^{1,p}(\mathscr{M})}. \tag{3.3}$$

Since $s+1 > p, q+1 > p$, it follows that there exists a positive number σ such that $\|u\|_{W^{1,p}(\mathscr{M})} \geqslant \sigma$. Hence, $\|u\|_{C^{1,\alpha}(\mathscr{M})} \geqslant C\sigma$. Choose $r < C\sigma$, condition (a) follows.

Next, we verify condition (b).

Define $H : [0,1] \times X \to X$ by $H(t,u) = K(N(u+t))$. Then, H is compact. Apparently, (b_1) holds.

Now, we show condition (b_2).

For $0 \leqslant t \leqslant 1$, the equation $H(t,u) = Tu$ is equivalent to the equation

$$-(\Delta_p)_g u + (u+t)^{p-1} = f(x, u+t, \nabla_g(u+t)) \quad \text{in } \mathscr{M}. \tag{3.4}$$

By Proposition 2.1, solutions of (3.4) are uniformly bounded, i.e. $\|u+t\|_{\mathscr{C}^{1,\alpha}} \leqslant C$ for some $C > 0$. Therefore, $\|u\|_{\mathscr{C}^{1,\alpha}} \leqslant C$. So we may choose $R > C > 0$ such that for $\|u\|_{\mathscr{C}^{1,\alpha}} = R$, $t \geqslant 0$, $H(t,u) \neq Tu$, i.e. condition (b_2) holds. Proposition 2.1 also implies (b_3).

By Lemma 3.1, T has a fixed point u in $D = \{u \in X : r < \|u\|_{\mathscr{C}^{1,\alpha}} < R\}$. The fixed point u is a positive weak solution of problem (1.1). The proof is complete.

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(Received November 1, 2009)
(Revised December 12, 2009)

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