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ON THE DIRICHLET PROBLEM OF LANDAU–LIFSHITZ–MAXWELL EQUATIONS

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Abstract. We prove the existence and uniqueness of non-trivial stable solutions to Landau-Lifshitz-Maxwell equations with Dirichlet boundary condition for large anisotropies and small non-simply connected domains.

1. Introduction

We seek for a solution $u=(u_1,u_2,u_3):\Omega\to S^2\subset\mathbb{R}^3$ of the Landau-Lifshitz-Maxwell equations with the Dirichlet boundary condition

$$\begin{cases}\Delta u+|\nabla u|^2u+H-(H\cdot u)u-\lambda(W_u(u)-(W_u(u)\cdot u)u)=0 \ \text{ in } \ \Omega,\\ u=g \ \text{ on } \ \partial\Omega,\\ \text{curl H}=0, \ \operatorname{div}(H+u\chi_\Omega)=0 \ \text{ in } \ \mathbb{R}^3,\end{cases} \tag{1.1}$$

where Ω is a non-simply connected bounded domain in $\mathbb{R}^3$ with uniformly C^4 boundary, $\lambda>0$ is a parameter, $g\in C^{3+\alpha_0}(\partial\Omega,S^2\cap\{u_3=0\})$ $(0<\alpha_0<1)$ and χ_Ω is the characteristic function of the domain $\overline{\Omega}$,

$$\chi_\Omega(x)=\begin{cases}1, & \forall x\in\overline{\Omega},\\ 0, & \forall x\notin\overline{\Omega}.\end{cases}$$

For thin films with in plane magnetization (c.f. [19]), as a first approximation, we can assume $W(u)=u_3^2$ and denote $W_u(u)=(0,0,2u_3)$.

H is the demagnetizing field generated by the magnetization u and determined by the Maxwell's equation. The (1.1) is the Euler-Lagrange equation of the Landau-Lifshitz energy functional

$$E_\lambda(u)=\int_\Omega\left[\frac{1}{2}|\nabla u|^2-\frac{1}{2}u\cdot H+\lambda W(u)\right]dx \tag{1.2}$$

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on $H_g^1(\Omega,S^2)$. Here

$$H_g^1(\Omega,S^2):=\{u-g\in H_0^1(\Omega,\mathbb{R}^3)|\; u(\Omega)\subset S^2\}$$

is well defined for Ω and g. The boundary conditions are influenced by surface anisotropy as well as by interface coupling phenomena. The Dirichlet boundary condition g is induced from the interface magnetization of the adjacent medium (c.f. [19]).

Functional (1.2) was first derived for ferromagnetic problem by Landau and Lifshitz [28] in 1935. The equation (1.1) is the static equivalent of the time-dependent Landau-Lifshitz-Maxwell equations (c.f.[37]-[45])

$$\begin{cases} \dfrac{\partial u}{\partial t}=-u\times(u\times(\Delta u-\lambda W_u(u)+H)) \\ \qquad\qquad\qquad\qquad +\gamma u\times(\Delta u-\lambda W_u(u)+H)\ \text{ in }\ \Omega\times(0,\infty), \\ u=g\ \text{ on }\ \partial\Omega\times(0,\infty), \\ u\in S^2\ \text{ in }\ \Omega\times(0,\infty), \\ \operatorname{curl} H=0,\ \operatorname{div}(H+u\chi_\Omega)=0\ \text{ in }\ \mathbb{R}^3. \end{cases} \tag{1.3}$$

Here $\gamma\geqq 0$ is called the gyromagnetic factor.

Afterhere, a solution $u(x)$ will be expressed in

$$u(x)=(\cos\xi(x)\cos\theta(x),\cos\xi(x)\sin\theta(x),\sin\xi(x)),$$

where $-\pi/2\leqq\xi\leqq\pi/2$, $\theta\in S^1=\mathbb{R}/(2\pi\mathbb{Z})$ and

$$g(x)=(\cos\theta_g(x),\sin\theta_g(x),0).$$

For convenience, we also use the notations:

$$\begin{aligned} &u(\theta,\xi)=(\cos\xi(x)\cos\theta(x),\cos\xi(x)\sin\theta(x),\sin\xi(x)),\\ &u_\theta(\theta,\xi)=(-\cos\xi\sin\theta,\cos\xi\cos\theta,0),\\ &u_\xi(\theta,\xi)=(-\sin\xi\cos\theta,-\sin\xi\sin\theta,\cos\xi). \end{aligned}$$

In [37], the existence and non-existence of non-trivial stable solutions to Landau-Lifshitz equation (the first equation of (1.1) with constant H) with the Dirichlet boundary condition were obtained. This paper is the continuation of [37]. In this paper, we study the Landau-Lifshitz-Maxwell equations (1.1) in a non-simply connected bounded domain of $\mathbb{R}^3$. The existence and uniqueness of non-trivial solution and its stability are obtained. Precisely, we have:

THEOREM 1. *There exists* $d_0>0$ *such that for any non-simply connected domain* Ω *if* $|\Omega|\leqslant d_0$, *then there is* $\lambda_0>0$ *such that for* $\lambda>\lambda_0$, *there exists a unique solution* (u_λ,H_λ),

$$\begin{aligned} &u_\lambda(x)=(\cos\xi_\lambda(x)\cos\theta_\lambda(x),\cos\xi_\lambda(x)\sin\theta_\lambda(x),\sin\xi_\lambda(x))\in C^{2+\alpha}(\overline{\Omega}),\\ &H_\lambda\in C^{1+\alpha}(\overline{\Omega})\cap(BMO\cap L^p)(\mathbb{R}^3),\ \ \forall p\in(1,\infty), \end{aligned}$$

$(0 < \alpha < 1)$ to (1.1) corresponding to the homotopy class $[\theta_g]$ of continuous maps from Ω to $S^2 \cap \{u_3 = 0\}$. Moreover, θ_λ is homotopic to θ_g and

$$\|\xi_\lambda\|_{C^\alpha(\overline{\Omega})} \leqq \frac{C}{\sqrt{\lambda}},$$

where the constant C is independent of λ.

A steady state solution $u_\lambda(x)$ is called exponentially asymptotically stable (c.f.[27]) if there are $\mu_0 > 0$ and $\varepsilon, C > 0$ such that for all $\bar{u} \in W^{2,2}(\Omega, S^2)$, if $\|\bar{u} - u_\lambda\|_{W^{2,2}(\Omega)} \leqslant \varepsilon$, then there exists a unique global solution $(u(x,t), H)$ of (1.3) with the initial data $u(x,0) = \bar{u}(x)$, and

$$\|u(x,t) - u_\lambda(x)\|_{W^{2,2}(\Omega)} \leqslant Ce^{-\mu_0 t}\|\bar{u}(x) - u_\lambda(x)\|_{W^{2,2}(\Omega)}, \quad \forall t \geqslant 0.$$

THEOREM 2. *Assume that Ω is not simply connected and $|\Omega| \leqslant d_0$ as in Theorem 1. Then there exists a $\lambda_0 > 0$ and for $\lambda \geqq \lambda_0$, there exists a $\gamma_0 = \gamma_0(\lambda) > 0$ such that, for $\lambda \geqq \lambda_0$ and $\gamma \in [0, \gamma_0]$, the solution (u_λ, H_λ) obtained in Theorem 1 is exponentially asymptotically stable steady state solutions of the time-dependent Landau-Lifshitz-Maxwell equation* (1.3).

To prove the theorems, we first solve the Maxwell's equation and express H in u and a Calderón-Zygmund operator. Then we consider the limit case of $\lambda \to \infty$ and get a solution in the homotopy class $[\theta_g]$ of continuous mappings from $\overline{\Omega}$ to S^1 by Schauder fixed point theorem. We search solutions for large λ in the neighborhood of the limit case solutions by the Schauder fixed point theorem again. In the last we analyze the spectrum of the linearized operator in a detailed way by using the Kato's perturbation theory (c.f. [25]) and to prove that the solutions are exponentially asymptotically stable steady state solutions of (1.3) by using nonlinear parabolic equations theory (c.f.[27]).

In [45], we proved the existence of non-trivial stable solutions to Landau-Lifshitz-Maxwell equations with Neumann boundary condition for large anisotropies and small domains that are non-simply connected and rotationally invariant around an axis.

Remark that in the Theorem 1 and Theorem 2 of this paper we do not need the assumption that the domain Ω is rotation invariant as in [45] for Neumann boundary condition. Moreover the assumption of diam$(\Omega) \leqslant d_0$ used in [45] is replaced by the volume $|\Omega| \leqslant d_0$.

On the other hand, in [37] and [45], the key estimate

$$\|\xi\|_{C^{\alpha_0}(\overline{\Omega})} \leqslant \frac{C}{\lambda}$$

was obtained by using the Campanato inequality. But for the Dirichlet problem considered in this paper, the Campanato inequality can not be used, because the right of (5.6) may be non-zero on the boundary of Ω (c.f. [8]). Here we shall apply [27] Theorem 3.1.3 to (5.6) to get a similar estimate

$$\|\xi\|_{C^{\alpha_0}(\overline{\Omega})} \leqslant \frac{C}{\sqrt{\lambda}}.$$

Related works can also be founded in [20], [34], [5], [26], [16], [33], [6], [32], [14], [15], [18] and [35]-[45] and the papers cited in there. Visintin [34] proved the existence of a kind of weak solutions to Landau-Lifshitz-Maxwell equations. In two dimensional case, the existence, uniqueness and partial regularity of (1.3) were considered in [15]. Also in the two dimension, Gustafson and Shatah studied the existence and stability of localized periodic solution to the Landau-Lifshitz equation, and Chang, Shatah, Uhlenbeck studied the well-posedness of the Cauchy problem for Schrödinger maps (there are not H and the first term in the right of the first equation of (1.3)). In three dimension, Ball, Taheri and Winter constructed local energy minimizers around a fixed constant solution to the model of micromagnetics. Their results are different from mine.

This paper consists of five sections. The Maxwell equation is studied in Section 2. In Section 3, the Landau-Lifshitz-Maxwell equations are expressed in spherical coordinates. In Section 4, we consider the limit case of $\lambda \to \infty$. The main theorems of this paper are proved in Sections 5-6.

2. Maxwell equations

The magnetization u and the demagnetizing field H are related by the Maxwell's equation

$$\begin{cases} \operatorname{curl} H = 0 \ \text{ in } \mathbb{R}^3, \\ \operatorname{div}(H + u\chi_\Omega) = 0 \ \text{ in } \mathbb{R}^3. \end{cases} \tag{2.1}$$

First we recall a lemma proved in [45]. For reader's convenience, we also give its proof here.

LEMMA 2.1. *There exists a continuous linear map $\mathscr{L}$,*

$$L^2(\Omega;\mathbb{R}^3) \ni u \mapsto \nabla v, \ \ v \in V = \left\{ v \in H^1_{loc}(\mathbb{R}^3;\mathbb{R}) : \ \nabla v \in L^2(\mathbb{R}^3;\mathbb{R}^3), \int_\Omega v dx = 0 \right\}$$

such that $H = \nabla v = \mathscr{L}(u)$ is the unique solution of (2.1) in V. Moreover, $\mathscr{L}$ is bounded from

$$\begin{aligned} &\mathscr{H}^1 (\text{Hardy}) \ \text{ to } \ L^1(\mathbb{R}^3), \\ &L^\infty(\Omega) \ \text{ to } \ BMO, \\ &L^p(\Omega) \ \text{ to } \ L^p(\mathbb{R}^3), \ \forall p \in (1,\infty). \end{aligned}$$

Proof. From [20], we know that there exists a continuous linear map

$$L^2(\Omega;\mathbb{R}^3) \ni u \mapsto v \in \{ v \in H^1_{loc}(\mathbb{R}^3;\mathbb{R}) : \ \nabla v \in L^2(\mathbb{R}^3;\mathbb{R}^3), \int_\Omega v dx = 0 \}$$

such that $H = \nabla v = \mathscr{L}(u)$ is the unique solution of (2.1) in V.

Note that from

$$\operatorname{div}(\nabla v + \chi_\Omega u) = 0 \ \text{ in } \ \mathbb{R}^3,$$

for any domain $D \supset\supset \Omega$,

$$0 = \int_D \operatorname{div}(\nabla v + \chi_\Omega u)dy = \int_{\partial D} \frac{\partial}{\partial \nu} v(y)dS(y).$$

Then for any $x \in \mathbb{R}^3$, for any ball $B_R(x) \supset\supset \Omega$, by the Green's representation formula

$$\begin{aligned}
v(x) &= \int_{\partial B_R(x)} v(y)\frac{\partial}{\partial \nu}\frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-y|}\frac{\partial}{\partial \nu}v(y)dS(y) \\
&\quad - \int_{B_R} \frac{1}{4\pi|x-y|}\operatorname{div}(\chi_\Omega(y)u(y))dy \\
&= \frac{-1}{4\pi}\int_{|\omega|=1} v(x+R\omega)d\omega - \frac{1}{4\pi R}\int_{\partial B_R(x)} \frac{\partial}{\partial \nu}v(y)dS(y) \\
&\quad - \int_{B_R} \frac{1}{4\pi|x-y|}\operatorname{div}(\chi_\Omega(y)u(y))dy \\
&= \frac{-1}{4\pi}\int_{|\omega|=1} v(x+R\omega)d\omega - \int_{B_R} \frac{1}{4\pi|x-y|}\operatorname{div}(\chi_\Omega(y)u(y))dy.
\end{aligned}$$

Notice that from $\nabla v \in L^2(\mathbb{R}^3)$, we have

$$\lim_{R\to\infty} \nabla_x \int_{|\omega|=1} v(x+R\omega)d\omega = 0.$$

So

$$\begin{aligned}
\nabla v(x) &= \frac{-1}{4\pi}\nabla\int_{\mathbb{R}^3} \frac{1}{|x-y|}\operatorname{div}(\chi_\Omega(y)u(y))dy \\
&= \frac{1}{4\pi}\nabla\int_\Omega u(y)\cdot\nabla\frac{1}{|x-y|}dy.
\end{aligned}$$

We have (see [30, Theorem 2.6.2]),

$$\nabla v(x) = \mathscr{L}(u) = Au(x)\chi_\Omega(x) - \lim_{\rho\to 0}\frac{1}{4\pi}\int_{\Omega\setminus B_\rho(x)} \nabla_x(u(y)\cdot\nabla_x\frac{1}{|x-y|})dy,$$

for $u \in C^\alpha_{loc}(\Omega)$, where $A = (A_{jk})_{jk}$ is a constant matrix

$$A_{jk} = \frac{-1}{4\pi}\int_{\partial B_1(0)} \nu_j(y)\partial_k\frac{1}{|-y|}dS(y).$$

It is easy to check that $\mathscr{L} - A$ is a Calderón-Zygmund operator ([31, Chapter 7]). Then it is bounded from

$$\begin{aligned}
&\mathscr{H}^1(\text{Hardy}) \text{ to } L^1(\mathbb{R}^3), \\
&L^\infty(\Omega) \text{ to BMO}, \\
&L^p(\Omega) \text{ to } L^p(\mathbb{R}^3),\ \forall p \in (1,\infty).
\end{aligned}$$

Thus the same is true for $\mathscr{L}$. □

3. Expressing Landau-Lifshitz-Maxwell equations in spherical coordinates

Denote

$$u(x) = (\cos\xi(x)\cos\theta(x), \cos\xi(x)\sin\theta(x), \sin\xi(x)),$$

where $-\pi/2 \leqq \xi \leqq \pi/2$, $\theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and

$$g(x) = (\cos\theta_g(x), \sin\theta_g(x), 0).$$

For convenience, we also use the notations:

$$\begin{aligned}
&u(\theta,\xi) = (\cos\xi(x)\cos\theta(x), \cos\xi(x)\sin\theta(x), \sin\xi(x)),\\
&u_\theta(\theta,\xi) = (-\cos\xi\sin\theta, \cos\xi\cos\theta, 0),\\
&u_\xi(\theta,\xi) = (-\sin\xi\cos\theta, -\sin\xi\sin\theta, \cos\xi),
\end{aligned}$$

and etc. By these notations, the energy functional E_λ is rewritten as

$$E_\lambda(\theta,\xi) = \int_\Omega \left[\frac{1}{2}|\nabla\xi|^2 + \frac{\cos^2\xi}{2}|\nabla\theta|^2 - \frac{1}{2}u(\theta,\xi)\cdot\nabla v + \lambda\sin^2\xi\right]dx. \tag{3.1}$$

The Euler-Lagrange equation of (3.1) can be written as

$$\begin{cases}\Delta\xi - (\lambda - \dfrac{|\nabla\theta|^2}{2})\sin 2\xi + u_\xi(\theta,\xi)\cdot\nabla v = 0 \text{ in } \Omega,\\ \xi = 0 \text{ on } \partial\Omega,\end{cases} \tag{3.2}$$

and

$$\begin{cases}\operatorname{div}(\cos^2\xi\nabla\theta) + u_\theta(\theta,\xi)\cdot\nabla v = 0 \text{ in } \Omega,\\ \theta = \theta_g \text{ on } \partial\Omega,\\ \Delta v = (-1)\operatorname{div}\{u(\theta,\xi)\chi_\Omega\} \text{ on } \mathbb{R}^3.\end{cases} \tag{3.3}$$

4. Limit case

Let $\lambda \to \infty$ in (3.1) and consider the limit functional

$$E_\infty(\theta) = \int_\Omega \left[\frac{1}{2}|\nabla\theta|^2 - \frac{1}{2}u(\theta,0)\cdot\nabla v\right]dx.$$

Its critical points are the maps to S^1 which satisfy

$$\begin{cases}\Delta\theta + u_\theta(\theta,0)\cdot\nabla v = 0 \text{ in } \Omega,\\ \theta = \theta_g \text{ on } \partial\Omega,\\ \Delta v = (-1)\operatorname{div}\{u(\theta,0)\chi_\Omega\} \text{ on } \mathbb{R}^3.\end{cases} \tag{4.1}$$

LEMMA 4.1. *Assume that $\partial\Omega$ is uniformly C^2. There is a constant $d_0 > 0$ such that if Ω is not simply connected, $|\Omega| \leqslant d_0$, then in the homotopy class $[\theta_g]$ of continuous mappings from $\overline{\Omega}$ into S^1 there exists a unique solution $\theta_* \in C^{1+\alpha}(\overline{\Omega}, S^1)$ and $\nabla v_* \in L^p(\mathbb{R}^3) \cap BMO(\mathbb{R}^3)$ $(1 < \forall p < \infty)$ to (4.1).*

Proof. Step 1: Existence. A classical proof of existence can be obtained by applying the direct method of calculus of variations to the minimization problem. Here we give another proof that can provide higher regularity.

For any given $\nabla v \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)(1 < p < \infty)$, there is a minimizing sequence of E_∞ which converges to a minimizer θ of E_∞. Let Θ denote the map from $\nabla v \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ to θ by

$$\begin{cases} \Delta\theta + u_\theta(\theta, 0) \cdot \nabla v = 0 \ \text{ in } \ \Omega, \\ \theta = \theta_g \ \text{ on } \ \partial\Omega. \end{cases}$$

Note that $\Delta\theta \in L^p(\Omega)$ $(1 < p < \infty)$. From [27, Theorem 3.1.1], $\theta \in W^{2,p}(\Omega)$. By the Sobolev embedding theorem ([1]), for any $\alpha \in (0,1)$, $\theta \in C^{1+\alpha}(\overline{\Omega})$.

For any ∇v_1 and $\nabla v_2 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, let $\theta_i = \Theta(\nabla v_i)$ $(i = 1,2)$. Then

$$\begin{aligned} &\int_\Omega |\nabla(\theta_1 - \theta_2)|^2 dx \\ &\qquad = \int_\Omega (\theta_1 - \theta_2)\big\{(u_\theta(\theta_1, 0) - u_\theta(\theta_2, 0)) \cdot \nabla v_1 \\ &\qquad\qquad\qquad\qquad\qquad + u_\theta(\theta_2, 0) \cdot (\nabla v_1 - \nabla v_2)\big\} dx \\ &\qquad \leqq \|\nabla v_1\|_{L^{3/2}(\Omega)} \|\theta_1 - \theta_2\|^2_{L^6(\Omega)} + \|\nabla(v_1 - v_2)\|_{L^2(\Omega)} \|\theta_1 - \theta_2\|_{L^2(\Omega)}. \end{aligned} \tag{4.2}$$

From the Sobolev inequality and the Poincaré inequality,

$$\begin{aligned} \|\theta_1 - \theta_2\|^2_{L^6(\Omega)} &\leqslant C_1 \|\nabla(\theta_1 - \theta_2)\|^2_{L^2(\Omega)}, \\ \|\theta_1 - \theta_2\|_{L^2(\Omega)} &\leqslant C|\Omega|^{1/3} \|\nabla(\theta_1 - \theta_2)\|_{L^2(\Omega)}, \end{aligned} \tag{4.3}$$

where the constants C_1, C are independent of Ω. See [17], we may take

$$C_1 = \left(\frac{1}{3\pi}\right)^{1/2} \left(\frac{3}{\Gamma(5/2)}\right)^{1/3}.$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla v|^2 dx &= (-1) \int_\Omega u(\theta, \xi) \cdot \nabla v dx \\ &\leqslant |\Omega|^{1/2} \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}. \end{aligned}$$

So we have

$$\|\nabla v\|_{L^2(\mathbb{R}^3)} \leqslant |\Omega|^{1/2} \tag{4.4}$$

and

$$\|\nabla v\|_{L^{3/2}(\Omega)} \leqslant |\Omega|^{1/6} \|\nabla v\|_{L^2(\Omega)} \leqslant |\Omega|^{2/3}. \tag{4.5}$$

From (4.2), (4.3) and (4.5), there is $d_0 > 0$ such that if $|\Omega| \leqslant d_0$, then

$$\int_\Omega |\nabla(\theta_1 - \theta_2)|^2 dx \leqslant C\|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)} \tag{4.6}$$

which implies that Θ is a continuous map from $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ to $C^{1+\alpha}(\overline{\Omega}, S^1)$.

Let $[\theta_g]$ denote the homotopy class of θ_g in $C^\alpha(\overline{\Omega}, S^1)$. Note that $[\theta_g]$ is convex and closed. From Lemma 2.1, $\nabla v \in L^p(\mathbb{R}^3) \cap BMO(\mathbb{R}^3)$ for $p \in (1,\infty)$, then the map $\Theta\mathscr{L}u(\theta,0)$ is continuous from $\theta \in [\theta_g]$ to $[\theta_g]$, and $\Theta\mathscr{L}u([\theta_g],0)$ is pre-compact in $[\theta_g] \subset C^\alpha(\overline{\Omega}, S^1)$. Then by the Schauder fixed point theorem, we proved the existence of solutions to (4.1).

Step 2: Uniqueness. Because,

$$\int_{\mathbb{R}^3} |\nabla(v_1 - v_2)|^2 dx = (-1)\int_\Omega (u(\theta_1,0) - u(\theta_2,0)) \cdot \nabla(v_1 - v_2) dx,$$

we have

$$\begin{aligned}\Big(\int_{\mathbb{R}^3} |\nabla(v_1 - v_2)|^2 dx\Big)^{1/2} &\leqslant \Big(\int_\Omega |u(\theta_1,0) - u(\theta_2,0)|^2 dx\Big)^{1/2} \\ &\leqslant |\Omega|^{1/3}\Big(\int_\Omega |u(\theta_1,0) - u(\theta_2,0)|^6 dx\Big)^{1/6} \\ &\leqslant 2C_1|\Omega|^{1/3}\|\nabla(\theta_1 - \theta_2)\|_{L^2(\Omega)}.\end{aligned}$$

From (4.6), we have

$$\Big(\int_\Omega |\nabla(\theta_1 - \theta_2)|^2 dx\Big)^{1/2} \leqslant CC_1|\Omega|^{1/3}\Big(\int_\Omega |\nabla(\theta_1 - \theta_2)|^2 dx\Big)^{1/2}.$$

So there is $d_0 > 0$ such that if $|\Omega| \leqslant d_0$, then $\theta_1 = \theta_2$. $\square$

LEMMA 4.2. *Assume that $\partial\Omega$ is uniformly C^4. Then the solution obtained in Lemma 4.1 satisfies*

$$\nabla v_* \in C^{3+\alpha}(\overline{\Omega}), \quad \theta_* \in C^{3+\alpha}(\overline{\Omega}).$$

Here the derivatives on the boundary $\partial\Omega$ take the inner limit of the derivatives in Ω respectively.

Proof. Since $u_*(x) = (\cos\theta_*(x), \sin\theta_*(x), 0) \in W^{1,p}(\Omega)$, by [1, Theorem 4.26], there is an extension $U_* \in W^{1,p}(B_R)$ of u_* for $B_R \supset\supset \Omega$, and

$$\|U_*\|_{W^{1,p}(B_R)} \leqslant C\|u_*\|_{W^{1,p}(\Omega)},$$

where C is independent of u_*.

Consider the equation

$$\Delta V = (-1)\mathrm{div}(U_*\chi_{B_R}) \text{ in } \mathbb{R}^3. \tag{4.7}$$

By elliptic equation theory ([17]), we have

$$\|V\|_{W^{2,p}_{loc}(\mathbb{R}^3)} \leqslant C\|U_*\|_{W^{1,p}(B_R)} \leqslant C\|u_*\|_{W^{1,p}(\Omega)}.$$

Since on Ω, $(-1)\mathrm{div}\,(U_*\chi_{B_R}) = (-1)\mathrm{div}(u_*\chi_\Omega) = \Delta v_*$, we have $\nabla v_* \in W^{1,p}(\Omega)$, and from (4.1), $\theta_* \in W^{3,p}(\Omega)$ and so is u_*. Using [1, Theorem 4.26] and (4.7) again,

$$\|U_*\|_{W^{3,p}(B_R)} \leqslant C\|u_*\|_{W^{3,p}(\Omega)}, \ \ V \in W^{4,p}(\mathbb{R}^3),$$

so $\nabla v_* \in W^{4,p}(\Omega)$, and from the Sobolev inequality, $\nabla v_* \in C^{3+\alpha}(\overline{\Omega})$. From (4.1), $\theta_* \in C^{3+\alpha}(\overline{\Omega})$. □

5. Proof of Theorem 1

Let $\alpha_0 \in (0,1)$ and define

$$\begin{aligned} M(\theta_*,\nabla v_*) = \big\{(\theta,\nabla v)|\ \theta \in & C^{1+\alpha_0}(\overline{\Omega}),\ \theta|_{\partial\Omega} = \theta_g, \\ & \theta \in [\theta_g],\ \|\theta-\theta_*\|_{C^{1+\alpha_0}(\overline{\Omega})} \leqslant 1, \\ & v \in V,\ \nabla v \in C^{\alpha_0}(\overline{\Omega}),\ \|\nabla v - \nabla v_*\|_{C^{\alpha_0}(\overline{\Omega})} \leqslant 1\big\}. \end{aligned}$$

LEMMA 5.1. *Let $\partial\Omega$ be uniformly C^2. For any given $(\theta,\nabla v) \in M(\theta_*,\nabla v_*)$, there exists a continuous map Ξ_λ such that $\xi_\lambda = \Xi_\lambda(\theta,\nabla v)$ is a solution to (3.2) which satisfies,*

$$\|\xi_\lambda\|_{W^{2,q}(\Omega)} \leqslant C_1,\ \ \forall q \in (1,\infty) \tag{5.1}$$

and

$$\|\xi_\lambda\|_{W^{1,q}(\Omega)} \leqslant \frac{C_2}{\sqrt{\lambda}}, \tag{5.2}$$

$$\|\xi_\lambda\|_{C^\alpha(\overline{\Omega})} \leqq \frac{C_2}{\sqrt{\lambda}}, \tag{5.3}$$

provided λ is large enough. Here the constants $C_i = C_i(\|\theta\|_{C^{1+\alpha_0}(\overline{\Omega})}, \|\nabla v\|_{C^{\alpha_0}(\overline{\Omega})})$ $(i = 1,2)$ are independent of λ.

Proof. Let $\eta := \xi + C/\lambda$, where C is a constant to be determined in the proof. The equation for η is written as

$$\begin{cases} (-1)\Delta\eta = -(\lambda - \dfrac{|\nabla\theta|^2}{2})\sin 2(\eta - \dfrac{C}{\lambda}) + u_\xi(\theta, \eta - \dfrac{C}{\lambda})\cdot\nabla v \ \text{ in } \ \Omega, \\ \eta = \dfrac{C}{\lambda} \ \text{ on } \ \partial\Omega. \end{cases} \tag{5.4}$$

Let

$$F(\eta) = -(\lambda - \frac{|\nabla\theta|^2}{2})\sin 2(\eta - \frac{C}{\lambda}) + u_\xi(\theta, \eta - \frac{C}{\lambda})\cdot\nabla v.$$

It is easy to check that there exists a constant $C = C(\|\theta\|_{C^{1+\alpha_0}(\overline{\Omega})}, \|\nabla v\|_{C^{\alpha_0}(\overline{\Omega})})$ and $\lambda_0(>0)$ such that

$$F(0) \geqslant 0 \text{ and } F(\frac{2C}{\lambda}) \leqslant 0$$

for $\lambda \geqslant \lambda_0$. From [4], there exists a non-negative solution η_λ to (5.4) which satisfies

$$0 \leqslant \eta_\lambda \leqslant \frac{2C}{\lambda},$$

provided $\lambda \geqslant \lambda_0$. Let $\xi_\lambda = \eta_\lambda - \frac{C}{\lambda}$. Thus ξ_λ is a solution of (3.2) and

$$-\frac{C}{\lambda} \leqslant \xi_\lambda \leqslant \frac{C}{\lambda}, \tag{5.5}$$

provided $\lambda \geqslant \lambda_0$.

Rewrite (3.2) as

$$-\Delta\xi + 2\lambda\xi = \lambda(2\xi - \sin 2\xi) + \frac{|\nabla\theta|^2}{2}\sin 2\xi + u_\xi(\theta,\xi)\cdot\nabla v, \tag{5.6}$$

and use [27, Theorem 3.1.3] to obtain $\forall q \in (1,\infty)$:

$$\|\xi\|_{W^{1,q}(\Omega)} \leqslant \frac{C}{\sqrt{\lambda}}(\lambda\|2\xi - \sin 2\xi\|_{L^q(\Omega)} + \|\nabla\theta\|^2_{L^q(\Omega)} + \|\nabla v\|_{L^q(\Omega)}),$$
$$\|\xi\|_{W^{2,q}(\Omega)} \leqslant C(\lambda\|2\xi - \sin 2\xi\|_{L^q(\Omega)} + \|\nabla\theta\|^2_{L^q(\Omega)} + \|\nabla v\|_{L^q(\Omega)}).$$

Note that for any $\delta > 0$, there exists $\lambda(\delta) > 0$ such that

$$\|2\xi_\lambda - \sin 2\xi_\lambda\|_{L^q(\Omega)} \leqslant \frac{\delta}{\lambda}\|\xi_\lambda\|_{L^q(\Omega)},$$

provided $\lambda \geqslant \lambda(\delta)$. So we have (5.1)-(5.2).

Using (5.2) for $q > 3$ and the Sobolev embedding theorem, we get (5.3) with $\alpha \leqslant 1 - \frac{3}{q}$. □

LEMMA 5.2. *Assume that $\partial\Omega$ is uniformly C^4. There exists $d_0 > 0$ such that if $|\Omega| \leqslant d_0$, then for $\xi_\lambda = \Xi_\lambda(\theta,\nabla v)$ obtained in Lemma 5.1, there exists a continuous map $\bar{\Theta}$ such that $(\bar{\theta},\nabla\bar{v}) = \bar{\Theta}(\xi_\lambda)$ is a solution to (3.3). Moreover, there is $\lambda_0 > 0$ such that*

$$\|\bar{\theta} - \theta_*\|_{C^{2+\alpha_0}(\overline{\Omega})},\ \|\nabla\bar{v} - \nabla v_*\|_{C^{1+\alpha_0}(\overline{\Omega})}$$

are bounded provided that $\lambda \geqslant \lambda_0$ and

$$\|\bar{\theta} - \theta_*\|_{C^{1+\alpha_0}(\overline{\Omega})} \to 0 \text{ and } \|\nabla\bar{v} - \nabla v_*\|_{C^{\alpha_0}(\overline{\Omega})} \to 0 \text{ as } \lambda \to \infty \tag{5.7}$$

uniformly for $(\theta,\nabla v) \in M(\theta_*,\nabla v_*)$.

Proof. The proof of existence part is similar to Lemma 4.1. Moreover, as in the proof of Lemma 4.2, if $(\bar{\theta},\nabla\bar{v}) \in C^{1+\alpha}(\overline{\Omega}) \times L^p(\mathbb{R}^3)$ $(1 < p < \infty)$ is a solution to (3.3), then $u(\bar{\theta},\xi_\lambda) \in W^{1,p}(\Omega)$, and for $B_R \supset\supset \Omega$, there is an extension $U \in W^{1,p}(B_R)$ of u. From the Maxwell equation

$$\Delta V = (-1)\mathrm{div}(U\chi_{B_R}) \text{ in } \mathbb{R}^3,$$

we have $V \in W^{2,p}(\mathbb{R}^3)$. So $\nabla\bar{v} \in W^{1,p}(\Omega)$, and from (3.3), $\bar{\theta} \in W^{3,p}(\Omega)$. By the Sobolev embedding theorem, $\bar{\theta} \in C^{2+\alpha_0}(\overline{\Omega})$. Noting that $u(\bar{\theta},\xi_\lambda) \in W^{2,p}(\Omega)$, by using the Maxwell equation again, we have $V \in W^{3,p}(\mathbb{R}^3)$ and $\nabla\bar{v} \in W^{2,p}(\Omega)$. Then $\nabla\bar{v} \in C^{1+\alpha_0}(\overline{\Omega})$.

So we only need to prove (5.7). From (3.3) and (4.1), we obtain the equations for $\bar{\theta} - \theta_*$:

$$\begin{cases} \mathrm{div}(\cos^2\xi_\lambda \nabla(\bar{\theta}-\theta_*)) \\ \qquad = -\nabla\cos^2\xi_\lambda \cdot \nabla\theta_* + (1-\cos^2\xi_\lambda)\Delta\theta_* + (u_\theta(\bar{\theta},0) - u_\theta(\bar{\theta},\xi_\lambda)) \cdot \nabla\bar{v} \\ \qquad\qquad + (u_\theta(\theta_*,0) - u_\theta(\bar{\theta},0)) \cdot \nabla\bar{v} - u_\theta(\theta_*,0)\cdot(\nabla\bar{v} - \nabla v_*) \text{ in } \Omega, \\ \bar{\theta} - \theta_* = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.8}$$

and the equation for $\bar{v} - v_*$,

$$\int_{\mathbb{R}^3} |\nabla(\bar{v}-v_*)|^2 dx = (-1)\int_\Omega (u(\bar{\theta},\xi_\lambda) - u(\theta_*,0))\cdot\nabla(\bar{v}-v_*)dx. \tag{5.9}$$

Multiplying (5.8) by $\bar{\theta} - \theta_*$, by standard elliptic equation theory as well as (5.9) and Lemma 5.1, as in the proof of Lemma 4.1, we can prove that there exists $d_0 > 0$ such that if $|\Omega| \leqslant d_0$ then

$$\|\nabla(\bar{\theta}-\theta_*)\|_{L^2(\Omega)} \to 0 \text{ as } \lambda \to \infty, \tag{5.10}$$

uniformly for $(\theta,\nabla v) \in M(\theta_*,\nabla v_*)$. Using (5.9) again, we have

$$\|\nabla\bar{v} - \nabla v_*\|_{L^2(\mathbb{R}^3)} \to 0 \text{ as } \lambda \to \infty, \tag{5.11}$$

uniformly for $(\theta,\nabla v) \in M(\theta_*,\nabla v_*)$.

As in the Lemma 4.1-4.2, we can prove that for λ large enough,

$$\|\bar{\theta}-\theta_*\|_{C^{2+\alpha_0}(\overline{\Omega})},\ \|\nabla\bar{v} - \nabla v_*\|_{C^{1+\alpha_0}(\overline{\Omega})}$$

are bounded uniformly for $(\theta,\nabla v) \in M(\theta_*,\nabla v_*)$. So we have (5.7). □

From Lemma 5.1-5.2, we obtain

LEMMA 5.3. *$\bar{\Theta}\Xi_\lambda(M(\theta_*,\nabla v_*))$ is pre-compact in $M(\theta_*,\nabla v_*)$ and $\bar{\Theta}\Xi_\lambda$ is continuous provided λ is large enough.*

LEMMA 5.4. *For any given $(\theta,\nabla v)\in M(\theta_*,\nabla v_*)$, if $\theta\in C^{3+\alpha_0}(\overline{\Omega})$ and $\nabla v\in C^{1+\alpha_0}(\overline{\Omega})$, then the solution $\xi_\lambda=\Xi_\lambda(\theta,\nabla v)$ of (3.2) obtained in Lemma 5.1 satisfies*

$$\|\xi_\lambda\|_{C^{2+\alpha_0}(\overline{\Omega})}\leqslant C_3, \tag{5.12}$$

and

$$\lim_{\lambda\to\infty}\|\xi_\lambda\|_{C^2(\overline{\Omega})}=0, \tag{5.13}$$

where the constant $C_3=C_3(\|\theta\|_{C^{2+\alpha_0}(\overline{\Omega})},\|\nabla v\|_{C^{1+\alpha_0}(\overline{\Omega})})$ is independent of λ.

Proof. Take the derivative $\partial_j=\frac{\partial}{\partial x_j}$ in the equation (3.2), and rewrite it as

$$\begin{aligned}-\Delta\partial_j\xi+2\lambda\partial_j\xi&=2\lambda(1-\cos2\xi)\partial_j\xi+\nabla\theta\cdot\nabla\partial_j\theta\sin2\xi+|\nabla\theta|^2(\cos2\xi)\partial_j\xi\\&\quad+\partial_j u_\xi(\theta,\xi)\cdot\nabla v+u_\xi(\theta,\xi)\cdot\nabla\partial_j v=:f\ \text{ in }\ \Omega,\\ \xi&=0\ \text{ on }\ \partial\Omega.\end{aligned} \tag{5.14}$$

Applying [27, Theorem 3.1.3] to (5.14), we have for any $q\in(1,\infty)$, there is $\lambda_0(q)>0$ such that for $\lambda\geqslant\lambda_0$,

$$\|\partial_j\xi\|_{W^{2,q}(\Omega)}\leqslant C\|f\|_{L^q(\Omega)}.$$

By the Sobolev embedding theorem, we get (5.12). (5.2) and (5.12) imply (5.13). □

PROOF OF THEOREM 1. From Lemma 5.3 and Schauder fixed point theorem, $\bar{\Theta}\Xi_\lambda$ has a fixed point $(\theta_\lambda,\nabla v_\lambda)$ in $M(\theta_*,\nabla v_*)$ for large λ. Then we obtain a solution $(\theta_\lambda,\xi_\lambda)$ and v_λ to (3.2)-(3.3) which has the properties stated in Theorem 1 by Lemma 5.1-5.3 and Lemma 5.4. □

6. Proof of Theorem 2

The time developing Landau-Lifshitz-Maxwell equations (1.3) can be written as

$$\begin{cases}\partial_t\theta=\dfrac{1}{\cos^2\xi}\{\operatorname{div}(\cos^2\xi\nabla\theta)+u_\theta(\theta,\xi)\cdot\nabla v\}\\ \qquad\qquad-\dfrac{\gamma}{\cos\xi}\{\Delta\xi+(\dfrac{|\nabla\theta|^2}{2}-\lambda)\sin2\xi+u_\xi(\theta,\xi)\cdot\nabla v\}\ \text{ in }\ \Omega\times\mathbb{R}^+,\\ \partial_t\xi=\Delta\xi+(\dfrac{|\nabla\theta|^2}{2}-\lambda)\sin2\xi+u_\xi(\theta,\xi)\cdot\nabla v\\ \qquad\qquad+\dfrac{\gamma}{\cos\xi}\{\operatorname{div}(\cos^2\xi\nabla\theta)+u_\theta(\theta,\xi)\cdot\nabla v\}\ \text{ in }\ \Omega\times\mathbb{R}^+,\\ \theta=\theta_g,\ \ \xi=0\ \text{ on }\ \partial\Omega\times\mathbb{R}^+,\\ \operatorname{div}\{\nabla v+\chi_\Omega u(\theta,\xi)\}=0\ \text{ in }\ \mathbb{R}^3\times\mathbb{R}^+.\end{cases} \tag{6.1}$$

For simplicity, we denote the solution $(\theta_\lambda,\xi_\lambda)$ obtained in Section 4 by (θ,ξ) and

$$\nabla v=\mathscr{L}(u(\theta,\xi)).$$

The linearized operator A_λ of the right terms of (6.1) is written as

$$A_\lambda = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} = \Delta + \frac{\nabla \cos^2\xi \cdot \nabla}{\cos^2\xi} - \frac{\gamma \sin 2\xi}{\cos\xi} \nabla\theta \cdot \nabla \\ + \frac{1}{\cos^2\xi}\big\{u_{\theta\theta}(\theta,\xi)\cdot\nabla v + u_\theta(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\cdot)\big\} \\ - \frac{\gamma}{\cos\xi}\big\{u_{\xi\theta}(\theta,\xi)\cdot\nabla v + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\cdot)\big\}, \end{aligned}$$

$$\begin{aligned} a_{12} = \frac{2}{\cos^2\xi}\nabla\xi\cdot\nabla\theta - \frac{2\sin\xi}{\cos\xi}\nabla\theta\cdot\nabla \\ + \frac{1}{\cos^2\xi}\big\{u_{\theta\xi}(\theta,\xi)\cdot\nabla v + u_\theta(\theta,\xi)\cdot\mathscr{L}(u_\xi(\theta,\xi)\cdot)\big\} \\ + \frac{2\sin\xi}{\cos^3\xi}u_\theta(\theta,\xi)\cdot\nabla v \\ - \frac{\gamma\sin\xi}{\cos^2\xi}\Big\{\Delta\xi + (\frac{|\nabla\theta|^2}{2} - \lambda)\sin 2\xi + u_\xi(\theta,\xi)\cdot\nabla v\Big\} \\ - \frac{\gamma}{\cos\xi}(\Delta + (|\nabla\theta|^2 - 2\lambda)\cos 2\xi) \\ - \frac{\gamma}{\cos\xi}\big\{u_{\xi\xi}(\theta,\xi)\cdot\nabla v + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\xi(\theta,\xi)\cdot)\big\}, \end{aligned}$$

$$\begin{aligned} a_{21} = \sin 2\xi\nabla\theta\cdot\nabla + \frac{\gamma}{\cos\xi}(\cos^2\xi\Delta + \nabla\cos^2\xi\cdot\nabla) + u_{\xi\theta}(\theta,\xi)\cdot\nabla v \\ + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\cdot) \\ + \frac{\gamma}{\cos\xi}\big\{u_{\theta\theta}(\theta,\xi)\cdot\nabla v + u_\theta(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\cdot)\big\}, \end{aligned}$$

$$\begin{aligned} a_{22} = \Delta + (|\nabla\theta|^2 - 2\lambda)\cos 2\xi + u_{\xi\xi}(\theta,\xi)\cdot\nabla v + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\xi(\theta,\xi)\cdot) \\ + \frac{\gamma}{\cos\xi}\big\{-\sin 2\xi\Delta\theta - 2\cos 2\xi\nabla\xi\cdot\nabla\theta - \sin 2\xi\nabla\theta\cdot\nabla\big\} \\ + \frac{\gamma\sin\xi}{\cos^2\xi}\operatorname{div}(\cos^2\xi\nabla\theta) + \gamma u_\theta(\theta,0)\cdot\mathscr{L}(u_\xi(\theta,\xi)\cdot). \end{aligned}$$

Decompose the operator A_λ into $\bar{A}_\lambda$ and the perturbation G:

$$G = A_\lambda - \bar{A}_\lambda$$

$$= \begin{pmatrix} \frac{\nabla \cos^2 \xi \cdot \nabla}{\cos^2 \xi} - \frac{\gamma \sin 2\xi \nabla\theta \cdot \nabla}{\cos \xi} & \left\{ \begin{array}{c} -\dfrac{\sin 2\xi \nabla\theta \cdot \nabla}{\cos^2 \xi} - \dfrac{\gamma \Delta}{\cos \xi} \\ +\dfrac{2\gamma\lambda}{\cos \xi}(\cos 2\xi + \sin^2 \xi) \end{array} \right\} \\ G_{21} & -\frac{\gamma \sin 2\xi \nabla\theta \cdot \nabla}{\cos \xi} \end{pmatrix}, \tag{6.2}$$

where

$$G_{21} = a_{21} - u_\xi(\theta, \xi) \cdot \mathscr{L}(u_\theta(\theta, \xi) \cdot)$$

We consider the spectrum of the operator A_λ as the perturbation one of the operator $\bar{A}_\lambda$. As in [37], [45], following Propositions 5.1-5.3 can be proved in the same way.

PROPOSITION 6.1. *Let $T = \beta I - \bar{A}_\lambda$. For $\delta > 0$, there exist $\beta > 0$, $\lambda_0 = \lambda_0(\delta)$, and for $\lambda \geqq \lambda_0$ there exists $\gamma_0(\lambda) > 0$ such that for $\lambda \geqq \lambda_0$ and $\gamma \in [0, \gamma_0]$, we have*

$$\|G\Phi\|_H \leqq \delta^{1/2}((\beta+1)\|\Phi\|_H + \|T\Phi\|_H) \ \textit{for} \ \Phi \in D(T).$$

That is, G is T-bounded with T-bound $b : b \leqq \delta^{1/2}$.

For $j = 1, 2$, let

$$Y_j = \left\{ \begin{pmatrix} \theta \\ \xi \end{pmatrix} \in W^{j,2}(\Omega, \mathbb{R}^2) \ : \ \begin{pmatrix} \theta \\ \xi \end{pmatrix} = \begin{pmatrix} \theta_g \\ 0 \end{pmatrix} \text{ on } \partial\Omega \right\},$$

where Y_j are endowed with the norm and inner product of $W^{j,2}(\Omega, \mathbb{R}^2)$. Note that Y_j are Hilbert spaces. As in [37], [35], we have

PROPOSITION 6.2. *There exist $d_0 > 0$, $\lambda_0 > 0$, and for $\lambda \geqslant \lambda_0$ there exists $\gamma_0(\lambda) > 0$ such that $A_\lambda : D(A_\lambda)(\subset L^2(\Omega, \mathbb{R}^2)) \to L^2(\Omega, \mathbb{R}^2)$ is a sectorial operator with $D(A_\lambda) = Y_2$, provided that $|\Omega| \leqslant d_0$, $\lambda \geqq \lambda_0$ and $\gamma \in [0, \gamma_0(\lambda)]$. Moreover the norm of Y_2 is equivalent to the graph norm of A_λ,*

$$\left\| \begin{pmatrix} \theta \\ \xi \end{pmatrix} \right\|_{Y_2} \sim \left\| \begin{pmatrix} \theta \\ \xi \end{pmatrix} \right\|_{L^2(\Omega, \mathbb{R}^2)} + \left\| A_\lambda \begin{pmatrix} \theta \\ \xi \end{pmatrix} \right\|_{L^2(\Omega, \mathbb{R}^2)},$$

and for any $\mu \in \rho(A_\lambda)$, $(\mu - A_\lambda)^{-1}$ is compact.

Let $\bar{\mu}_1(\lambda), \bar{\mu}_2(\lambda), \ldots, \bar{\mu}_k(\lambda), \ldots$ and

$$(\bar{\phi}_k^\lambda, \bar{\psi}_k^\lambda) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega), \ \|\bar{\phi}_k^\lambda\|^2_{L^2(\Omega)} + \|\bar{\psi}_k^\lambda\|^2_{L^2(\Omega)} = 1, \ k = 1, 2, \ldots$$

denote the eigenvalues and eigenfunctions of $\bar{A}_\lambda$, respectively. Assume

$$Re\bar{\mu}_1(\lambda) \leqslant Re\bar{\mu}_2(\lambda) \leqslant \ldots \leqslant Re\bar{\mu}_k(\lambda) \leqslant \ldots.$$

PROPOSITION 6.3. *There exist $d_0 > 0$, $\lambda_0 > 0$, $\bar{\gamma} > 0$ and $C > 0$ such that $Re\bar{\mu}_k(\lambda) \geqq -C$ provided that $|\Omega| \leqslant d_0$, $\lambda \geqq \lambda_0$ and $\gamma \in [0, \bar{\gamma}]$. Moreover, if there is $\{\lambda_j\}_j$ such that*

$$\limsup_{\lambda_j \to \infty} Re\bar{\mu}_k(\lambda_j) < \infty,$$

then

$$\limsup_{\lambda_j \to \infty} \lambda_j \int_\Omega (\bar{\psi}_k^{\lambda_j})^2 dx < \infty.$$

The eigenvalue problem for $\bar{A}_\lambda$ can be written as:

$$\begin{cases} \Delta\phi + \dfrac{2}{\cos^2\xi}(\nabla\xi \cdot \nabla\theta)\psi - \dfrac{\gamma}{\cos\xi}\left(|\nabla\theta|^2(\cos 2\xi)\psi\right) \\ \qquad - \dfrac{\gamma \sin\xi}{\cos^2\xi}\left(\Delta\xi + \dfrac{|\nabla\theta|^2}{2}\sin 2\xi\right)\psi \\ \qquad + \dfrac{1}{\cos^2\xi}\{u_{\theta\theta}(\theta,\xi)\cdot\nabla v\phi + u_\theta(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\phi)\} \\ \qquad - \dfrac{\gamma}{\cos\xi}\{u_{\xi\theta}(\theta,\xi)\cdot\nabla v\phi + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\phi)\} \\ \qquad + \dfrac{1}{\cos^2\xi}\{u_{\theta\xi}(\theta,\xi)\cdot\nabla v\psi + u_\theta(\theta,\xi)\cdot\mathscr{L}(u_\xi(\theta,\xi)\psi)\} \\ \qquad + \dfrac{2\sin\xi}{\cos^3\xi}u_\theta(\theta,\xi)\cdot\nabla v\psi - \dfrac{\gamma\sin\xi}{\cos^2\xi}\{u_\xi(\theta,\xi)\cdot\nabla v\psi\} \\ \qquad - \dfrac{\gamma}{\cos\xi}\{u_{\xi\xi}(\theta,\xi)\cdot\nabla v\psi + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\xi(\theta,\xi)\psi)\} \\ \quad = -\bar{\mu}\phi \ \text{ in } \ \Omega, \\ \phi = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{6.3}$$

and

$$\begin{cases} \Delta\psi + (|\nabla\theta|^2 - 2\lambda)(\cos 2\xi)\psi \\ \qquad + \dfrac{\gamma}{\cos\xi}(-\sin 2\xi\Delta\theta - 2\cos 2\xi\nabla\xi\cdot\nabla\theta)\psi \\ \qquad + \dfrac{\gamma\sin\xi}{\cos^2\xi}\operatorname{div}(\cos^2\xi\nabla\theta)\psi + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\theta(\theta,\xi)\phi) \\ \qquad + (u_{\xi\xi}(\theta,\xi)\cdot\nabla v)\psi + u_\xi(\theta,\xi)\cdot\mathscr{L}(u_\xi(\theta,\xi)\psi) \\ \qquad + \gamma u_\theta(\theta,0)\cdot\mathscr{L}(u_\xi(\theta,\xi)\psi) \\ \quad = -\bar{\mu}\psi \ \text{ in } \ \Omega, \\ \psi = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{6.4}$$

LEMMA 6.4. *Suppose there is $\{\lambda_j\}_j$ such that*

$$\limsup_{\lambda_j \to \infty} Re\bar{\mu}_k(\lambda_j) < \infty.$$

Then there exist $\lambda_0 > 0$ *and* $\bar{\gamma} > 0$ *such that*

$$\|\bar{\psi}_k^{\lambda_j}\|_{C^\alpha(\overline{\Omega})} \leqslant \frac{C}{\sqrt{\lambda_j}}, \tag{6.5}$$

$$\|\bar{\psi}_k^{\lambda_j}\|_{C^{2+\alpha}(\overline{\Omega})} \leqslant C, \tag{6.6}$$

and

$$\|\bar{\phi}_k^{\lambda_j}\|_{C^{2+\alpha}(\overline{\Omega})} \leqslant C, \tag{6.7}$$

provided $\lambda_j \geqq \lambda_0$ *and* $\gamma \in [0, \bar{\gamma}]$*, where the constant* C *only depends on* k *and the* $C^{2+\alpha}(\overline{\Omega})$ *norm of* (ξ, θ)*.*

Proof. Step 1. Since ϕ, $\psi \in L^2(\Omega)$, from Lemma 2.1 we have

$$\mathscr{L}(u_\theta(\theta,\xi)\phi),\ \mathscr{L}(u_\xi(\theta,\xi)\psi) \in L^2(\mathbb{R}^3).$$

From (6.3)-(6.4) and [27, Theorem 3.1.3], we get ϕ, $\psi \in W^{2,2}(\Omega)$. Since Ω is regular, by the Sobolev embedding theorem, ϕ, $\psi \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. So, from Lemma 2.1, we have for $p \in (1,\infty)$

$$\mathscr{L}(u_\theta(\theta,\xi)\phi),\ \mathscr{L}(u_\xi(\theta,\xi)\psi) \in L^p(\mathbb{R}^3).$$

By using (6.3)-(6.4) and [27, Theorem 3.1.3] again, we obtain

$$\phi,\ \psi \in W^{2,p}(\Omega),\ \ \forall p \in (1,\infty) \tag{6.8}$$

and there is constant C such that

$$\|\phi\|_{W^{2,p}(\Omega)},\ \|\psi\|_{W^{2,p}(\Omega)} \leqslant C \text{ uniformly for } \lambda_j \geqslant \lambda_0.$$

By the Sobolev embedding theorem,

$$\phi,\ \psi \in C^{1+\alpha}(\overline{\Omega}).$$

As in the proof of Lemma 4.2, we have

$$\mathscr{L}(u_\theta(\theta,\xi)\phi),\ \mathscr{L}(u_\xi(\theta,\xi)\psi) \in C^{1+\alpha}(\overline{\Omega}). \tag{6.9}$$

Considering the equations for $\partial_{x_i}\phi$ and $\partial_{x_i}\psi$, similarly we have ϕ, $\psi \in C^{2+\alpha}(\overline{\Omega})$ and

$$\|\phi\|_{C^{2+\alpha}(\overline{\Omega})},\ \|\psi\|_{C^{2+\alpha}(\overline{\Omega})}$$

are bounded uniformly for $\lambda_j \geqslant \lambda_0$.

Step 2. Applying [27, Theorem 3.1.3] to (6.4), we obtain that for all $q \in (1,\infty)$, there exist $\bar{\gamma} > 0$ and $\lambda_0 > 0$ such that for $\gamma \in [0,\bar{\gamma}]$ and $\lambda_j \geqslant \lambda_0$,

$$\|\psi\|_{W^{1,q}(\Omega)} \leqslant \frac{C}{\sqrt{\lambda_j}}, \tag{6.10}$$

where the constant C is independent of λ_j, γ. By the Sobolev embedding theorem we get (6.5). $\square$

LEMMA 6.5. *Assume* $|\Omega| \leqslant d_0$. *There exist* $\lambda_0 > 0$ *and* $\bar{\gamma} > 0$, *such that for* $\lambda \geqq \lambda_0$, $\gamma \in [0, \bar{\gamma}]$,

$$Re\bar{\mu}_1(\lambda) \geqslant \mu_0 (> 0),$$

where μ_0 *is independent of* λ.

Proof. If not, there is $\{\lambda_j\}_j$ such that

$$\limsup_{\lambda_j \to \infty} Re\bar{\mu}_1(\lambda_j) < \mu_0.$$

From Lemma 6.4, (6.3) and (6.4) converge to the eigenvalue problem

$$\begin{cases} \Delta\phi + u_{\theta\theta}(\theta_*, 0)\cdot(\nabla v_*)\phi + u_\theta(\theta_*, 0) - \gamma(0,0,1)\cdot h = -\mu\phi \ \text{ in } \ \Omega, \\ h = \mathscr{L}(u_\theta(\theta_*, 0)\phi), \ \ (0,0,1)\cdot h = 0 \ \text{ in } \ \Omega, \\ \phi = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{6.11}$$

Integrating over Ω and using

$$-\int_\Omega \phi u_\theta(\theta_*, 0)\cdot h dx = \int_{\mathbb{R}^3} |h|^2 dx,$$

we have

$$\begin{aligned} Re\mu \int_\Omega \phi^2 dx &= Re \int_\Omega |\nabla\phi|^2 dx - \phi^2 u_{\theta\theta}(\theta_*, 0)\cdot \nabla v_* dx + \int_{\mathbb{R}^3} |h|^2 dx \\ &\geqslant \int_\Omega |\nabla\phi|^2 dx - (\int_\Omega |\nabla v_*|^{3/2} dx)^{2/3} (\int_\Omega |\phi|^6 dx)^{1/3} + \int_{\mathbb{R}^3} |h|^2 dx \\ &\geqslant c_0 \int_\Omega \phi^2 dx \end{aligned}$$

provided that $|\Omega| \leqslant d_0$ and d_0 is small enough, where $c_0 > 0$ is a constant. Taking $\mu_0 = c_0/2$, we get a contradiction. So we proved this lemma. □

PROOF OF THEOREM 2. For fixed $\lambda \geqq \lambda_0$, we can choose $\gamma \in (0, \bar{\gamma})$ such that $\gamma\lambda$ is small enough. By a perturbation argument (c.f. [25]), we have that for $|\Omega| \leqslant d_0$ and for fixed $\lambda \geqq \lambda_0$, there exists $\gamma_0(\lambda) > 0$ such that for $\gamma \in [0, \gamma_0(\lambda)]$, the eigenvalues $\{\mu_k(\lambda)\}_k$ of operator A_λ have same behavior as $\bar{A}_\lambda$. Then

$$Re(\mu_1(\lambda)) \geqq \frac{\mu_0}{2}.$$

By using the result of the Proposition 6.2 and [27] (p.295, the remark behind the proof of Theorem 8.1.1), we have the local existence of solutions to (6.1) with initial data in a neighborhood of the steady state solutions obtained in the Theorem 1. Moreover by using [27, Theorem 9.1.2], the solutions which satisfy the conditions in the Theorem 2 are exponentially asymptotically stable. □

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