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ON A CLASS OF ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS INVOLVING ALMOST SECTORIAL OPERATORS

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Abstract. We study the existence of mild solutions for a class of abstract functional differential equations involving almost sectorial operators. A concrete application to delayed partial differential equations is presented.

1. Introduction

In this paper, we study the existence of "mild" solutions for a class of abstract functional differential equations of the form

$$x'(t) = Ax(t) + f(t, x_t), \qquad t \in [0,a], \tag{1.1}$$

$$x_0 = \varphi \in \Omega \subset \mathscr{B}, \tag{1.2}$$

where $A : D(A) \subset X \to X$ is an almost sectorial operator, $(X, \|\cdot\|)$ is a Banach space, $\mathscr{B}$ is the phase, $\Omega \subset \mathscr{B}$ is open and $f : [0,a] \times \Omega \to X$ is a suitable function.

There exists an extensive literature treating on the existence and qualitative properties of solutions for abstract systems as (1.1)-(1.2) for the case in which A is the infinitesimal generator of a C_0-semigroup of bounded linear operators, see [3, 4, 5, 6, 11, 12, 14, 15] and the references therein. This type of operators appears frequently in applications since many elliptic differential operators are generators of C_0-semigroup when they are considered in L^p-spaces, see [7]. However, if we look at spaces of more regular functions such as the spaces of Hölder continuous functions, we see that these elliptic operators are not generator of C_0-semigroups, see [7, Example 3.1.33] and [9]. Nevertheless, in many cases, estimates such as

$$\| (\lambda - A)^{-1} \| \leqslant \frac{M}{|\lambda|^{1-\alpha}}, \qquad \lambda \in \Sigma_{\omega,\vartheta} = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \vartheta\}, \tag{1.3}$$

with $\alpha \in (0,1)$, $\omega \in \mathbb{R}$ and $\vartheta \in (\pi/2, \pi)$ are available (see [9]), which allows to define an "associated analytic semigroup" by means of the Dunford integral

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} (\lambda - A)^{-1} d\lambda, \qquad t > 0, \tag{1.4}$$

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where $\Gamma_\theta = \{te^{i\theta} : t \in \mathbb{R} \setminus \{0\}\}$, $\theta \in (\vartheta, \pi/2)$.

In the literature, a linear operator $A : D(A) \subset X \to X$ which satisfy the condition (1.3) is called almost sectorial and the operator family $\{T(t), T(0) = I : t \geqslant 0\}$ is said the "semigroup of growth α" generated by A. The operator family $(T(t))_{t\geqslant 0}$ has properties similar at those of analytic semigroup which allow to study some classes of partial differential equations via the useful methods of semigroup theory. Concerning almost sectorial operators, semigroups of growth α and applications to partial differential equations we refer the reader to [1, 2, 9, 10, 8, 13] and the references therein.

To the best of our knowledge, the study of the existence of solutions of abstract system as (1.1)-(1.2) for which the operator A is almost sectorial is an untreated topic in the literature. This fact is the main motivation of this paper.

Next, we introduce some notations and technicalities. Let $(Z, \| \cdot \|_Z)$ be a Banach space. In this paper, $\mathscr{L}(Z, W)$ represents the space of bounded linear operators from Z into W endowed with norm of operators denoted $\| \cdot \|_{\mathscr{L}(Z,W)}$, and we write $\mathscr{L}(Z)$ and $\| \cdot \|_{\mathscr{L}(Z)}$ when $Z = W$. In addition, $B_l(z, Z)$ denotes the closed ball with center at $z \in Z$ and radius $l > 0$ in Z. As usual, $C([c,d], Z)$ represents the space formed by all the continuous functions from I into Z endowed with the sup-norm denoted by $\| \cdot \|_{C([c,d],Z)}$ and $L^p([c,d], X)$, $p \geqslant 1$, denotes the space formed by all the classes of Lebesgue-integrable functions from $[c,d]$ into X endowed with the norm

$$\| h \|_{L^p([c,d],X)} = \left(\int_{[c,d]} \| h(s) \|^p \, ds \right)^{\frac{1}{p}}.$$

Throughout this paper, $(X, \| \cdot \|)$ is a Banach space, $A : D(A) \subset X \to X$ is an almost sectorial operator and $(T(t))_{t\geqslant 0}$ is the semigroup of growth α generated by A. For simplicity, next we assume $\omega = 0$. The next lemma consider some properties of the operator family $(T(t))_{t\geqslant 0}$.

LEMMA 1.1. [2, 9] *Under the above conditions, the followings properties are satisfied.*

(a) *The operator A is closed, $T(t+s) = T(t)T(s)$ and $AT(t)x = T(t)Ax$ for all $t, s \in [0, \infty)$ and each $x \in D(A)$.*

(b) *$T(\cdot) \in C((0,\infty), X) \bigcap C^1((0,\infty), X)$ and $\frac{d}{dt}T(t) = AT(t)$ for all $t > 0$.*

(c) *For $n \in \mathbb{N} \cup \{0\}$, $A^n T(\cdot) \in C((0,\infty), X)$ and there exits $D_n > 0$ and a constant $\gamma > 0$, which is independent of n, such that $\| A^n T(t) \|_{\mathscr{L}(X)} \leqslant D_n e^{\gamma t} t^{-(n+\alpha)}$ for all $t > 0$.*

We include now some remarks on the abstract Cauchy problem

$$x'(t) = Ax(t) + \xi(t), \qquad t \in [0,a], \tag{1.5}$$

$$x(0) = x \in X, \tag{1.6}$$

where $\xi \in L^p([0,a], X)$ and $p > \frac{1}{1-\alpha}$. From [2, 10], we adopt the following concept of solution.

DEFINITION 1.1. A function $u : [0,b] \to X$, $0 < b \leqslant a$, is said a mild solution of (1.5)-(1.6) on $[0,b]$ if $u \in C((0,a],X)$ and

$$u(t) = T(t)x + \int_0^t T(t-s)\xi(s)ds, \quad \forall t \in [0,b]. \tag{1.7}$$

This paper has three sections. By considering the behavior near of zero of the function $t \to T(t)\varphi(0)$, in the Section 2 we discuss the existence of "mild" solutions for (1.1)-(1.2) for the case in which $\mathscr{B} = C([-r,0],X)$ and $\mathscr{B} = L^p([-r,0],X)$. In the last section, a concrete application to partial differential equations with delay is considered.

2. Existence of mild solutions

In this section we study the existence of mild solutions for the system (1.1)-(1.2). By considering Definition 1.1, we introduce the following concept of solution.

DEFINITION 2.1. A function $u : [-r,b] \to X$, $0 < b \leqslant a$, is called a mild solution of the abstract system (1.1)-(1.2) on $[-r,b]$ if $u_0 = \varphi$, $u_{|_{(0,b]}} \in C((0,b],X)$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s,u_s)ds, \ \forall t \in [0,b].$$

REMARK 2.1. In the remainder of this paper, $\varphi : [-r,0] \to X$ is a given function and $y : [-r,a] \to X$ is the function defined by $y(\theta) = \varphi(\theta)$ for $\theta \leqslant 0$ and $y(t) = T(t)\varphi(0)$ for $t > 0$. In addition, C_n, $n \in \mathbb{N}$, are positive constants such that

$$\| A^n T(t) \|_{\mathscr{L}(X)} \leqslant C_n t^{-(n+\alpha)} \text{ for all } t \in (0,a],$$

and for a bounded set $B \subset X$, we use the notation $\mathrm{Diam}_X(B)$ for

$$\mathrm{Diam}_X(B) = \sup_{a,b \in B} \| a - b \| .$$

To prove our results, we introduce the following conditions. In the next assumptions, $q \in (\frac{1}{1-\alpha}, \infty)$ or $q = \infty$ and $q' = \frac{p}{p-1}$ for $q < \infty$ and $q' = 1$ if $q = \infty$.

(H_1) The function $f(\cdot,\psi)$ is strongly measurable on $[0,a]$ for all $\psi \in \Omega$ and $f(t,\cdot) \in C(\Omega,X)$ for each $t \in [0,a]$. There are $m_f \in L^q([0,a],\mathbb{R}^+)$ and a non-decreasing function $W_f \in C([0,\infty),(0,\infty))$ such that

$$\| f(t,\psi) \| \leqslant m_f(t) W_f(\| \psi \|_{\mathscr{B}}) \text{ for all } (t,\psi) \in [0,a] \times \Omega.$$

(H_2) The function f is continuous and for all $l > 0$ with $[0,l] \times B_l(\varphi,\mathscr{B}) \subset [0,a] \times \Omega$, there exists $L_{f,l} \in L^q([0,a],\mathbb{R}^+)$ such that

$$\|f(s,\psi_1) - f(s,\psi_2)\| \leqslant L_{f,l}(s) \|\psi_1 - \psi_2\|_{\mathscr{B}}, \qquad \forall (s,\psi_i) \in [0,l] \times B_l(\varphi,\mathscr{B}).$$

Our development depends on the behavior of the function $y(\cdot)$ at zero. For this reason, next we consider separately the cases in which $\mathscr{B} = C([-r,0],X)$ and $\mathscr{B} = L^p([-r,0],X)$.

2.1. The case $\mathscr{B} = \mathbf{C}([-\mathbf{r},\mathbf{0}],\mathbf{X})$.

In this section we study the existence of mild solutions for (1.1)-(1.2) assuming that $\mathscr{B} = C([-r,0],X)$. Next, to simplify we write $\|\cdot\|_{\mathscr{B}}$ in place of $\|\cdot\|_{C([-r,0],X)}$. Now, we can establish our first existence result.

THEOREM 2.1. *Assume the condition* (H_1) *is satisfied,* $T(\cdot)\varphi(0) \in C([0,a],X)$ *and* $T(t)$ *is compact for all* $t > 0$. *Then there exists a mild solution of* (1.1)-(1.2) *on* $[-r,b]$, *for some* $0 < b \leqslant a$.

Proof. Let $0 < b_1 < a$ and $C > 0$ such that $B_{b_1}(\varphi,\mathscr{B}) \subseteq \Omega$ and $W_f(\|\psi\|_{\mathscr{B}}) \leqslant C$ for all $\psi \in B_{b_1}(\varphi,\mathscr{B})$. Since $\varphi \in C([-r,0],X)$ and $T(\cdot)\varphi(0) \in C([0,\infty),X)$, we have that the function $t \to y_t$ belongs to $C([0,a],\mathscr{B})$. Using this fact, we can select $0 < b < b_1$ such that

$$\sup_{s\in[0,b]} \|y_s - \varphi\|_{\mathscr{B}} \leqslant \frac{b_1}{2} \quad \text{and} \quad CC_0 \frac{\| m_f \|_{L^q([0,b])}\, b^{\frac{1}{q'}-\alpha}}{(1-q'\alpha)^{\frac{1}{q'}}} \leqslant \frac{b_1}{2}.$$

On the space

$$B_{\frac{b_1}{2}}(0,S(b)) = \{u \in C([-r,b],X) : u_0 = 0, \|u\|_{C([0,b],X)} \leqslant \frac{b_1}{2}\}$$

endowed with the norm $\|\cdot\|_{C([0,b],X)}$, we define the map

$$\Gamma : B_{\frac{b_1}{2}}(0,S(b)) \to C([-r,b],X)$$

by $(\Gamma u)_0 = 0$ and

$$\Gamma u(t) = \int_0^t T(t-s) f(s,u_s + y_s)ds, \qquad t \in [0,b]. \tag{2.1}$$

Next, we will prove that Γ is completely continuous from $B_{\frac{b_1}{2}}(0,S(b))$ into $B_{\frac{b_1}{2}}(0,S(b))$. To begin, we note that for $(s,u) \in [0,b] \times B_{\frac{b_1}{2}}(0,S(b))$,

$$\|u_s + y_s - \varphi\|_{\mathscr{B}} \leqslant \sup_{\theta\in[0,s]} \|u(\theta)\| + \|y_s - \varphi\|_{\mathscr{B}} \leqslant \frac{b_1}{2} + \frac{b_1}{2} \leqslant b_1, \tag{2.2}$$

which implies that $u_s + y_s \in B_{b_1}(\varphi,\mathscr{B})$ and $W_f(\| u_s + y_s \|_{\mathscr{B}}) \leqslant C$. Now, from the properties of $(T(t))_{t\geqslant 0}$ and f, the Bochner's criterion for integrable functions and the inequality

$$\| T(t-s) f(s,u_s + y_s) \| \leqslant \frac{C_0 m_f(s) W(\|u_s + y_s - \varphi\|_{\mathscr{B}})}{(t-s)^{\alpha}} \leqslant \frac{C_0 C m_f(s)}{(t-s)^{\alpha}}, \tag{2.3}$$

we infer that the function $s \to T(t-s)f(s,u_s + y_s)$ is integrable on $[0,t]$ for all $t \in [0,b]$, which implies that $\Gamma u \in C([-r,b],X)$ and Γ is well defined.

On the another hand, from the estimate

$$\|\Gamma u(t)\| \leqslant CC_0 \sup_{t\in[0,b]} \int_0^t \frac{m_f(s)}{(t-s)^\alpha} ds \leqslant CC_0 \frac{\| m_f \|_{L^q([0,b])} b^{\frac{1}{q'}-\alpha}}{(1-q'\alpha)^{\frac{1}{q'}}},$$

it follows that $\Gamma u \in B_{\frac{b_1}{2}}(0,S(b))$ and $\Gamma B_{\frac{b_1}{2}}(0,S(b)) \subset B_{\frac{b_1}{2}}(0,S(b))$. Moreover, a standard application of the Lebesgue dominated convergence Theorem proves that Γ is continuous. In the next steps, we prove that Γ is a compact map.

Step 1. The set $\Gamma B_{\frac{b_1}{2}}(0,S(b)) = \{\Gamma u(t) : u \in B_{\frac{b_1}{2}}(0,S(b))\}$ is relatively compact for all $t \in [-r,b]$.

The case $t \leqslant 0$ is trivial. Let $0 < \varepsilon < t < b$. For $u \in B_{\frac{b_1}{2}}(0,S(b))$ we get

$$\begin{aligned}\Gamma u(t) &= T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon) f(s,x_s+y_s)ds + \int_{t-\varepsilon}^t T(t-s) f(s,x_s+y_s)ds \\ &\in T(\varepsilon) B_{\frac{b_1}{2}}(0,X) + B_{l^1_\varepsilon}(0,X),\end{aligned}$$

where

$$l_{1,\varepsilon} = CC_0 \| m_f \|_{L^q([t-\varepsilon,t])} \varepsilon^{\frac{1}{q'}-\alpha} (1-q'\alpha)^{-\frac{1}{q'}}.$$

Since $T(\varepsilon)$ is compact, from the above we obtain that

$$\Gamma B_{\frac{b_1}{2}}(0,S(b))(t) \subset K_\varepsilon + C_\varepsilon,$$

where K_ε compact and $\mathrm{Diam}_X(C_\varepsilon) \to 0$ as $\varepsilon \to 0$. This proves that $\Gamma B_{\frac{b_1}{2}}(0,S(b))(t)$ is relatively compact in X.

Step 2. The set $\Gamma B_{\frac{b_1}{2}}(0,S(b)) = \{\Gamma u : u \in B_{\frac{b_1}{2}}(0,S(b))\}$ is equicontinuous.

Let $0 < 2\varepsilon < t < b$. By noting that $T(\cdot) \in C([\varepsilon,b],\mathscr{L}(X))$, we can select $0 < \delta < \varepsilon$ such that

$$\| T(\varepsilon+s) - T(\varepsilon) \|_{\mathscr{L}(X)} \leqslant \varepsilon \text{ for all } 0 < s \leqslant \delta.$$

Then, for $u \in B_{\frac{b_1}{2}}(0,S(b))$ and $0 < h < \delta$ such that $t+h < b$, we get

$$\begin{aligned}&\| \Gamma x(t+h) - \Gamma x(t) \| \\ &\quad = \| \int_0^{t-2\varepsilon} T(t+h-s) f(s,x_s+y_s)ds - \int_0^{t-2\varepsilon} T(t-s) f(s,x_s+y_s)ds \| \\ &\qquad + \| \int_{t-2\varepsilon}^{t+h} T(t+h-s) f(s,x_s+y_s)ds \| + \| \int_{t-2\varepsilon}^{t} T(t-s) f(s,x_s+y_s)ds \| \\ &\quad \leqslant \| (T(\varepsilon+h) - T(\varepsilon)) \int_0^{t-2\varepsilon} T(t-s-\varepsilon) f(s,x_s+y_s)ds \| \\ &\qquad + \int_{t-2\varepsilon}^{t+h} \frac{CC_0 m_f(s)}{(t+h-s)^\alpha} ds + \int_{t-2\varepsilon}^{t} \frac{CC_0 m_f(s)}{(t-s)^\alpha} ds\end{aligned}$$

$$\leqslant \varepsilon C C_0 \int_0^{t-2\varepsilon} \frac{m_f(s)}{(t-s-\varepsilon)^\alpha} ds$$
$$+ \frac{C C_0}{(1-q'\alpha)^{\frac{1}{q'}}} \parallel m_f \parallel_{L^q([0,b])} \left((h+2\varepsilon)^{\frac{1}{q'}-\alpha} + (2\varepsilon)^{\frac{1}{q'}-\alpha} \right)$$
$$\leqslant \frac{C C_0 \parallel m_f \parallel_{L^q([0,b])}}{(1-q'\alpha)^{\frac{1}{q'}}} \left[\varepsilon b^{\frac{1}{q'}-\alpha} + 2(3\varepsilon)^{\frac{1}{q'}-\alpha} \right],$$

which proves that $\Gamma B_{\frac{b_1}{2}}(0, S(b))$ is right equicontinuous at $t \in (0,b)$. Arguing as above, we can show that $\Gamma B_{\frac{b_1}{2}}(0, S(b))$ is left equicontinuous at $t \in (0,b]$ and equicontinuous at $t \leqslant 0$. We omit additional details. Thus, $\Gamma B_{\frac{b_1}{2}}(0, S(b))$ is equicontinuous on $[-r,b]$.

Finally, from the Schauder's fixed point criterion there exists a fixed point x of Γ, and by defining the function $u : [-r,b] \to X$ by $u = y + x$ we obtain a mild solution of (1.1)-(1.2) on $[-r,b]$. $\square$

THEOREM 2.2. *Assume the condition* (H_2) *is satisfied and*

$$T(\cdot)\varphi(0) \in C([0,a],X).$$

Then there exists a unique mild solution of (1.1)-(1.2) *on* $[-r,b]$*, for some* $0 < b \leqslant a$.

Proof. Let $0 < b_1 < a$ and $C > 0$ such that $B_{b_1}(\varphi, \mathscr{B}) \subseteq \Omega$ and

$$\parallel f(t,\psi) \parallel \leqslant C \text{ for all } (t,\psi) \in [0,b_1] \times B_{b_1}(\varphi, \mathscr{B}).$$

By using that the function $t \to y_t$ belongs to $C([0,a], \mathscr{B})$ (here is used that $T(\cdot)\varphi(0) \in C([0,a],X)$) we can select $0 < b < b_1$ such $\sup_{s\in[0,b]} \|y_s - \varphi\|_{\mathscr{B}} \leqslant b_1/2$ and

$$C_0 \left[(b_1+1) \parallel L_{f,b_1} \parallel_{L^q([0,b])} \frac{b^{\frac{1}{q'}-\alpha}}{(1-q'\alpha)^{\frac{1}{q'}}} + \sup_{s\in[0,b]} \parallel f(s,\varphi) \parallel \frac{b^{1-\alpha}}{1-\alpha} \right] < \min\{\frac{b_1}{2}, 1\}. \quad (2.4)$$

Let $\Gamma : B_{\frac{b_1}{2}}(0, S(b)) \to C([-r,b],X)$ be the operator introduced in the proof of Theorem 2.1. Proceeding as in the proof of Theorem 2.1, it is easy to see that Γ is well defined. Next, we prove that Γ is a contraction on $B_{\frac{b_1}{2}}(0, S(b))$. At first, we note that for $(s,u) \in [0,b] \times B_{\frac{b_1}{2}}(0, S(b))$,

$$\|u_s + y_s - \varphi\|_{\mathscr{B}} \leqslant \sup_{\theta\in[0,s]} \|u(\theta)\| + \|y_s - \varphi\|_{\mathscr{B}} \leqslant \frac{b_1}{2} + \frac{b_1}{2} \leqslant b_1,$$

from which we have that

$$\parallel f(s, u_s + y_s) \parallel \leqslant C.$$

Using this fact, for $u \in B_{\frac{b_1}{2}}(0,S(b))$ we find that

$$\begin{aligned}\|\Gamma u(t)\| &\leqslant C_0 \sup_{t\in[0,b]} \int_0^t \frac{\| f(s,u_s+y_s)-f(s,\varphi)\|}{(t-s)^\alpha} ds + C_0 \int_0^t \frac{\| f(s,\varphi)\|}{(t-s)^\alpha} ds \\ &\leqslant C_0 \left[b_1 \| L_{f,b_1} \|_{L^q([0,b])} \frac{b^{\frac{1}{q'}-\alpha}}{(1-q'\alpha)^{\frac{1}{q'}}} + \sup_{s\in[0,b]} \| f(s,\varphi)\| \frac{b^{1-\alpha}}{1-\alpha} \right],\end{aligned}$$

which implies that $\Gamma u \in B_{\frac{b_1}{2}}(0,S(b))$ and $\Gamma B_{\frac{b_1}{2}}(0,S(b)) \subset B_{\frac{b_1}{2}}(0,S(b))$ since u is arbitrary. Moreover, from (2.4) and the estimate

$$\begin{aligned}\|\Gamma u(t)-\Gamma v(t)\| &\leqslant C_0 \int_0^t \frac{L_{f,b_1}(s)}{(t-s)^\alpha} \| u_s - v_s \|_{C([-r,0],X)}\, ds \\ &\leqslant C_0 \| L_{f,b_1} \|_{L^q([0,b])} \frac{b^{\frac{1}{q'}-\alpha}}{(1-q'\alpha)^{\frac{1}{q'}}} \|u-v\|_{C([0,b],X)},\end{aligned}$$

it follows that Γ is a contraction on $B_{\frac{b_1}{2}}(0,S(b))$ and there exists a unique fixed point $x \in B_{\frac{b_1}{2}}(0,S(b))$ of Γ. Finally, by defining the function $u:[-r,b]\to X$ by $u=x+y$, we obtain a mild solution of (1.1)-(1.2) on $[-r,b]$. This completes the proof. □

2.2. The case $\mathscr{B} = \mathbf{L^p}([-\mathbf{r},\mathbf{0}],\mathbf{X})$.

The results in the Section 2.1 are proved assuming that $T(\cdot)\varphi(0) \in C([0,a],X)$, which implies that $t \to y_t$ belongs to $C([0,a],C([-r,0],X))$. To treat the general case in which $\varphi(0)\in X$ is arbitrary, we use as phase space the space $\mathscr{B} = L^p([-r,0],X)$ with $p \in [1,\frac{1}{\alpha})$. The following Lemma establishes a well known property of integrable functions. In the remainder of this section, $\mathscr{B} = L^p([-r,0],X)$ for some $p\in[1,\frac{1}{\alpha})$ and $\|\cdot\|_{\mathscr{B}}$ denotes the norm in $L^p([-r,0],X)$.

LEMMA 2.1. *If $u \in L^p([-r,b],X)$, $b>0$, then the function $t\to u_t$ belongs to $C([0,b],\mathscr{B})$.*

The proof of Theorem 2.3 below, follows with minor modifications from the proof of Theorem 2.1. We include a briefly sketch of the proof for completeness.

THEOREM 2.3. *Assume the condition (H_1) is satisfied and $T(t)$ is compact for all $t>0$. Then there exists a mild solution of (1.1)-(1.2) on $[-r,b]$, for some $0<b\leqslant a$.*

Proof. Using that $p\in[1,\frac{1}{\alpha})$ we can prove that $y \in L^p([-r,a],X)$ and the function $t\to y_t$ belongs to $C([0,a],\mathscr{B})$, see Lemma 2.1. Let $b, b_1, C, B_{\frac{b_1}{2}}(0,S(b))$ and Γ be defined as in the proof of Theorem 2.2 and assume that $b<1$.

Under the above conditions, for $u \in B_{\frac{b_1}{2}}(0, S(b))$ and $s \in (0, b]$, we see that

$$\|u_s + y_s - \varphi\|_{\mathscr{B}} \leqslant \Big(\int_0^b \| u(s) \|^p \, ds\Big)^{\frac{1}{p}} + \|y_s - \varphi\|_{\mathscr{B}} \leqslant \Big(\frac{b_1}{2}\Big) b^{\frac{1}{p}} + \frac{b_1}{2} \leqslant b_1,$$

which implies that

$$u_s + y_s \in B_{b_1}(\varphi, \mathscr{B}) \quad \text{and} \quad W_f(\| u_s + y_s \|_{\mathscr{B}}) \leqslant C.$$

Now, from the Bochner's criterion for integrable and the estimate

$$\| T(t-s) f(s, u_s + y_s) \| \leqslant \frac{C_0 C m_f(s)}{(t-s)^{\alpha}},$$

we obtain that the function $s \to T(t-s) f(s, u_s + y_s)$ is integrable on $[0, t]$ for all $t \in [0, b]$. This shows that $\Gamma u \in C([-r, b], X)$ and Γ is well defined.

The remainder of the proof can be completed proceeding as in the proof of Theorem 2.1. We omit additional details. $\square$

Arguing as in the proof of Theorem 2.2, we can prove the following result.

THEOREM 2.4. *Assume the condition* (H_2) *is satisfied. Then there exists a unique mild solution of* (1.1)-(1.2) *on* $[-r, b]$ *for some* $0 < b \leqslant a$.

3. Application

In this section we consider an application of our abstract results. At first, we include some technicalities. Next, $\mathfrak{U} \subset \mathbb{R}^n$ is a open bounded set with smooth boundary $\partial\mathfrak{U}$, $\eta \in (0, 1)$ and $X = C^{\eta}(\overline{\mathfrak{U}}, \mathbb{R}^n)$. We note that $C^{\eta}(\overline{\mathfrak{U}}, \mathbb{R}^n)$ is the space formed by all the η-Hlder continuous functions from $\overline{\mathfrak{U}}$ into $\mathbb{R}^n$ endowed with the norm

$$\| \xi \|_{C^{\eta}(\overline{\mathfrak{U}}, \mathbb{R}^n)} = \| \xi \|_{C(\overline{\mathfrak{U}}, \mathbb{R}^n)} + [| \xi |]_{C^{\eta}(\overline{\mathfrak{U}}, \mathbb{R}^n)},$$

where $\| \cdot \|_{C(\overline{\mathfrak{U}}, \mathbb{R}^n)}$ is the sup-norm on $\overline{\mathfrak{U}}$,

$$[| \xi |]_{C^{\eta}(\overline{\mathfrak{U}}, \mathbb{R}^n)} = \sup_{x, y \in \overline{\mathfrak{U}}, x \neq y} \frac{| \xi(x) - \xi(y) |}{| x - y |^{\eta}}$$

and $| \cdot |$ is the Euclidean norm in $\mathbb{R}^n$.

On the space X, we consider the operator $A : D(A) \subset X \to X$ given by $Au = \Delta u$ with domain

$$D(A) = \{u \in C^{2+\eta}(\overline{\mathfrak{U}}, \mathbb{R}^n) : u_{|_{\partial\mathfrak{U}}} = 0\}.$$

From [9], we know that A is an almost sectorial operator which verifies (1.3) with $\alpha = \frac{\eta}{2}$ and A is not sectorial. In the remainder of this section, $(T(t))_{t \geqslant 0}$ represents the analytic semigroup of growth α generated by A.

Consider the delayed partial differential equation

$$\frac{\partial}{\partial t}u(t,\xi) = \Delta u(t,\xi) + \int_{t-r}^{t} \beta(s-t)u(s,\xi)ds, \qquad \forall (t,\xi) \in [0,a] \times \overline{\mathfrak{U}}, \tag{3.1}$$

$$u(t,\xi) = 0, \qquad \forall (t,\xi) \in [0,a] \times \partial\mathfrak{U}, \tag{3.2}$$

$$u(\tau,\xi) = \psi(\tau,\xi), \quad \forall (\tau,\xi) \in [-r,0] \times \overline{\mathfrak{U}}, \tag{3.3}$$

where $\psi : [-r,0] \times \overline{\mathfrak{U}} \to \mathbb{R}^n$ and $\beta : [-r,0] \to \mathbb{R}$ are suitable functions.

To represent this system in the abstract form (1.1)-(1.2), we introduce the function $f : [0,a] \times \mathscr{B} \to X$ given by

$$f(t,\psi)(\xi) = \int_{-r}^{0} \beta(s)\psi(s,\xi)ds.$$

If $\beta \in L^1([0,a],\mathbb{R})$ and $\mathscr{B} = C([-r,0],X)$, then $f \in C([0,a],\mathscr{L}(\mathscr{B},X))$ and

$$\| f(t,\cdot) \|_{\mathscr{L}(\mathscr{B},X)} \leqslant \| \beta \|_{L^1([0,a])} \text{ for all } t \in [0,a].$$

Similarly, if $\mathscr{B} = L^p([-r,0],X)$ and $\beta \in L^{p'}([0,a],\mathbb{R})$, then $f \in C([0,a],\mathscr{L}(\mathscr{B},X))$ and

$$\| f(t,\cdot) \|_{\mathscr{L}(\mathscr{B},X)} \leqslant \| \beta \|_{L^{p'}([0,a])} \text{ for all } t \in [0,a].$$

In the next results, which are consequences of Theorems 2.2 and 2.4, we said that a function $u : [-r,b] \times \overline{\mathfrak{U}} \to \mathbb{R}^n$ is a mild solution of (3.1)-(3.3) on $[-r,b]$ if the function $u : [0,b] \to X$ given by $u(t)(\xi) = u(t,\xi)$ is a mild solution of the associated abstract system (1.1)-(1.2). Next, for $\theta \in [-r,0]$, we denote by $\varphi(\theta)$ the function $\varphi(\theta) : \overline{\mathfrak{U}} \to \mathbb{R}^n$ given by $\varphi(\theta)(\xi) = \psi(\theta,\xi)$.

PROPOSITION 3.1. *Assume $\mathscr{B} = C([-r,0],X)$, the function $\theta \to \varphi(\theta)$ belongs to $C([-r,0],X)$ and $T(\cdot)\varphi(0) \in C([0,a],X)$. Then there exists a unique mild solution of* (3.1)-(3.3) *on $[-r,b]$, for some $0 < b \leqslant a$.*

PROPOSITION 3.2. *Assume $\mathscr{B} = L^p([-r,0],X)$ with $p \in (1,\frac{1}{\eta})$ and the function $\theta \to \varphi(\theta)$ belongs to $L^p([-r,0],X)$. Then there exists a unique mild solution of* (3.1)-(3.3) *on $[-r,b]$, for some $0 < b \leqslant a$.*

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