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STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS AND DISTRIBUTED DELAYS

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Abstract. We consider the scalar equation

$$\dot{x}(t)+\sum_{j=1}^{m}a_j(t)\int_0^h x(t-s)dr_j(s)=0 \ \ (h=const>0,\ \dot{x}=dx/dt),$$

where $r_j(s)$ are nondecreasing functions. Besides, we do not require that $a_j(t)$ are positive for all $t\geqslant 0$. So the function

$$z+\sum_{j=1}^{m}a_j(t)\int_0^h e^{-zs}dr_j(s)$$

can have zeros in the right-hand plane for some $t\geqslant 0$. It is proved that the considered equation is exponentially stable, provided $a_j(t)=b_j+c_j(t)$, where b_j are positive constants, such that all the zeros of the function $z+\sum_{j=1}^{m}b_j\int_0^h e^{-zs}dr_j(s)$ are in the open left-hand plane, and the integrals $\int_0^t c_j(s)ds$ $(j=1,\ldots,m)$ are sufficiently small for all $t>0$.

1. Introduction and statement of the main result

The present paper deals with the equation

$$\dot{x}(t)+\sum_{j=1}^{m}a_j(t)\int_0^h x(t-s)dr_j(s)=0 \ \ (t>0,\ h=const>0,\ \dot{x}=dx/dt), \tag{1.1}$$

where $r_j(s)$ are nondecreasing functions having finite variations $\mathrm{var}(r_j)$, and $a_j(t)$ are piece-wise continuous real functions bounded on $[0,\infty)$.

The literature on the first order linear functional differential equations is very rich, cf. [1, 5, 7, 9, 11, 12, 15] and references therein, but mainly, the coefficients in (1.1) are assumed to be positive. The papers [2, 3, 16] are devoted to stability properties of differential equations with several (not distributed) delays and an arbitrary number of positive and negative coefficients. In particular, the papers [2, 3] give us explicit stability tests in the iterative and limit forms. Besides the main tool is the comparison method based on the Bohl-Perron type theorem. The sharp stability condition for the first order functional-differential equation with one variable delay was established by A.D. Myshkis (the so called 3/2-stability theorem) in his celebrated paper [10] (see also

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[6]). The similar result was established by J. Lillo [8]. The 3/2-stability theorem was generalized to nonlinear equations and equations with unbounded delays in the papers [13, 14, 15].

In the prsesent paper *we do not require that* $a_j(t)$ *are positive for all* $t \geqslant 0$ *and therefore the function*

$$z+\sum_{j=1}^{m} a_j(t)\int_0^h e^{-zs}dr_j(s)$$

can have zeros in the right-hand plane for some $t \geqslant 0$. In addition, we improve the 3/2-stability theorem in the case of the constant delay.

Let $a_j(t) = b_j + c_j(t)$ $(j = 1,...,m)$, where b_j are positive constants, such that *all the zeros of the function*

$$K(z) := z+\sum_{j=1}^{m} b_j\int_0^h e^{-zs}dr_j(s)$$

are in the open left-hand plane, and functions $c_j(t)$ have the property

$$w_j := \sup_{t\geqslant 0}|\int_0^t c_j(t)dt| < \infty \ \ (j = 1,...,m).$$

The function

$$W(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{zt}dz}{K(z)}$$

is the fundamental solution to the equation

$$\dot{y}(t) = -\sum_{j=1}^{m} b_j\int_0^h y(t-s)dr_j(s). \tag{1.2}$$

Without loss of the generality assume that

$$\mathrm{var}(r_j) = 1 \ \ (j = 1,...,m).$$

Furthermore, for a function f defined and bounded on $[0,\infty)$ (not necessarily continuous) introduce the norm $\|f\|_C = \sup_{t\geqslant 0}|f(t)|$. So $\|a_k\|_C = \sup_{t\geqslant 0}|a_k(t)|$. In addition, put

$$\|W\|_{L^1} = \int_0^\infty |W(t)|dt.$$

Now we are in a position to formulate our main result.

THEOREM 1.1. *Let*

$$(\sum_{j=1}^{m} w_j)\left[1+\|W\|_{L^1}\sum_{k=1}^{m}(b_k+\|a_k\|_C)\right] < 1. \tag{1.3}$$

Then equation (1.1) *is exponentially stable.*

This theorem is proved in the next section. It is sharp. Namely, if $a_j(t) \equiv b_j$ $(j = 1, \ldots, m)$, then $w_j = 0$ and condition (1.3) is automatically fulfilled.

Put

$$\hat{r}(s) = \sum_{j=1}^{m} b_j r_j(s).$$

Then (1.2) takes the form

$$\dot{y}(t) = -\int_0^h y(t-s)d\hat{r}(s). \tag{1.4}$$

Besides,

$$\operatorname{var}(\hat{r}) = \sum_{j=1}^{m} b_j \operatorname{var}(r_j) = \sum_{j=1}^{m} b_j.$$

For instance, let

$$eh \sum_{j=1}^{m} b_j = eh \operatorname{var}(\hat{r}) < 1. \tag{1.5}$$

Then $W(t) \geqslant 0$ and equation (1.2) is exponentially stable, cf. [4]. Now, integrating (1.2), we have

$$\begin{aligned} 1 = W(0) &= \int_0^\infty \int_0^h W(t-s)d\hat{r}(s)\, dt \\ &= \int_0^h \int_0^\infty W(t-s)\, dt\, d\hat{r}(s) \\ &= \int_0^h \int_{-s}^\infty W(t)\, dt\, d\hat{r}(s) \\ &= \int_0^h \int_0^\infty W(t)\, dt\, d\hat{r}(s) = \operatorname{var}(\hat{r})\|W\|_{L^1}. \end{aligned}$$

So,

$$\|W\|_{L^1} = \frac{1}{\sum_{k=1}^m b_k}. \tag{1.6}$$

Thus, Theorem 1.1 implies

COROLLARY 1.2. *Let the conditions* (1.5) *and*

$$\sum_{j=1}^{m} w_j < \frac{\sum_{k=1}^m b_k}{\sum_{k=1}^m (2b_k + \|a_k\|_C)} \tag{1.7}$$

hold. Then equation (1.1) *is exponentially stable.*

Furthermore, let

$$a_j(t) = b_j + u_j(\omega_j t) \ \ (\omega_j > 0;\ j = 1, \ldots, m) \tag{1.8}$$

with a piece-wise continuous functions $u_j(t)$, such that

$$\nu_j := \sup_t \left| \int_0^t u_j(s)ds \right| < \infty.$$

Then we have $\|a_j\|_C \leqslant b_j + \|u_j\|_C$ and

$$w_j = \sup_t \left| \int_0^t u_j(\omega_j s)ds \right| = \nu_j/\omega_j.$$

For example if $u_j(t) = \sin(t)$, then $\nu_j = 2$. Now Theorem 1.1 and (1.7) imply our next result.

COROLLARY 1.3. *Let the conditions* (1.5), (1.8) *and*

$$\sum_{k=1}^{m} \frac{\nu_j}{\omega_j} < \frac{\sum_{k=1}^{m} b_k}{\sum_{k=1}^{m}(3b_k + \|u_k\|_C)} \tag{1.9}$$

hold. Then equation (1.1) *is exponentially stable.*

EXAMPLE 1.4. Consider the equation

$$\dot{x} = -\sum_{j=1}^{m}(b_j + \tau_j \sin(\omega_j t)) \int_0^1 x(t-s)d_j(s)ds \ \ (\tau_j = const > 0), \tag{1.10}$$

where $d_j(s)$ are positive and bounded on $[0,1]$ functions, satisfying the condition

$$\int_0^1 d_j(s)ds = 1.$$

Assume that (1.5) holds with $h = 1$. Then $\nu_j = 2\tau_j$ and condition (1.9) takes the form

$$\sum_{k=1}^{m} \frac{2\tau_j}{\omega_j} < \frac{\sum_{k=1}^{m} b_k}{\sum_{k=1}^{m}(3b_k + \tau_k)}.$$

So for arbitrary τ_j, there are ω_j, such that equation (1.10) is exponentially stable. In particular, consider the equation

$$\dot{x} = -(b + \tau_0 \sin(\omega t))x(t-1) \ \ (b < e^{-1}; \tau_0, \omega = const > 0). \tag{1.11}$$

Then according to condition (1.9) for any τ_0, there is an ω, such that equation (1.11) is exponentially stable. At the same time the $3/2$-stability theorem requires the condition $h(\tau_0 + b) < 3/2$ for all ω.

2. Proof of Theorem 1.1

Due to the Variation of Constants Formula the equation

$$\dot{x}(t) = -\sum_{k=1}^{m} b_j \int_0^h x(t-s)dr_j(s) + f(t) \ \ (t>0)$$

with a given function f and the zero initial condition

$$x(t) = 0 \ (t \leqslant 0)$$

is equivalent to the equation

$$x(t) = \int_0^t W(t-\tau)f(\tau)d\tau. \tag{2.1}$$

Recall that a differentiable in t function $G(t,s)$ $(t \geqslant s \geqslant 0)$ is the fundamental solution to (1.1) if it satisfies that equation in t and the initial conditions

$$G(s,s) = 1, \ G(t,s) = 0 \ (t < s, \ s \geqslant 0).$$

Put $G(t,0) = G(t)$. Subtracting (1.2) from (1.1) we have

$$\frac{d}{dt}(G(t) - W(t))$$
$$= -\sum_{k=1}^{m} b_j \int_0^h (G(t-s) - W(t-s))dr_j(s) - \sum_{j=1}^{m} c_j(t) \int_0^h G(t-s)dr_j(s). \tag{2.2}$$

Now (2.1) implies

$$G(t) = W(t) - \int_0^t W(t-\tau) \sum_{j=1}^{m} c_j(\tau) \int_0^h G(\tau-s)dr_j(s)\, d\tau. \tag{2.3}$$

We need the following simple lemma.

LEMMA 2.1. *Let $f(t), u(t)$ and $v(t)$ be scalar functions defined on a finite segment $[a,b]$ of the real axis. Assume that $f(t)$ and $v(t)$ are boundedly differentiable and $u(t)$ is integrable on $[a,b]$. Then with the notation*

$$j_u(t) = \int_a^t u(s)ds \ \ (a < t \leqslant b),$$

the equality

$$\int_a^t f(s)u(s)v(s)ds = f(t)j_u(t)v(t) - \int_a^t [f'(s)j_u(s)v(s) + f(s)j_u(s)v'(s)]ds$$

is valid.

Proof. Clearly,

$$\frac{d}{dt} f(t) j_u(t) v(t) = f'(t) j_u(t) v(t) + f(t) u(t) v(t) + f(t) j_u(t) v'(t).$$

Integrating this equality and taking into account that $j_u(a) = 0$, we arrive at the required result. □

Put

$$J_j(t) := \int_0^t c_j(s) ds.$$

By the previous lemma,

$$\int_0^t W(t-\tau) c_j(\tau) G(\tau - s) d\tau = W(0) J_j(t) G(t-s) \\ - \int_0^t \Big[\frac{dW(t-\tau)}{d\tau} J_j(\tau) G(\tau - s) + W(t-\tau) J_j(\tau) \frac{dG(\tau-s)}{d\tau}\Big] d\tau. \quad (2.4)$$

But

$$\frac{dG(\tau - s)}{d\tau} = -\sum_{k=1}^m a_k(\tau - s) \int_0^h G(\tau - s - s_1) dr_k(s_1)$$

and

$$\frac{dW(t-\tau)}{d\tau} = -\frac{dW(t-\tau)}{dt} = \sum_{j=1}^m b_j \int_0^h W(t-\tau-s_1) dr_j(s_1) \\ = \int_0^h W(t-\tau-s_1) d\hat{r}_j(s_1).$$

Thus,

$$\int_0^t W(t-\tau) c_j(\tau) G(\tau - s) d\tau = Z_j(t,s),$$

where

$$Z_j(t,s) := J_j(t) G(t-s) + \int_0^t J_j(\tau) [- \int_0^h W(t-\tau-s_1) d\hat{r}(s_1) G(\tau - s) \\ + W(t-\tau) \sum_{k=1}^m a_k(\tau - s) \int_0^h G(\tau - s - s_1) dr_k(s_1)] d\tau.$$

Now (2.3) implies

LEMMA 2.2. *The equality*

$$G(t) = W(t) - \int_0^h \sum_{j=1}^m Z_j(t,s) dr_j(s)$$

is true.

We have

$$\sup_{t\geqslant 0}|Z_j(t,s)| \leqslant w_j\|G\|_C\Big[1+\int_0^t\int_0^h |W(t-\tau-s_1)|d\hat{r}(s_1)d\tau + \sum_{k=1}^m \|a_k\|_C\int_0^t |W(t-\tau)|d\tau\Big].$$

But

$$\int_0^h\int_0^t |W(t-\tau-s)|d\tau\, d\hat{r}(s) = \int_0^h\int_{-s}^{t-s} |W(t-\tau)|d\tau\, d\hat{r}(s) \leqslant \operatorname{var}(\hat{r})\int_0^\infty |W(\tau)|d\tau.$$

Thus,

$$\|Z_j(t,s)\|_C \leqslant w_j\|G\|_C(1+\sum_{k=1}^m (b_k+\|a_k\|_C)\|W\|_{L^1}).$$

From the previous lemma we get $\|G\|_C \leqslant \|W\|_C + \gamma\|G\|_C$, where

$$\gamma := (\sum_{k=1}^m w_j)\big[1+\sum_{k=1}^m (b_k+\|a_k\|_C)\|W\|_{L^1}\big].$$

Condition (1.3) means that $\gamma < 1$. We thus have proved the following result.

LEMMA 2.3. *Let condition (1.3) hold. Then*

$$\|G\|_C \leqslant \frac{\|W\|_C}{1-\gamma}. \tag{2.5}$$

The previous lemma implies the stability of (1.1). Substituting

$$x_\varepsilon(t) = e^{-\varepsilon t}x(t) \tag{2.6}$$

with $\varepsilon > 0$ into (1.1), we have the equation

$$\dot{x}_\varepsilon(t) = \varepsilon x_\varepsilon(t) - \sum_{k=1}^m a_k(t)\int_0^h e^{\varepsilon s}x_\varepsilon(t-s)dr_k(s). \tag{2.7}$$

If $\varepsilon > 0$ is sufficiently small, then according to (2.5) we easily obtain that $\|x_\varepsilon\|_C < \infty$ for any solution x_ε of (2.7). Hence (2.6) implies $|x(t)| \leqslant e^{-\varepsilon t}\|x_\varepsilon\|_C$ for any solution x of (1.1). $\square$

3. Additional stability results

In this section we suggest a bound for the norm of $W(t)$ under the condition

$$eh\sum_{j=1}^{m} b_j < \xi \ \ (1 < \xi < 2). \tag{3.1}$$

Due to Theorem 1.1 that bound gives us explicit stability conditions, provided (3.1) holds. Consider the equation

$$d\tilde{y}(t)/dt + \sum_{j=1}^{m} \tilde{b}_j \int_0^h \tilde{y}(t-s)dr_j(s) = 0, \tag{3.2}$$

where $\tilde{b}_j = b_j/\xi$. Let $\tilde{W}$ be the fundamental solution to the equation (3.2). Subtracting (3.2) from (1.4), we obtain

$$\frac{d}{dt}(W(t)-\tilde{W}(t)) + \sum_{j=1}^{m} \tilde{b}_j \int_0^h (W(t-s)-\tilde{W}(t-s))dr_j(s)$$
$$= -\sum_{j=1}^{m} (b_j-\tilde{b}_j) \int_0^h W(t-s)dr_j(s).$$

Due to the Variation of Constants Formula,

$$W(t)-\tilde{W}(t) = -\int_0^t \tilde{W}(t-\tau) \sum_{j=1}^{m} (b_j-\tilde{b}_j) \int_0^h W(\tau-s)dr_j(s)d\tau.$$

Hence, taking into account that $\operatorname{var}(r_j)=1$, by simple calculations we get

$$\|W-\tilde{W}\|_{L^1} \leqslant \|\tilde{W}\|_{L^1}\|W\|_{L^1} \sum_{j=1}^{m} (b_j-\tilde{b}_j).$$

If

$$\psi := \|\tilde{W}\|_{L^1} \sum_{j=1}^{m} (b_j-\tilde{b}_j) < 1,$$

then $\|W\|_{L^1} \leqslant \|\tilde{W}\|_{L^1}(1-\psi)^{-1}$. But condition (3.1) implies (1.5) with $\tilde{b}_k$ instead of b_k. So according to (1.6) we have

$$\|\tilde{W}\|_{L^1} = \frac{1}{\sum_{k=1}^{m} \tilde{b}_k} = \frac{\xi}{\sum_{k=1}^{m} b_k}.$$

Consequently, $\psi = \xi - 1$ and $\|W\|_{L^1} \leqslant \|\tilde{W}\|_{L^1}(2-\xi)^{-1}$. Thus we have proved the following result.

LEMMA 3.1. *Let conditions* (3.1) *and* $\operatorname{var}(r_j)=1$ $(j=1,\dots,m)$ *hold. Then*

$$\|W\|_{L^1} \leqslant \frac{\xi}{(2-\xi)\sum_{k=1}^{m} b_k}.$$

Now we can directly apply Theorem 1.1.

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