Differential
Equations
& Applications
Volume 3, Number 1 (2011), 21–41

IMPLICIT DIFFERENCE INEQUALITIES CORRESPONDING TO PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

MILENA NETKA

(*Communicated by L. Berezansky*)

Abstract. We give theorems on implicit difference inequalities generated by initial-boundary value problems for parabolic functional differential equations. We apply this result for the investigation of the stability of difference schemes. Classical solutions of mixed problems are approximated in the paper by solutions of suitable implicit difference methods. The proofs of the convergence of difference methods are based on a comparison technique and the results on difference functional inequalities are used. Numerical examples are presented.

1. Introduction

Differential inequalities found applications in several topics concerning differential or functional differential equations. Such problems as: estimates of solutions of ordinary or partial differential or functional differential equations, estimates of the domain of the existence of classical or generalized solutions, criteria of uniqueness and continuous dependence, are classical examples, however not the only ones. Moreover, discrete versions of differential inequalities, the so called difference inequalities, are frequently used to prove the convergence of numerical methods.

Parabolic functional differential equations have the following property: difference methods for suitable initial or initial-boundary value problems consist in replacing partial derivatives with difference operators. Moreover, because differential equations contain a functional variable which is an element of the space of continuous functions defined on a finite dimensional space, we need some interpolating operators. This leads to nonlinear difference functional problems which satisfy consistency conditions on all sufficiently regular solutions of functional differential equations. The main task in these consideration is to find a finite difference approximation of an original problem which is stable. The methods of difference inequalities are used in the investigation of the stability of nonlinear difference functional equations generated by initial or initial-boundary value problems.

In recent years, a number of papers concerning implicit numerical methods for functional partial differential equations have been published. Difference approximations of classical solutions to first order partial functional differential equations were

Mathematics subject classification (2010): 35R10, 65M12.
Keywords and phrases: functional differential equations, implicit difference methods, nonlinear estimates of the Perron type.

Paper DEA-03-03

considered in [3], [5]. Initial problems on the Haar pyramid and initial-boundary value problems were considered. Implicit difference schemes for nonlinear parabolic equations with initial-boundary conditions of the Dirichlet type were studied in [1], [9]. The convergence of a class of implicit difference methods for parabolic equations with initial-boundary conditions of the Neumann type were investigated in [7], [8]. Monotone iterative methods and implicit difference schemes for computing approximate solutions to parabolic equations with time delay were analyzed in [11], [12], [19]. The stability and convergence of numerical method of lines for initial boundary value problems were investigated in [10].

The aim of the paper is to show theorems on implicit difference inequalities corresponding to parabolic functional differential equations with general initial-boundary conditions. We give also applications of theorems on implicit difference inequalities. More precisely, we propose implicit difference schemes for the numerical solving of functional differential equations. We give a complete converegence analysis for the methods and we show by examples that new difference schemes are considerably better than classical methods.

Results presented in the paper are new also in the case of differential equations without the functional dependence.

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of parabolic functional differential problems can be found in [2], [6], [13], [14], [17], [18]. The monograph [20] gives an exposition of the theory of parabolic functional differential equations. We use in the paper general ideas for finite difference equations which were introduced in [15], [16].

We formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E_0 = [-b_0, 0] \times [-b, b] \text{ and } E = [0, a] \times [-b, b],$$

where $a > 0$, $b_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = (0, +\infty)$ and $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $i = 1, \ldots, n$. Set $r_0 = b_0 + a$, $r = 2b$ and $B = [-r_0, 0] \times [-r, r], \Sigma = [-r_0, a] \times [-b - r, b + r]$. For a function $z : \Sigma \to \mathbb{R}$ and for a point $(t,x) \in E$ we define a function $z_{(t,x)} : B \to \mathbb{R}$ by $z_{(t,x)}(\tau, y) = z(t + \tau, x + y)$, $(\tau, y) \in B$. For $(t,x) \in E$ we put

$$D[t,x] = \{ (\tau, y) \in \mathbb{R}^{1+n} : \tau \leqslant 0, \ (t + \tau, x + y) \in E_0 \cup E \}.$$

It is clear that $D[t,x] = [-b_0 - t, 0] \times [-b - x, b - x]$ and $D[t,x] \subset B$ for $(t,x) \in E$. Let $M_{n \times n}$ be the class of all $n \times n$ matrices with real elements. Write $\Xi = E \times C(B, \mathbb{R}) \times \mathbb{R}^n \times M_{n \times n}$ and suppose that $F : \Xi \to \mathbb{R}$ is a given function. We will say that F satisfies the condition (V) if for each $(t, x, w, q, s) \in \Xi$ and $\tilde{w} \in C(B, \mathbb{R})$ such that $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in D[t,x]$ we have $F(t, x, w, q, s) = F(t, x, \tilde{w}, q, s)$. Note that the condition (V) means that the value of F at the point $(t, x, w, q, s) \in \Xi$ depends on (t, x, q, s) and on the restriction of w to the set $D[t,x]$ only. Let us denote by z an unknown function of the variables (t,x), $x = (x_1, \ldots, x_n)$. We consider the functional differential equation

$$\partial_t z(t,x) = F\big(t,x,z_{(t,x)},\partial_x z(t,x),\partial_{xx} z(t,x)\big), \tag{1.1}$$

where $\partial_x z = (\partial_{x_1} z,\ldots,\partial_{x_n} z)$, $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$. We assume that F satisfies the condition (V) and we consider classical solutions of (1.1). Now we formulate initial-boundary conditions for (1.1). Write

$$S_i = \{x\in[-b,b] : x_i = b_i\},\ \ S_{n+i} = \{x\in[-b,b] : x_i = -b_i\},\ i=1,\ldots,n$$
$$Q_1^+ = S_1,\quad Q_i^+ = S_i\setminus \cup_{j=1}^{i-1} S_j,\quad Q_i^- = S_{n+i}\setminus\cup_{j=1}^{n+i-1} S_j,\quad i=1,\ldots,n.$$

Set

$$\begin{aligned}&\partial_0 E_i^+ = [0,a)\times Q_i^+,\\ &\partial_0 E_i^- = [0,a)\times Q_i^-,\ i=1,\ldots,n,\\ &\partial_0 E = \cup_{i=1}^n(\partial_0 E_i^+\cup\partial_0 E_i^-).\end{aligned}$$

Suppose that $\beta,\ \gamma,\ \Psi : \partial_0 E\to\mathbb{R},\ \psi : E_0\to\mathbb{R}$ are given functions. The following initial-boundary conditions are associated with (1.1):

$$z(t,x) = \psi(t,x)\quad \text{on } E_0, \tag{1.2}$$
$$\beta(t,x)z(t,x)+\gamma(t,x)\partial_{x_i}z(t,x) = \Psi(t,x)\quad \text{on } \partial_0 E_i^+,\ i=1,\ldots,n, \tag{1.3}$$
$$\beta(t,x)z(t,x)-\gamma(t,x)\partial_{x_i}z(t,x) = \Psi(t,x)\quad \text{on } \partial_0 E_i^-,\ i=1,\ldots,n. \tag{1.4}$$

A function $z : E_0\cup E\to\mathbb{R}$ will be called the function of class C_* if z is continuous on $E_0\cup E$, the partial derivatives $\partial_t z$, $\partial_x z = (\partial_{x_1} z,\ldots,\partial_{x_n} z)$, $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$ exist on E and the functions $\partial_t z,\partial_x z,\partial_{xx} z$ are continuous on E. We consider solutions of (1.1)-(1.4) of class C_*.

For spaces X and Y we denote by $\mathscr{F}(X,Y)$ the class of all functions defined on X and taking values in Y. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. We define a mesh in $\mathbb{R}^{1+n}$ in the following way. Let $\mathbf{h} = (h_0,h)$, $h = (h_1,\ldots,h_n)$, stand for steps of the mesh. For $(r,m)\in\mathbb{Z}^{1+n}$, $m = (m_1,\ldots,m_n)$ we define nodal points as follows

$$t^{(r)} = rh_0,\quad x^{(m)} = (m_1h_1,\ldots,m_nh_n) = \big(x_1^{(m_1)},\ldots,x_n^{(m_n)}\big).$$

Let us denote by $\mathbb{H}$ the set of all $\mathbf{h}$ for which there exist $(M_1,\ldots,M_n) = M\in\mathbb{Z}^n$ and $M_0\in\mathbb{Z}$ such that $M_ih_i = b_i$ for $i=1,\ldots,n$, $M_0h_0 = b_0$. For $\mathbf{h}\in\mathbb{H}$ we put $\|\mathbf{h}\| = h_0+h_1+\ldots+h_n$. Let $K\in\mathbb{N}$ be defined by relations $Kh_0\leqslant a<(K+1)h_0$. For $\mathbf{h}\in\mathbb{H}$ we put

$$\mathbb{R}^{1+n}_{\mathbf{h}} = \{(t^{(r)},x^{(m)}) : (r,m)\in\mathbb{Z}^{1+n}\},$$

and

$$E_{0.\mathbf{h}} = E_0\cap\mathbb{R}^{1+n}_{\mathbf{h}},\ \ E_{\mathbf{h}} = E\cap\mathbb{R}^{1+n}_{\mathbf{h}},\ \ B_{\mathbf{h}} = B\cap\mathbb{R}^{1+n}_{\mathbf{h}},$$
$$\partial_0 E^+_{\mathbf{h}.i} = \partial_0 E_i^+\cap\mathbb{R}^{1+n}_{\mathbf{h}},\ \partial_0 E^-_{\mathbf{h}.i} = \partial_0 E_i^+\cap\mathbb{R}^{1+n}_{\mathbf{h}}.$$

Difference operators are defined in the following way. Let $e_i \in \mathbb{R}^n$ defined by $e_i = (0,\dots,0,1,0,\dots,0)$ with 1 standing on the $i-$th place. Write

$$J = \{(i,j):\ i,j = 1,\dots,n,\ i \neq j\}.$$

Suppose that we have defined the sets $J_+,\ J_- \subset J$ such that $J_+ \cup J_- = J,\ J_+ \cap J_- = \emptyset$. We assume that $(i,j) \in J_+$ if $(j,i) \in J_+$. In particular, it may happen that $J_+ = \emptyset$ or $J_- = \emptyset$. Relations between equation (1.1) and the sets J_+, J_- are given in Remark 2.1.

Given $z \in \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$ and $(r,m) \in \mathbb{Z}^{1+n}$, $0 \leqslant r \leqslant K-1$, $-(M-1) \leqslant m \leqslant M-1$, where $M-1 = (M_1-1,\dots,M_n-1)$. Write

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\left[z^{(r+1,m)} - z^{(r,m)}\right], \tag{1.5}$$

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i}[z^{(r,m+e_i)} - z^{(r,m)}],\ i = 1,\dots,n,$$

$$\delta_i^- z^{(r,m)} = \frac{1}{h_i}[z^{(r,m)} - z^{(r,m-e_i)}],\ i = 1,\dots,n,$$

and $\delta z^{(r,m)} = (\delta_1 z^{(r,m)},\dots,\delta_n z^{(r,m)})$ where

$$\delta_i z^{(r,m)} = \frac{1}{2}[\delta_i^+ z^{(r,m)} + \delta_i^- z^{(r,m)}],\ i = 1,\dots,n. \tag{1.6}$$

The difference operator $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\dots,n}$, is defined in the following way:

$$\delta_{ii}^{(2)} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)} \quad \text{for } i = 1,\dots,n \tag{1.7}$$

and

$$\delta_{ij}^{(2)} z^{(r,m)} = \frac{1}{2}\left[\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)}\right] \quad \text{for } (i,j) \in J_-, \tag{1.8}$$

$$\delta_{ij}^{(2)} z^{(r,m)} = \frac{1}{2}\left[\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)}\right] \quad \text{for } (i,j) \in J_+. \tag{1.9}$$

Solutions of difference functional equations are elements of the space $\mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$. Since equation (1.1) contains the functional variable $z_{(t,x)}$ which is an element of the space $C(D[t,x],\mathbb{R})$, we need an interpolating operator $T_{\mathbf{h}} : \mathcal{F}(B_{\mathbf{h}},\mathbb{R}) \to C(B,\mathbb{R})$. We adopt additional assumption on $T_{\mathbf{h}}$ in Section 3. For a function $z : \Sigma_{\mathbf{h}} \to \mathbb{R}$ and for a point $(t^{(r)}, x^{(m)}) \in E_{\mathbf{h}}$ we define a function $z_{[r,m]} : B_{\mathbf{h}} \to \mathbb{R}$ by

$$z_{[r,m]}(\tau,y) = z(t^{(r)} + \tau, x^{(m)} + y),\ \ (\tau,y) \in B_{\mathbf{h}}.$$

Set

$$F_{\mathbf{h}}[z]^{(r,m)} = F\left(t^{(r)}, x^{(m)}, T_{\mathbf{h}} z_{[r,m]}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}\right)$$

and

$$\Lambda_{\mathbf{h}.i}^+[z]^{(r,m)} = \beta^{(r,m)} z^{(r,m)} + \gamma^{(r,m)} \delta_i^- z^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}.i}^+,$$

$$\Lambda_{\mathbf{h}.i}^-[z]^{(r,m)} = \beta^{(r,m)} z^{(r,m)} - \gamma^{(r,m)} \delta_i^+ z^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}.i}^-,$$

where $i = 1, \cdots, n$. Write

$$\partial_0 E^+_{\mathbf{h}} = \cup_{i=1}^n \partial_0 E^+_{\mathbf{h}.i}, \quad \partial_0 E^-_{\mathbf{h}} = \cup_{i=1}^n \partial_0 E^-_{\mathbf{h}.i}, \quad \partial_0 E_{\mathbf{h}} = \partial_0 E^+_{\mathbf{h}} \cup \partial_0 E^-_{\mathbf{h}}.$$

For a function $z : E_{0.\mathbf{h}} \cup E_{\mathbf{h}} \to \mathbb{R}$ we define a function $\Lambda_{\mathbf{h}}[z] : \partial_0 E_{\mathbf{h}} \to \mathbb{R}$ in the following way:

$$\Lambda_{\mathbf{h}}[z]^{(r,m)} = \Lambda^+_{\mathbf{h}.i}[z]^{(r,m)} \quad \text{if} \quad (t^{(r)}, x^{(m)}) \in \partial_0 E^+_{\mathbf{h}.i},$$
$$\Lambda_{\mathbf{h}}[z]^{(r,m)} = \Lambda^-_{\mathbf{h}}[z]^{(r,m)} \quad \text{if} \quad (t^{(r)}, x^{(m)}) \in \partial_0 E^-_{\mathbf{h}.i}.$$

Suppose that $\psi_{\mathbf{h}} : E_{0.\mathbf{h}} \to \mathbb{R}$ and $\Psi_{\mathbf{h}} : \partial_0 E_{\mathbf{h}} \to \mathbb{R}$ are given functions. We consider the difference functional equation

$$\delta_0 z^{(r,m)} = F_{\mathbf{h}}[z]^{(r,m)} \tag{1.10}$$

with the initial-boundary conditions:

$$z^{(r,m)} = \psi^{(r,m)}_{\mathbf{h}} \quad \text{on } E_{0.\mathbf{h}}, \tag{1.11}$$
$$\Lambda_{\mathbf{h}}[z]^{(r,m)} = \Psi^{(r,m)}_{\mathbf{h}} \quad \text{on } \partial_0 E_{\mathbf{h}}. \tag{1.12}$$

REMARK 1.1. Note that the values $z^{(r+1,m+\lambda)}$ appear in the expressions $\delta z^{(r+1,m)}$ and $\delta^{(2)} z^{(r+1,m)}$, where $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \{-1, 0, 1\}$, $i = 1, \dots, n$ and $\|\lambda\| \leqslant 2$. Then (1.10)-(1.12) is an implicit difference scheme for (1.1)-(1.4).

Our motivations for the construction of implicit difference schemes are the following. Two type assumptions are needed in theorems on the stability of difference schemes corresponding to (1.1)-(1.4). The first type conditions concern regularity of F. It is assumed that the function F of the variables (t, x, w, q, s), $q = (q_1, \dots, q_n)$, $s = [s_{ij}]_{i,j=1,\dots,n}$, is of class C^1 with respect to (q, s) and the functions:

$$\partial_q F = (\partial_{q_1} F, \dots, \partial_{q_n} F) \text{ and } \partial_s F = [\partial_{s_{ij}} F]_{i,j=1,\dots,n}$$

are bounded. It is assumed also that F satisfies the Perron type estimate with respect to the functional variable w.

The second type conditions concern the mesh. It is required that

$$1 - 2h_0 \sum_{i=1}^n \frac{1}{h_i^2} \partial_{s_{ii}} F(P) + h_0 \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{h_i h_j} \left| \partial_{s_{ij}} F(P) \right| \geqslant 0, \quad P = (t, x, w, q, s) \in \Xi. \tag{1.13}$$

It is clear that strong assumptions on relations between h_0 and $h = (h_1, \dots, h_n)$ are required in (1.13). It is important in our considerations that assumption (1.13) is omitted in theorems on difference functional inequalities and in theorems on the convergence of implicit difference methods for (1.1)-(1.4).

2. Difference functional inequalities

The following assumption will be needed throughout the paper.

ASSUMPTION **H[** F **]**. The function $F:\Xi\to\mathbb{R}$ of the variables (t,x,w,q,s), $q=(q_1,\ldots,q_n)$, $s=[s_{ij}]_{i,j=1,\ldots,n}$ satisfies the condition (V) and:

1) $F\in C(\Xi,\mathbb{R})$ and the derivatives $\partial_q F=(\partial_{q_1}F,\ldots,\partial_{q_n}F)$, $\partial_s F=[\partial_{s_{ij}}F]_{i,j=1,\ldots,n}$ exist and the functions $\partial_q F:\Xi\to\mathbb{R}^n$, $\partial_s F:\Xi\to M_{n\times n}$ are continuous and bounded,

2) the matrix $\partial_s F$ is symmetric and

$$-\frac{1}{2}\left|\partial_{q_i}F(P)\right|+\frac{1}{h_i}\partial_{s_{ii}}F(P)-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{h_j}\left|\partial_{s_{ij}}F(P)\right|\geqslant 0,\quad i=1,\ldots,n, \tag{2.1}$$

and

$$\partial_{s_{ij}}F(P)\geqslant 0 \text{ for } (i,j)\in J_+,\quad \partial_{s_{ij}}F(P)\leqslant 0 \text{ for } (i,j)\in J_-, \tag{2.2}$$

where $P\in\Xi$,

3) $h\in\mathbb{H}$ and there is $\varepsilon_0>0$ such that for $0<h_0<\varepsilon_0$ and $w,\bar{w}\in\mathscr{F}(B_{\mathbf{h}},\mathbb{R})$ if $w(\tau,y)\leqslant\bar{w}(\tau,y)$ for $(\tau,y)\in B_{\mathbf{h}}$, then

$$w^{(0,\theta)}+h_0F(t,x,T_{\mathbf{h}}w,q,s)\leqslant\bar{w}^{(0,\theta)}+h_0F(t,x,T_{\mathbf{h}}\bar{w},q,s),$$

4) $\beta:\partial_0E_{\mathbf{h}}\to(0,\infty)$, $\gamma:\partial_0E_{\mathbf{h}}\to\mathbb{R}_+$ and $\mathbf{h}\in\mathbb{H}$, $h_0\leqslant\varepsilon_0$, $T_{\mathbf{h}}:\mathscr{F}(B_{\mathbf{h}},\mathbb{R})\to C(B,\mathbb{R})$.

REMARK 2.1. We have assumed that for each $(i,j)\in J$, the functions

$$g_{ij}(P)=\operatorname{sign}\partial_{s_{ij}}F(P),\ P\in\Xi,$$

are constant. Relations (2.2) can be considered as definitions of the sets J_+ and J_-.

REMARK 2.2. Suppose that $f:E\times\mathbb{R}\times C(B,\mathbb{R})\times\mathbb{R}^n\times M_{n\times n}\to\mathbb{R}$ is a given function of the variables (t,x,p,w,q,s) and $F(t,x,w,q,s)=f(t,x,w(0,\theta),w,q,s)$ on Ξ, where $\theta=(0,\ldots,0)\in\mathbb{R}^n$. Suppose that:

1) f is nondecreasing with respect to the functional variable w,

2) there exist the derivative $\partial_p f$ and the function $\partial_p f$ is continuous and bounded.

Then F satisfies condition 3) of Assumption **H[** F **]**.

We prove a theorem on difference inequalities generated by problem (1.10)-(1.12).

THEOREM 2.1. *Suppose that Assumption **H[** F **]** is satisfied and*

1) *the functions $u,v:E_{0.\boldsymbol{h}}\cup E_{\boldsymbol{h}}\to\mathbb{R}$ satisfy the differential difference inequality*

$$\delta_0u^{(r,m)}-F_{\boldsymbol{h}}[u]^{(r,m)}\leqslant\delta_0v^{(r,m)}-F_{\boldsymbol{h}}[v]^{(r,m)}\quad \text{on } E_{\boldsymbol{h}}, \tag{2.3}$$

2) *the initial estimate* $u^{(r,m)} \leqslant v^{(r,m)}$ *on* $E_{0.\boldsymbol{h}}$ *and boundary inequalities* $\Lambda_{\boldsymbol{h}}[u]^{(r,m)} \leqslant \Lambda_{\boldsymbol{h}}[v]^{(r,m)}$ *on* $\partial_0 E_{\boldsymbol{h}}$ *are satisfied.*

Then

$$u^{(r,m)} \leqslant v^{(r,m)} \quad \text{on } E_{\boldsymbol{h}}. \tag{2.4}$$

Proof. Notice that for $r=0$ inequality (2.4) is satisfied. Suppose that

$$u^{(i,m)} \leqslant v^{(i,m)} \text{ for } (t^{(i)}, x^{(m)}) \in (E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n).$$

We prove that $u^{(r+1,m)} \leqslant v^{(r+1,m)}$ for $-M \leqslant m \leqslant M$. Suppose by contradiction that above inequality fails to be true. Let $\mu = (\mu_1, \ldots, \mu_n)$ be defined by relation

$$(u-v)^{(r+1,\mu)} = \max\left\{(u-v)^{(r+1,m)} :\ -M \leqslant m \leqslant M\right\}. \tag{2.5}$$

We thus get

$$(u-v)^{(r+1,\mu)} > 0. \tag{2.6}$$

Then two possibilities can happen,

$$\text{either } (i):\ \left(t^{(r+1)}, x^{(\mu)}\right) \in \partial_0 E_{\mathbf{h}} \quad \text{or} \quad (ii):\ \left(t^{(r+1)}, x^{(\mu)}\right) \in E_{\mathbf{h}} \setminus \partial_0 E_{\mathbf{h}}.$$

Let us consider the first case. Then there is $i \in \{1, \ldots, n\}$ such that $x_i^{(\mu_i)} = b_i$ or $x_i^{(\mu_i)} = -b_i$. If $x_i^{(\mu_i)} = b_i$ (the other case can be treated similary) then $\delta_i^-(u-v)^{(r+1,\mu)} \geqslant 0$. But from assumption 2) it follows that

$$\beta^{(r+1,\mu)}(u-v)^{(r+1,\mu)} + \gamma^{(r+1,\mu)}\delta_i^-(u-v)^{(r+1,\mu)} \leqslant 0.$$

Hence $\beta^{(r+1,\mu)}(u-v)^{(r+1,\mu)} \leqslant 0$ which contradicts condition (2.6). Hence

$$(t^{(r+1)}, x^{(\mu)}) \in E_{\mathbf{h}} \setminus \partial_0 E_{\mathbf{h}}.$$

Write

$$A^{(r,\mu)} = (u-v)^{(r,\mu)} + h_0\Big[F_{\mathbf{h}}[u]^{(r,m)} - F\big(t^{(r)}, x^{(\mu)}, T_{\mathbf{h}} v_{[r,\mu]}, \delta u^{(r+1,\mu)}, \delta^{(2)} u^{(r+1,\mu)}\big)\Big],$$
$$B^{(r,\mu)} = h_0\Big[F\big(t^{(r)}, x^{(\mu)}, T_{\mathbf{h}} v_{[r,\mu]}, \delta u^{(r+1,\mu)}, \delta^{(2)} u^{(r+1,\mu)}\big) - F_{\mathbf{h}}[v]^{(r,m)}\Big].$$

It follows form (2.3) that

$$(u-v)^{(r+1,\mu)} \leqslant A^{(r,\mu)} + B^{(r,\mu)}. \tag{2.7}$$

Set

$$Q^{(r,m)}(\tau) = \Big(t^{(r)}, x^{(m)}, T_{\mathbf{h}} v_{[t,m]}, \delta v^{(r+1,m)} + \tau\delta(u-v)^{(r+1,m)},$$
$$\delta^{(2)} v^{(r+1,m)} + \tau\delta^{(2)}(u-v)^{(r+1,m)}\Big),$$

where $0 \leqslant \tau \leqslant 1$. Write

$$\tilde{S}_i^{(r,m)} = \frac{h_0}{h_i^2} \int_0^1 \partial_{s_{ii}} F(Q^{(r,m)}(\tau)) d\tau - h_0 \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{h_j h_i} \int_0^1 \left| \partial_{s_{ij}} F(Q^{(r,m)}(\tau)) \right| d\tau$$

and

$$\begin{aligned} S_0^{(r,m)} = -2h_0 \sum_{j=1}^n \frac{1}{h_j^2} \int_0^1 \partial_{s_{jj}} F(Q^{(r,m)}(\tau)) d\tau & \\ + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} \int_0^1 \left| \partial_{s_{ij}} F(Q^{(r,m)}(\tau)) \right| d\tau, & \qquad (2.8) \end{aligned}$$

$$S_{i.+}^{(r,m)} = \frac{h_0}{2h_i} \int_0^1 \partial_{q_i} F(Q^{(r,m)}(\tau)) d\tau + \tilde{S}_i^{(r,m)}, \tag{2.9}$$

$$S_{i.-}^{(r,m)} = -\frac{h_0}{2h_i} \int_0^1 \partial_{q_i} F(Q^{(r,m)}(\tau)) d\tau + \tilde{S}_i^{(r,m)}, \tag{2.10}$$

where $i = 1, \ldots, n$ and

$$S_{ij}^{(r,m)} = \frac{h_0}{2h_i h_j} \int_0^1 \partial_{s_{ij}} F(Q^{(r,m)}(\tau)) d\tau, \quad (i,j) \in J. \tag{2.11}$$

It follows from Assumption **H[**F**]** that:

$$S_{i.+}^{(r,m)} \geqslant 0 \quad S_{i.-}^{(r,m)} \geqslant 0 \quad \text{for } i = 1, \ldots, n, \tag{2.12}$$

$$S_{ij}^{(r,m)} \geqslant 0 \quad \text{for } (i,j) \in J_+, \quad S_{ij}^{(r,m)} \leqslant 0 \quad \text{for } (i,j) \in J_-, \tag{2.13}$$

$$S_0^{(r,m)} + \sum_{i=1}^n S_{i.+}^{(r,m)} + \sum_{i=1}^n S_{i.-}^{(r,m)} + 2 \sum_{(i,j) \in J} \left| S_{ij}^{(r,m)} \right| = 0. \tag{2.14}$$

We conclude from Hadamard mean value theorem that

$$\begin{aligned} B^{(r,\mu)} = S_0^{(r,\mu)} (u-v)^{(r+1,\mu)} & \\ + \sum_{i=1}^n S_{i.+}^{(r,\mu)} (u-v)^{(r+1,\mu+e_i)} + \sum_{i=1}^n S_{i.-}^{(r,\mu)} (u-v)^{r+1,\mu-e_i)} & \\ + \sum_{(i,j) \in J_+} S_{ij}^{(r,\mu)} \left[(u-v)^{(r+1,\mu+e_i+e_j)} + (u-v)^{(r+1,\mu-e_i-e_j)} \right] & \\ - \sum_{(i,j) \in J_-} S_{ij}^{(r,\mu)} \left[(u-v)^{(r+1,\mu+e_i-e_j)} + (u-v)^{(r+1,\mu-e_i+e_j)} \right]. & \end{aligned}$$

This gives $B^{(r,\mu)} \leqslant 0$. The functions $w = u_{[r,m]}$, $\bar{w} = v_{[r,m]}$ satisfy the condition $w(\tau,y) \leqslant \bar{w}(\tau,y)$ for $(\tau,y) \in B_{\mathbf{h}}$. We conclude from condition 3) of Assumption **H[**F**]** that $A^{(r,\mu)} \leqslant 0$. According to (2.7) we have $(u-v)^{(r+1,\mu)} \leqslant 0$, which contradicts (2.6). Hence inequality (2.4) is proved on $E_{\mathbf{h}}$.

Put $\Xi_0 = E \times C(B,\mathbb{R}) \times \mathbb{R}^n$ and suppose that

$$f : E \to M_{n\times n},\ f = [f_{ij}]_{i,j=1,\dots,n}, \quad G : \Xi_0 \to \mathbb{R}$$

are given functions. We consider the functional differential equation

$$\partial_t z(t,x) = \sum_{i,j=1}^{n} f_{ij}(t,x)\partial_{x_i x_j} z(t,x) + G(t,x,z_{(t,x)},\partial_x z(t,x)) \tag{2.15}$$

with initial-boundary conditions (1.2)-(1.4). Let $\delta_0,\ \delta,\ \delta^{(2)}$ be the difference operators defined by (1.5)-(1.9) respectively. If we apply Theorem 2.1 to the difference equation

$$\delta_0 z^{(r,m)} = \sum_{i,j=1}^{n} f_{ij}^{(r,m)} \delta_{ij} z^{(r+1,m)} + G\big(t^{(r)},x^{(m)},T_{\mathbf{h}} z_{[r,m]},\delta z^{(r+1,m)}\big), \tag{2.16}$$

where $f_{ij}^{(r,m)} = f_{ij}(t^{(r)},x^{(m)}),\ 1 \leqslant i,j \leqslant n,$ then we need the following assumption on f: for each $(i,j) \in J$ the functions $\tilde{f}_{ij}(t,x) = \operatorname{sign} f_{ij}(t,x),\ (t,x) \in E,$ are constant on E, see assumption (2.2). We prove that this condition can be omitted if we modify the definitions of $\delta_{ij} z^{(r+1,m)}$ for $(i,j) \in J$. More precisely, we consider problem (2.16), (1.11), (1.12) with $\delta_0,\ \delta,\ \delta_{ii},\ 1 \leqslant i \leqslant n,$ given by (1.5)-(1.7) and we define $\delta_{ij} z$ for $(i,j) \in J$ in the following way

$$\begin{aligned}
&\text{if} \quad f_{ij}^{(r,m)} < 0, \quad \text{then} \quad \delta_{ij} z^{(r+1,m)} = \frac{1}{2}\big[\delta_i^+ \delta_j^- z^{(r+1,m)} + \delta_i^- \delta_j^+ z^{(r+1,m)}\big],\\
&\text{if} \quad f_{ij}^{(r,m)} \geqslant 0, \quad \text{then} \quad \delta_{ij} z^{(r+1,m)} = \frac{1}{2}\big[\delta_i^+ \delta_j^+ z^{(r+1,m)} + \delta_i^- \delta_j^- z^{(r+1,m)}\big].
\end{aligned}$$

Set

$$G_{\mathbf{h}}[z]^{(r,m)} = \sum_{i,j=1}^{n} f_{ij}^{(r,m)} \delta_{ij} z^{(r+1,m)} + G\big(t^{(r)},x^{(m)},T_{\mathbf{h}} z_{[r,m]},\delta z^{(r+1,m)}\big).$$

We consider the difference functional equation corresponding to (2.15)

$$\delta_0 z^{(r,m)} = G_{\mathbf{h}}[z]^{(r,m)} \tag{2.17}$$

with the initial-boundary conditions (1.11)-(1.12).

ASSUMPTION **H[f,G]** The functions $f : E \to M_{n\times n},\ G : \Xi_0 \to \mathbb{R}^n$ are continuous and:

1) G satisfies the condition (V) and there exist the derivatives $\partial_q G = (\partial_{q_1} G,\dots,\partial_{q_n} G)$ and the function $\partial_q G : \Xi_0 \to \mathbb{R}^n$ is continuous and bounded,

2) the matrix f is symmetric and

$$-\frac{1}{2}\big|\partial_{q_i} G(P)\big| + \frac{1}{h_i} f_{ii}(t,x) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_j}\big|f_{ij}(t,x)\big| \geqslant 0, \quad i = 1,\dots,n,$$

where $P = (t,x,w,q) \in \Xi_0$,

3) $\mathbf{h} \in \mathbb{H}$ and there is $\varepsilon_0 > 0$ such that for $0 < h_0 < \varepsilon_0$ and $w, \bar{w} \in \mathscr{F}(B_{\mathbf{h}}, \mathbb{R})$ if $w(\tau, y) \leqslant \bar{w}(\tau, y)$ for $(\tau, y) \in B_{\mathbf{h}}$, then

$$w^{(0,\theta)} + h_0 G(t, x, T_{\mathbf{h}} w, q) \leqslant \bar{w}^{(0,\theta)} + h_0 G(t, x, T_{\mathbf{h}} \bar{w}, q),$$

4) $\beta : \partial_0 E_{\mathbf{h}} \to (0, \infty)$, $\gamma : \partial_0 E_{\mathbf{h}} \to \mathbb{R}_+$ and $T_{\mathbf{h}} : \mathscr{F}(B_{\mathbf{h}}, \mathbb{R}) \to C(B, \mathbb{R})$.

THEOREM 2.2. *Suppose that Assumption* ***H[f, G]*** *is satisfied and*:

1) *the functions $u, v : E_{0.\boldsymbol{h}} \cup E_{\boldsymbol{h}} \to \mathbb{R}$ satisfy the differential difference inequality*

$$\delta_0 u^{(r,m)} - G_{\boldsymbol{h}}[u]^{(r,m)} \leqslant \delta_0 v^{(r,m)} - G_{\boldsymbol{h}}[v]^{(r,m)} \quad \text{on } E_{\boldsymbol{h}},$$

2) *the initial estimate $u^{(r,m)} \leqslant v^{(r,m)}$ on $E_{0.\boldsymbol{h}}$ and boundary inequalities $\Lambda_{\boldsymbol{h}}[u]^{(r,m)} \leqslant \Lambda_{\boldsymbol{h}}[v]^{(r,m)}$ on $\partial_0 E_{\boldsymbol{h}}$ are satisfied.*

Then

$$u^{(r,m)} \leqslant v^{(r,m)} \quad \text{on } E_{\boldsymbol{h}}. \tag{2.18}$$

Proof. It is easy to see that for $r = 0$ inequality (2.18) is satisfied. Suppose that

$$u^{(i,m)} \leqslant v^{(i,m)} \text{ for } (t^{(i)}, x^{(m)}) \in (E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n).$$

We prove that

$$u^{(r+1,m)} \leqslant v^{(r+1,m)} \quad \text{for } -M \leqslant m \leqslant M.$$

Suppose by contradiction that above inequality fails to be true. Let μ be defined by relation (2.5). Then condition (2.6) is satisfied. It follows easily that $(t^{(r+1)}, x^{(\mu)}) \in E_{\mathbf{h}} \setminus \partial_0 E_{\mathbf{h}}$. We conclude from assumption 2) that

$$(u - v)^{(r+1,\mu)} \leqslant (u - v)^{(r,\mu)} + h_0 \big[G_{\mathbf{h}}[u]^{(r,\mu)} - G_{\mathbf{h}}[v]^{(r,\mu)} \big].$$

It follows form condition 3) of Assumption **H[f, G]** and from the above inequality that

$$\begin{aligned}
&(u - v)^{(r+1,\mu)} \\
&\quad \leqslant h_0 \sum_{i,j=1}^{n} f_{ij}^{(r,\mu)} \delta_{ij} (u - v)^{(r+1,\mu)} \\
&\qquad + h_0 \Big[G\big(t^{(r)}, x^{(\mu)}, (T_{\mathbf{h}} v)_{[r,\mu]}, \delta u^{(r+1,\mu)}\big) - G\big(t^{(r)}, x^{(\mu)}, (T_{\mathbf{h}} v)_{[r,\mu]}, \delta v^{(r+1,\mu)}\big) \Big].
\end{aligned}$$

Write

$$J_+^{(r,\mu)} = \big\{ (i, j) \in J : f_{ij}^{(r,\mu)} \geqslant 0 \big\}, \;\; J_-^{(r,\mu)} = J \setminus J_+^{(r,\mu)},$$

and

$$S_0^{(r,\mu)} = -2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} f_{ii}^{(r,\mu)} + \sum_{(i,j) \in J} \frac{1}{h_i h_j} \big| f_{ij}^{(r,\mu)} \big|$$

$$S_{i.+}^{(r,\mu)} = \frac{1}{h_i^2} f_{ii}^{(r,\mu)} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{h_j h_i} \big| f_{ij}^{(r,\mu)} \big| + \frac{1}{2h_i} \int_0^1 \partial_{q_i} G(P^{(r,\mu)}(\tau)) d\tau$$

$$S_{i.-}^{(r,\mu)} = \frac{1}{h_i^2} f_{ii}^{(r,\mu)} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{h_j h_i} \big| f_{ij}^{(r,\mu)} \big| - \frac{1}{2h_i} \int_0^1 \partial_{q_i} G(P^{(r,\mu)}(\tau)) d\tau$$

where $i = 1, \ldots, n$ and

$$P^{(r,\mu)}(\tau) = \big(t^{(r)}, x^{(\mu)}, (T_{\mathbf{h}} v)_{[r,\mu]}, \delta v^{(r+1,m)} + \tau \delta (u-v)^{(r+1,m)}\big).$$

We conclude from the Hadamard mean value theorem that

$$\begin{aligned}
&(u-v)^{(r+1,\mu)}(1 - S_0^{(r,\mu)}) \\
&\quad \leqslant h_0 \Big[\sum_{i=1}^{n} S_{i.+}^{(r,\mu)} (u-v)^{(r+1,\mu+e_i)} + \sum_{i=1}^{n} S_{i.-}^{(r,\mu)} (u-v)^{(r+1,\mu-e_i)} \\
&\qquad + h_0 \sum_{(i,j)\in J_+^{(r,\mu)}} \frac{1}{2h_j h_i} f_{ij}^{(r,\mu)} \big[(u-v)^{(r+1,\mu+e_i+e_j)} + (u-v)^{(r+1,\mu-e_i-e_j)} \big] \\
&\qquad - \sum_{(i,j)\in J_-^{(r,\mu)}} \frac{1}{2h_j h_i} f_{ij}^{(r,\mu)} \big[(u-v)^{(r+1,\mu+e_i-e_j)} + (u-v)^{(r+1,\mu-e_i+e_j)} \big] \Big].
\end{aligned} \tag{2.19}$$

It is easily seen that $S_{i.+}^{(r,\mu)},\ S_{i.-}^{(r,\mu)} \geqslant 0$, for $i = 1, \ldots, n$ and

$$S_0^{(r,\mu)} + h_0 \sum_{i=1}^{n} S_{i.+}^{(r,\mu)} + h_0 \sum_{i=1}^{n} S_{i.-}^{(r,\mu)} + h_0 \sum_{(i,j)\in J} \frac{1}{h_j h_i} \big| f_{ij}^{(r,\mu)} \big| = 0. \tag{2.20}$$

From (2.19), (2.20) we conclude that $(u-v)^{(r+1,\mu)} \leqslant 0$ which contradicts (2.6). Hence inequality (2.18) is proved on $E_{\mathbf{h}}$.

3. Implicit difference schemes

We first prove a theorem on the existence and uniqueness of solutions to (1.10)-(1.12).

THEOREM 3.1. *Suppose that Assumption **H[F]** is satisfied and*

$$\psi_{\boldsymbol{h}} : E_{0.\boldsymbol{h}} \to \mathbb{R},\ \ \Psi_{\boldsymbol{h}} : \partial_0 E_{\boldsymbol{h}} \to \mathbb{R},\ \ \beta : \partial_0 E_{\boldsymbol{h}} \to (0, +\infty),\ \ \textit{and}\ \ \gamma : \partial_0 E_{\boldsymbol{h}} \to \mathbb{R}_+.$$

Then there is exactly one solution $u_{\boldsymbol{h}} : E_{0.\boldsymbol{h}} \cup E_{\boldsymbol{h}} \to \mathbb{R}$ of problem (1.10) - (1.12).

Proof. Suppose that $0 \leqslant r < K$ is fixed and that the solution $u_{\mathbf{h}}$ to problem (1.10)-(1.12) is given on the set $(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$. We prove that the values

$u_{\mathbf{h}}^{(r+1,m)}$, $-M \leqslant m \leqslant M$, exist and that they are unique. It is sufficient to show that there exists exactly one solution of the system of equations

$$z^{(r+1,m)} = u_{\mathbf{h}}^{(r,m)} + h_0 F\left(t^{(r)}, x^{(m)}, T_{\mathbf{h}}(u_{\mathbf{h}})_{[r,m]}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}\right), \tag{3.1}$$

where $-(M-1) \leqslant m \leqslant M-1$, and

$$\Lambda[z]^{(r+1,m)} = \Psi_{\mathbf{h}}^{(r+1,m)} \quad \text{on } \partial_0 E_{\mathbf{h}}. \tag{3.2}$$

It follows from Assumption $\mathbf{H}[F]$ that there is $A_{\mathbf{h}} \in \mathbb{R}_+$ such that

$$A_{\mathbf{h}} \geqslant 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{s_{ii}} F(P) - h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} \left|\partial_{s_{ij}} F(P)\right|, \quad P \in \Xi. \tag{3.3}$$

Write

$$\mathbb{G}_i^{(r,m)}[\tau] = \frac{h_i \Psi_{\mathbf{h}}^{(r,m)}}{h_i \beta^{(r,m)} + \gamma^{(r,m)}} + \frac{\gamma^{(r,m)}}{h_i \beta^{(r,m)} + \gamma^{(r,m)}} \tau, \quad i = 1, \ldots, n,$$

where $\tau \in \mathbb{R}$ and

$$\mathscr{G}_{\mathbf{h}}^{(r,m)}[q,s] = F\left(t^{(r)}, x^{(m)}, T_{\mathbf{h}}(u_{\mathbf{h}})_{[r,m]}, q, s\right),$$

where $q \in \mathbb{R}^n$, $s \in M_{n\times n}$. Difference problem (3.1), (3.2) is equivalent to the system

$$z^{(r+1,m)} = \frac{1}{1+A_{\mathbf{h}}} \left[A_{\mathbf{h}} z^{(r+1,m)} + u_{\mathbf{h}}^{(r,m)} + h_0 \mathscr{G}_{\mathbf{h}}^{(r,m)}[\delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}]\right], \tag{3.4}$$

where $-(M-1) \leqslant m \leqslant M-1$ and

$$\begin{cases} z^{(r+1,m)} = \mathbb{G}_i^{(r+1,m)}[z^{(r+1,m-e_i)}] & \text{on } \partial_0 E_{\mathbf{h}.i}^{+}, \\ z^{(r+1,m)} = \mathbb{G}_i^{(r+1,m)}[z^{(r+1,m+e_i)}] & \text{on } \partial_0 E_{\mathbf{h}.i}^{-}, \end{cases} \tag{3.5}$$

where $i = 1, \ldots, n$. Set $X_{\mathbf{h}} = \{x^{(m)} : -M \leqslant m \leqslant M\}$. For $\chi \in \mathscr{F}(X_{\mathbf{h}}, \mathbb{R})$ we write $\chi^{(m)} = \chi(x^{(m)})$,

$$\delta\chi^{(m)} = (\delta_1 \chi^{(m)}, \ldots, \delta_n \chi^{(m)}) \text{ and } \delta^{(2)}\chi^{(m)} = [\delta_{ij}\chi^{(m)}]_{i,j=1,\ldots,n},$$

where δ_i, δ_{ij}, $1 \leqslant i \leqslant n$, are defined in Section 2. The norm in the space $\mathbb{F}(X_{\mathbf{h}}, \mathbb{R})$ we define by

$$\|\chi\|_{X_{\mathbf{h}}} = \max\{\left|\chi^{(m)}\right| : x^{(m)} \in X_{\mathbf{h}}\}.$$

Set

$$Y_{\mathbf{h}} = \Big\{\chi \in \mathbb{F}(X_{\mathbf{h}}, \mathbb{R}) : \chi^{(m)} = \mathbb{G}_i^{(r+1,m)}[\chi^{(m-e_i)}] \text{ on } \partial_0 E_{\mathbf{h}.i}^{+} \quad \text{and}$$
$$\chi^{(m)} = \mathbb{G}_i^{(r+1,m)}[\chi^{(m+e_i)}] \text{ on } \partial_0 E_{\mathbf{h}.i}^{-},\ i = 1, \ldots, n\Big\}.$$

Let $W_{r.\mathbf{h}}$ be an operator defined on $Y_{\mathbf{h}}$ in the following way:

$$W_{r.\mathbf{h}}[\chi]^{(m)} = \frac{1}{1+A_{\mathbf{h}}} \left[A_{\mathbf{h}} \chi^{(m)} + u_{\mathbf{h}}^{(r,m)} + h_0 \mathscr{G}_{\mathbf{h}}^{(r,m)}[\delta\chi^{(m)}, \delta^{(2)}\chi^{(m)}]\right],$$

where $-(M-1) \leqslant m \leqslant M-1$ and

$$W_{r.\mathbf{h}}[\chi]^{(m)} = \mathbb{G}_i^{(r+1,m)}[W_{r.\mathbf{h}}[\chi]^{(m-e_i)}] \text{ on } \partial_0 E^+_{\mathbf{h}.i}, \tag{3.6}$$

$$W_{r.\mathbf{h}}[\chi]^{(m)} = \mathbb{G}_i^{(r+1,m)}[W_{r.\mathbf{h}}[\chi]^{(m+e_i)}] \text{ on } \partial_0 E^-_{\mathbf{h}.i}, \tag{3.7}$$

where $i = 1, \ldots, n$. It follows that $W_{r.\mathbf{h}} : Y_{\mathbf{h}} \to Y_{\mathbf{h}}$. It is clear that problem (3.4), (3.5) is equivalent to the equation

$$\chi = W_{r.\mathbf{h}}[\chi]. \tag{3.8}$$

Suppose that $\chi,\ \tilde{\chi} \in Y_{\mathbf{h}}$. Write

$$Q^{(r,m)}(\tau) = \Big(t^{(r)}, x^{(m)}, T_{\mathbf{h}}(u_{\mathbf{h}})_{[r,m]}, \delta\chi^{(m)} + \tau\delta(\tilde{\chi}-\chi)^{(m)}, \\ \delta^{(2)}\chi^{(m)} + \tau\delta^{(2)}(\tilde{\chi}-\chi)^{(m)}\Big).$$

Suppose that the numbers $S_0^{(r,m)},\ S_{i.+}^{(r,m)},,\ S_{i.-}^{(r,m)},\ i = 1, \ldots, n$ and $S_{ij}^{(r,m)}$, for $(i,j) \in J$ are defined by (2.8)-(2.11) with the above given $Q^{(r,m)}(\tau)$. By using the Hadamard mean value theorem to the difference

$$\mathscr{G}_{\mathbf{h}}^{(r,m)}[\delta\tilde{\chi}^{(m)}, \delta^{(2)}\tilde{\chi}^{(m)}] - \mathscr{G}_{\mathbf{h}}^{(r,m)}[\delta\chi^{(m)}, \delta^{(2)}\chi^{(m)}]$$

we get

$$\begin{aligned}\big[W_{r.\mathbf{h}}[\tilde{\chi}]^{(m)} - W_{r.\mathbf{h}}[\chi]^{(m)}\big](1+A_{\mathbf{h}}) &= \big(A_{\mathbf{h}} + S_0^{(r,m)}\big)(\tilde{\chi}-\chi)^{(m)} \\ &+ \sum_{i=1}^{n} S_{i.+}^{(r,m)}(\tilde{\chi}-\chi)^{(m+e_i)} + \sum_{i=1}^{n} S_{i.-}^{(r,m)}(\tilde{\chi}-\chi)^{(m-e_i)} \\ &+ \sum_{(i,j)\in J_+} S_{ij}^{(r,m)}\big[(\tilde{\chi}-\chi)^{(m+e_i+e_j)} + (\tilde{\chi}-\chi)^{(m-e_i-e_j)}\big] \\ &- \sum_{(i,j)\in J_-} S_{ij}^{(r,m)}\big[(\tilde{\chi}-\chi)^{(m+e_i-e_j)} + (\tilde{\chi}-\chi)^{(m-e_i+e_j)}\big],\end{aligned}$$

where $-(M-1) \leqslant m \leqslant M-1$. The above relations and (2.12)-(2.14) imply

$$\|W_{r.\mathbf{h}}[\tilde{\chi}]^{(m)} - W_{r.\mathbf{h}}[\chi]^{(m)}\| \leqslant \frac{A_{\mathbf{h}}}{1+A_{\mathbf{h}}}\|\tilde{\chi}-\chi\|_{X_{\mathbf{h}}} \quad \text{for } -(M-1) \leqslant m \leqslant M-1.$$

Suppose that $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_{\mathbf{h}}$. It follows from (3.6) and (3.7) that

$$W_{r.\mathbf{h}}[\tilde{\chi}]^{(m)} - W_{r.\mathbf{h}}[\chi]^{(m)} = \frac{\gamma^{(r,m)}}{h_i\beta^{(r,m)} + \gamma^{(r,m)}}\big\{W_{r.\mathbf{h}}[\tilde{\chi}]^{(m-e_i)} - W_{r.\mathbf{h}}[\chi]^{(m-e_i)}\big\}$$

for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E^+_{\mathbf{h}.i}$ and

$$W_{r.\mathbf{h}}[\tilde{\chi}]^{(m)} - W_{r.\mathbf{h}}[\chi]^{(m)} = \frac{\gamma^{(r,m)}}{h_i\beta^{(r,m)} + \gamma^{(r,m)}}\big\{W_{r.\mathbf{h}}[\tilde{\chi}]^{(m+e_i)} - W_{r.\mathbf{h}}[\chi]^{(m+e_i)}\big\}$$

for $(t^{(r+1)},x^{(m)})\in\partial_0 E^-_{\mathbf{h}.i}$, where $i=1,\dots,n$. The result is

$$\|W_{r.\mathbf{h}}[\tilde\chi]^{(m)}-W_{r.\mathbf{h}}[\chi]^{(m)}\|\leqslant\frac{A_\mathbf{h}}{1+A_\mathbf{h}}\|\tilde\chi-\chi\|_{X_\mathbf{h}}\tag{3.9}$$

for $(t^{(r+1)},x^{(m)})\in\partial_0 E_\mathbf{h}$. Hence for $\chi,\ \bar\chi\in Y_\mathbf{h}$ we have

$$\|W_{r.\mathbf{h}}[\tilde\chi]-W_{r.\mathbf{h}}[\chi]\|_{X_\mathbf{h}}\leqslant\frac{A_\mathbf{h}}{1+A_\mathbf{h}}\|\tilde\chi-\chi\|_{X_\mathbf{h}}.\tag{3.10}$$

The Banach fixed point theorem implies that there exists exactly one solution to (3.8). It follows that the values $u_\mathbf{h}^{(r+1,m)}$, $-M\leqslant m\leqslant M$, exist and that they are unique. Since $u_\mathbf{h}$ is given on $E_{0.\mathbf{h}}$, the proof is completed by induction.

ASSUMPTION **H[σ,F]**. There exists $\sigma:[0,a]\times\mathbb{R}_+\to\mathbb{R}_+$ such that:

1) σ is continous and $\sigma(t,0)=0$ for $t\in[0,a]$,

2) σ is nondecreasing with respect to both variables and the maximal solution of the Cauchy problem $\omega'(t)=\sigma(t,\omega(t))$, $\omega(0)=0$, is $\bar\omega(t)=0$, $t\in[0,a]$,

3) the estimate

$$F(t,x,w,q,s)-F(t,x,\bar w,q,s)\leqslant\sigma(t,\|w-\bar w\|_B)$$

is satisfied for $w,\bar w\in C(B,\mathbb{R})$, $w\geqslant\bar w$ and $(t,x,q,s)\in E\times\mathbb{R}^n\times M_{n\times n}$.

ASSUMPTION **H[$T_\mathbf{h}$]**. The operator $T_\mathbf{h}:\mathscr{F}(B_\mathbf{h},\mathbb{R})\to C(B,\mathbb{R})$ satisfies the conditions:

1) for $w,\ \tilde w\in\mathscr{F}(B_\mathbf{h},\mathbb{R})$ we have

$$\|T_\mathbf{h}[w]-T_\mathbf{h}[\tilde w]\|_B\leqslant\|w-\tilde w\|_{B_\mathbf{h}},$$

2) if $w:B\to\mathbb{R}$ is of class C^1 then there is $\gamma_*:\mathbb{H}\to\mathbb{R}_+$ such that

$$\|T_\mathbf{h}[w_\mathbf{h}]-w\|_B\leqslant\gamma_*(\mathbf{h}),\ \text{and}\ \lim_{\mathbf{h}\to0}\gamma_*(\mathbf{h})=0,$$

where $w_\mathbf{h}$ is the restriction of w to the set $B_\mathbf{h}$.

THEOREM 3.2. *Suppose that Assumptions **H[F]**, **H[σ,F]**, **H[T_h]** are satisfied and*

1) $u_h:E_{0.h}\cup E_h\to\mathbb{R}$ *is a solution of* (1.10)-(1.12),

2) $v:E_0\cup E\to\mathbb{R}$ *is a solution of* (1.1)-(1.4) *and* v *is of class* C_* *and* v_h *is restriction of* v *to the set* $E_{0.h}\cup E_h$,

3) *the functions* $\beta:\partial_0E\to(0,+\infty)$, $\gamma:\partial_0E\to\mathbb{R}_+$ *are continuous, and* $\beta(t,x)\geqslant1$ *on* ∂_0E,

4) *for* $\alpha_0 : \mathbb{H} \to \mathbb{R}_+$ *the following initial-boundary inequalities are satisfied*

$$\left|\psi_{\boldsymbol{h}}^{(r,m)} - \psi^{(r,m)}\right| \leqslant \alpha_0(\boldsymbol{h}) \quad \textit{on } E_{0.\boldsymbol{h}},$$
$$\left|\Psi_{\boldsymbol{h}}^{(r,m)} - \Psi^{(r,m)}\right| \leqslant \alpha_0(\boldsymbol{h}) \quad \textit{on } \partial_0 E_{\boldsymbol{h}}$$

and $\lim_{\boldsymbol{h}\to 0} \alpha_0(\boldsymbol{h}) = 0$.

Then there is $\alpha : \mathbb{H} \to \mathbb{R}_+$ *such that*

$$\left|(u_{\boldsymbol{h}} - v_{\boldsymbol{h}})^{(r,m)}\right| \leqslant \alpha(\boldsymbol{h}) \quad \textit{on } E_{\boldsymbol{h}} \tag{3.11}$$

and $\lim_{\boldsymbol{h}\to 0} \alpha(\boldsymbol{h}) = 0$.

Proof. There are

$$\Gamma_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R},\ \ \tilde{\Gamma} : \partial_0 E_{\mathbf{h}} \to \mathbb{R},\ \ \tilde{\beta}_1 : \mathbb{H} \to \mathbb{R}_+ \ \text{ and } \ \tilde{\beta}_2 : \mathbb{H} \to \mathbb{R}_+$$

such that

$$\delta_0 v_{\mathbf{h}}^{(r,m)} = F_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} + \Gamma_{\mathbf{h}}^{(r,m)} \quad \text{on } E_{\mathbf{h}}, \tag{3.12}$$
$$\Lambda_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} = \Psi_{\mathbf{h}}^{(r,m)} + \tilde{\Gamma}_{\mathbf{h}}^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}}$$

and

$$\left|\Gamma_{\mathbf{h}}^{(r,m)}\right| \leqslant \tilde{\beta}_1(\mathbf{h}) \quad \text{on } E_{\mathbf{h}}, \quad \left|\tilde{\Gamma}_{\mathbf{h}}^{(r,m)}\right| \leqslant \tilde{\beta}_2(\mathbf{h}) \quad \text{on } \partial_0 E_{\mathbf{h}}, \tag{3.13}$$
$$\lim_{\mathbf{h}\to 0} \tilde{\beta}_1(\mathbf{h}) = 0, \quad \lim_{\mathbf{h}\to 0} \tilde{\beta}_2(\mathbf{h}) = 0.$$

Let $\tilde{v}_{\mathbf{h}} = v_{\mathbf{h}} + \omega_{\mathbf{h}}$ where $\omega_{\mathbf{h}}$ is a maximal solution of the problem

$$\omega'(t) = \sigma(t, \omega(t)) + \tilde{\beta}_1(\mathbf{h}), \quad \omega(0) = \alpha_0(\mathbf{h}) + \tilde{\beta}_2(\mathbf{h}).$$

Then we have $\tilde{v}_{\mathbf{h}}^{(r,m)} \geqslant u_{\mathbf{h}}^{(r,m)}$ on $E_{0.\mathbf{h}}$ and

$$\begin{aligned}
\Lambda_{\mathbf{h}}[u_{\mathbf{h}} - \tilde{v}_{\mathbf{h}}]^{(r,m)} &= \Lambda_{\mathbf{h}}[u_{\mathbf{h}} - v_{\mathbf{h}} - \omega_{\mathbf{h}}]^{(r,m)} \\
&= \Psi_{\mathbf{h}}^{(r,m)} - \Lambda_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} - \beta^{(r,m)} \omega_{\mathbf{h}}^{(r)} \\
&\leqslant \tilde{\beta}_2(\mathbf{h}) - \beta^{(r,m)} \omega_{\mathbf{h}}^{(r)} \\
&\leqslant (1 - \beta^{(r,m)}) \omega_{\mathbf{h}}^{(r)} \leqslant 0.
\end{aligned}$$

We show that $\delta_0 \tilde{v}_{\mathbf{h}}^{(r,m)} - F_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)} \geqslant 0$ on $E_{\mathbf{h}} \setminus \partial_0 E_{\mathbf{h}}$. It follows from Assumption $\mathbf{H}[\sigma, F]$ that

$$\begin{aligned}
\delta_0 \tilde{v}_{\mathbf{h}}^{(r,m)} - F_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)} &= \delta_0 v_{\mathbf{h}}^{(r,m)} + \frac{1}{h_0}\left[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}\right] - F_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)} \\
&\quad + F_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} - F_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} \\
&\geqslant \delta_0 v_{\mathbf{h}}^{(r,m)} - F_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} + \frac{1}{h_0}\left[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}\right]
\end{aligned}$$

$$- \sigma(t^{(r)}, \omega_{\mathbf{h}}(t^{(r)})).$$

Since $\omega_{\mathbf{h}}$ is convex, we have that

$$\delta_0 \tilde{v}_{\mathbf{h}}^{(r,m)} - F_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)} \geqslant \frac{1}{h_0}\left[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}\right] - \tilde{\beta}_1(\mathbf{h}) - \sigma(t^{(r)}, \omega_{\mathbf{h}}(t^{(r)})) \geqslant 0.$$

From Theorem 2.1 follows that $u_{\mathbf{h}}^{(r,m)} \leqslant \tilde{v}_{\mathbf{h}}^{(r,m)}$ on $E_{\mathbf{h}}$. Analogously we prove that $u_{\mathbf{h}}^{(r,m)} \geqslant \tilde{w}_{\mathbf{h}}^{(r,m)}$ on $E_{\mathbf{h}}$, where

$$\tilde{w}_{\mathbf{h}}^{(r,m)} = v_{\mathbf{h}}^{(r,m)} - \omega_{\mathbf{h}}^{(r)} \text{ on } E_{\mathbf{h}}.$$

Consequently we have that

$$\left|u_{\mathbf{h}}^{(r,m)} - v_{\mathbf{h}}^{(r,m)}\right| \leqslant \omega_{\mathbf{h}}^{(r)} \text{ on } E_{\mathbf{h}}.$$

REMARK 3.1. Suppose that all the assumptions of Theorem 3.2 are satisfied and $\sigma : [0,a] \times \mathbb{R}_+ \times C(I,\mathbb{R}_+) \to \mathbb{R}_+$ is given by $\sigma(t,p) = Lp$ on $[0,a] \times \mathbb{R}_+$, where $L \in \mathbb{R}_+$. Then

$$\left|u_{\mathbf{h}}^{(r,m)} - v_{\mathbf{h}}^{(r,m)}\right| \leqslant \tilde{\alpha}(\mathbf{h}) \text{ on } E_{\mathbf{h}},$$

where

$$\tilde{\alpha}(\mathbf{h}) = (\alpha_0(\mathbf{h}) + \tilde{\beta}_2(\mathbf{h}))e^{La} + \frac{\tilde{\beta}_1(\mathbf{h})}{L}(e^{La} - 1) \quad \text{if } L > 0, \tag{3.14}$$

$$\tilde{\alpha}(\mathbf{h}) = \alpha_0(\mathbf{h}) + \tilde{\beta}_2(\mathbf{h}) + a\tilde{\beta}_1(\mathbf{h}) \quad \text{if } L = 0. \tag{3.15}$$

Now we formulate a result on the error estimate. For $x \in \mathbb{R}^n$, $X \in M_{n\times n}$, $X = [x_{ij}]_{i,j=1,\ldots,n}$, we put

$$\|x\| = \sum_{i=1}^{n} \left|x_i\right|, \quad \|X\| = \max\{\sum_{j=1}^{n} \left|x_{ij}\right| : 1 \leqslant i \leqslant n\}.$$

LEMMA 3.1. *Suppose that all the assumptions of Theorem* 3.2 *are satisfied with* $\sigma(t,p) = Lp$ *on* $[0,a] \times \mathbb{R}_+$, *where* $L \in \mathbb{R}_+$ *and:*

1) *for* $P = (t,x,w,q,s) \in \Xi$ *we have:* $\|\partial_q F(P)\| \leqslant L_0$, $\|\partial_s F(P)\| \leqslant L_0$,

2) $\psi_{\boldsymbol{h}} = \psi$ *on* $E_{0.\boldsymbol{h}}$ *and* $\Psi_{\boldsymbol{h}} = \Psi$ *on* $\partial_0 E_{\boldsymbol{h}}$ *and interpolating operator* $T_{\boldsymbol{h}} : \mathscr{F}_C(B_{\boldsymbol{h}},\mathbb{R}) \to C(B,\mathbb{R})$ *is given in* [4] *and* $\left|\gamma(t,x)\right| \leqslant N$ *on* $\partial_0 E$,

3) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of* (1.1) - (1.4) *and* v *is of class* C^2 *and there is* $\tilde{L} \in \mathbb{R}_+$ *such that*

$$\|\partial_{xx} v(t,x) - \partial_{xx} v(t,y)\| \leqslant \tilde{L}\|x - y\| \quad \text{on } E_0 \cup E. \tag{3.16}$$

Then

$$\left|(u_{\boldsymbol{h}} - v_{\boldsymbol{h}})^{(r,m)}\right| \leqslant \tilde{\alpha}(\boldsymbol{h}) \quad \text{on } E_{\boldsymbol{h}}, \tag{3.17}$$

where $u_{\boldsymbol{h}} : E_{0.\boldsymbol{h}} \cup E_{\boldsymbol{h}} \to \mathbb{R}$ *is a solution to* (1.10)-(1.12) *and* $v_{\boldsymbol{h}}$ *is the restriction of* v *to the set* $E_{0.\boldsymbol{h}} \cup E_{\boldsymbol{h}}$ *and* $\tilde{\alpha}$ *is given by* (3.14)-(3.15) *with*

$$\tilde{\beta}_1(\boldsymbol{h}) = \frac{1}{2}\tilde{C}h_0 + L_0(\tilde{C}+\tilde{L})\|\boldsymbol{h}\| + L_0\tilde{C}\|\boldsymbol{h}\|^2, \quad \tilde{\beta}_2(\boldsymbol{h}) = N\tilde{C}\|\boldsymbol{h}\|$$

and $\tilde{C} \in \mathbb{R}_+$ *is defined by the relations*

$$\left|\partial_{tt} v(t,x)\right| \leqslant \tilde{C}, \quad \|\partial_{xx} v(t,x)\| \leqslant \tilde{C} \quad \text{on } E_0 \cup E. \tag{3.18}$$

Proof. It follows from (3.16), (3.18) and from Theorem 5.27 in [4] that

$$\begin{aligned}
&\left|\partial_t v^{(r,m)} - \delta_0 v_{\mathbf{h}}^{(r,m)}\right| \leqslant \frac{1}{2}\tilde{C}h_0, \;\; \|\partial_x v^{(r,m)} - \delta v_{\mathbf{h}}^{(r,m)}\| \leqslant \tilde{C}\|\mathbf{h}\|,\\
&\|\partial_{xx} v^{(r,m)} - \delta^{(2)} v_{\mathbf{h}}^{(r,m)}\| \leqslant \tilde{L}\|\mathbf{h}\|, \;\; \|T_{\mathbf{h}}(v_{\mathbf{h}})_{[r,m]} - v_{(t^{(r)},x^{(m)})}\|_B \leqslant \tilde{C}\|\mathbf{h}\|^2
\end{aligned}$$

on $E_{\mathbf{h}}$. Then inequalities (3.13) are satisfied with the above given $\tilde{\beta}_1$, $\tilde{\beta}_2$. Then inequality (3.17) is a consequence of (3.14)-(3.15).

In the result on the error estimate, we need estimates of the derivatives of the solution v of problem (1.1)-(1.4). One may obtain them by the method of differential inequalities.

Now we consider implicit difference schemes for problem (2.15), (1.2)-(1.4).

ASSUMPTION $\mathbf{H}[\mathscr{G},\sigma]$ Suppose that there is $\sigma : [0,a]\times\mathbb{R}_+ \to \mathbb{R}_+$ such that conditions 1), 2) of Assumption $\mathbf{H}[\sigma,F]$ are satisfied and

$$\mathscr{G}(t,x,w,q) - \mathscr{G}(t,x,\bar{w},q) \leqslant \sigma(t,\|w-\bar{w}\|_B)$$

where $w,\bar{w} \in C(B,\mathbb{R})$, $w \geqslant \bar{w}$ and $(t,x) \in E$.

THEOREM 3.3. *Suppose that Assumptions* $\boldsymbol{H}[f,G]$, $\boldsymbol{H}[G,\sigma]$, $\boldsymbol{H}[T_{\boldsymbol{h}}]$ *are satisfied and:*

1) $\psi_{\boldsymbol{h}} : E_{0.\boldsymbol{h}} \to \mathbb{R}$, $\Psi_{\boldsymbol{h}} : \partial_0 E_{\boldsymbol{h}} \to \mathbb{R}$, $\beta \in C(\partial_0 E_{\boldsymbol{h}},(0,+\infty))$, $\gamma \in C(\partial_0 E_{\boldsymbol{h}},\mathbb{R}_+)$ *and* $\beta(t,x) \geqslant 1$ *on* $\partial_0 E$,

2) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of* (2.15), (1.2)-(1.4) *and* v *is of class* C_* *and* $v_{\boldsymbol{h}}$ *is restriction of* v *to the set* $E_{0.\boldsymbol{h}} \cup E_{\boldsymbol{h}}$,

3) *there is* $\alpha_0 : \mathbb{H} \to \mathbb{R}_+$ *such that*

$$\left|\psi_{\boldsymbol{h}}^{(r,m)} - \psi^{(r,m)}\right| \leqslant \alpha_0(\boldsymbol{h}) \quad \text{on } E_{0.\boldsymbol{h}}, \quad \left|\Psi_{\boldsymbol{h}}^{(r,m)} - \Psi^{(r,m)}\right| \leqslant \alpha_0(\boldsymbol{h}) \quad \text{on } \partial_0 E_{\boldsymbol{h}}$$

and $\lim\limits_{\boldsymbol{h}\to 0} \alpha_0(\boldsymbol{h}) = 0$. *Then we have:*

1) *there exists exactly one solution* $u_{\boldsymbol{h}} : E_{0.\boldsymbol{h}} \cup E_{\boldsymbol{h}} \to \mathbb{R}$ *of problem* (2.15), (1.2)-(1.4)*;*

2) *there is* $\alpha : \mathbb{H} \to \mathbb{R}_+$ *such that*

$$\left|(u_{\boldsymbol{h}} - v_{\boldsymbol{h}})^{(r,m)}\right| \leqslant \alpha(\boldsymbol{h}) \quad \text{on } E_{\boldsymbol{h}} \tag{3.19}$$

and $\lim\limits_{\boldsymbol{h}\to 0} \alpha(\boldsymbol{h}) = 0$.

Proof. Proceeding like as in the proof of Theorem 3.1 we deduce that there exists exactly one solution $u_{\mathbf{h}}$ of problem (2.15), (1.2)-(1.4). We will show that (3.19) is satisfied on $E_{\mathbf{h}}$. There are

$$\Gamma_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R},\;\; \tilde{\Gamma}_{\mathbf{h}} : \partial_0 E_{\mathbf{h}} \to \mathbb{R},$$
$$\tilde{\gamma}_1 : \mathbb{H} \to \mathbb{R}_+,\;\; \tilde{\gamma}_2 : \mathbb{H} \to \mathbb{R}_+,$$

such that for $-(M-1) \leqslant m \leqslant M-1,\ 0 \leqslant r \leqslant K-1$,

$$\begin{aligned} \delta_0 v_{\mathbf{h}}^{(r,m)} &= G_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} + \Gamma_{\mathbf{h}}^{(r,m)} \quad \text{on } E_{\mathbf{h}}, \\ \Lambda_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} &= \Psi_{\mathbf{h}}^{(r,m)} + \tilde{\Gamma}_{\mathbf{h}}^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}}, \end{aligned} \tag{3.20}$$

and

$$\left|\Gamma_{\mathbf{h}}^{(r,m)}\right| \leqslant \tilde{\gamma}_1(\mathbf{h}) \quad \text{on } E_{\mathbf{h}}, \quad \left|\tilde{\Gamma}_{\mathbf{h}}^{(r,m)}\right| \leqslant \tilde{\gamma}_2(\mathbf{h}) \quad \text{on } \partial_0 E_{\mathbf{h}},$$
$$\lim_{\mathbf{h}\to 0} \tilde{\gamma}_1(\mathbf{h}) = 0, \quad \lim_{\mathbf{h}\to 0} \tilde{\gamma}_2(\mathbf{h}) = 0.$$

Write $\tilde{v}_{\mathbf{h}} = v_{\mathbf{h}} + \omega_{\mathbf{h}}$ on $E_{\mathbf{h}}$, where $\omega_{\mathbf{h}}$ is a maximal solution of the problem

$$\omega'(t) = \sigma(t, \omega(t)) + \tilde{\gamma}_1(\mathbf{h}), \quad \omega(0) = \alpha_0(\mathbf{h}) + \tilde{\gamma}_2(\mathbf{h}).$$

Then we have $u_{\mathbf{h}}^{(r,m)} \leqslant \tilde{v}_{\mathbf{h}}^{(r,m)}$ on $E_{0.\mathbf{h}}$ and

$$\begin{aligned} \Lambda_{\mathbf{h}}[u_{\mathbf{h}} - \tilde{v}_{\mathbf{h}}]^{(r,m)} &= \Lambda_{\mathbf{h}}[u_{\mathbf{h}}]^{(r,m)} - \Lambda_{\mathbf{h}}[u_{\mathbf{h}}]^{(r,m)} - \Lambda_{\mathbf{h}}[\omega_{\mathbf{h}}]^{(r,m)} \\ &= \Psi_{\mathbf{h}}^{(r,m)} - \Lambda_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} - \beta^{(r,m)} \omega_{\mathbf{h}}^{(r)} \\ &\leqslant \tilde{\gamma}_2(\mathbf{h}) - \beta^{(r,m)} \omega_{\mathbf{h}}^{(r)} \leqslant 0. \end{aligned}$$

We show that

$$\delta_0 \tilde{v}_{\mathbf{h}}^{(r,m)} \geqslant G_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)} \quad \text{on } E_{\mathbf{h}} \setminus \partial_0 E_{\mathbf{h}}.$$

It follows from Assumption $\mathbf{H}[G, \sigma]$ and (3.20) that

$$\begin{aligned} \delta_0 \tilde{v}_{\mathbf{h}}^{(r,m)} &= \delta_0 v_{\mathbf{h}}^{(r,m)} + \frac{1}{h_0}[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}] \\ &= G_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} + \Gamma_{\mathbf{h}}^{(r,m)} + \frac{1}{h_0}[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}] \\ &\geqslant \sum_{i,j=1}^{n} f_{ij}^{(r,m)} \delta_{ij} v_{\mathbf{h}}^{(r+1,m)} + G(t^{(r)}, x^{(m)}, (T_{\mathbf{h}} v_{\mathbf{h}})_{[r,m]}, \delta v_{\mathbf{h}}^{(r+1,m)}) - \tilde{\gamma}_1(\mathbf{h}) \\ &\quad - G\big(t^{(r)}, x^{(m)}, (T_{\mathbf{h}} \tilde{v}_{\mathbf{h}})_{[r,m]}, \delta v_{\mathbf{h}}^{(r+1,m)}\big) + G\big(t^{(r)}, x^{(m)}, (T_{\mathbf{h}} \tilde{v}_{\mathbf{h}})_{[r,m]}, \delta v_{\mathbf{h}}^{(r+1,m)}\big) \\ &\quad + \frac{1}{h_0}[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}] \\ &\geqslant G_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)} - \tilde{\gamma}_1(\mathbf{h}) - \sigma(t^{(r)}, \omega_{\mathbf{h}}^{(r)}) + \frac{1}{h_0}[\omega_{\mathbf{h}}^{(r+1)} - \omega_{\mathbf{h}}^{(r)}] \geqslant G_{\mathbf{h}}[\tilde{v}_{\mathbf{h}}]^{(r,m)}. \end{aligned}$$

From Theorem (2.2) follows that $u_{\mathbf{h}}^{(r,m)} \leqslant \tilde{v}_{\mathbf{h}}^{(r,m)}$ on $E_{\mathbf{h}}$. Analogously we prove that $u_{\mathbf{h}}^{(r,m)} \geqslant \tilde{w}_{\mathbf{h}}^{(r,m)}$ on $E_{\mathbf{h}}$ where:

$$\tilde{w}_{\mathbf{h}}^{(r,m)} = v_{\mathbf{h}}^{(r,m)} - \omega_{\mathbf{h}}^{(r)} \text{ on } E_{\mathbf{h}} \quad \text{and} \quad \tilde{w}_{\mathbf{h}}^{(r,m)} = v_{\mathbf{h}}^{(r,m)} - \alpha_0(\mathbf{h}) \text{ on } E_{0.\mathbf{h}}.$$

Consequently we have that $\left|(u_{\mathbf{h}} - v_{\mathbf{h}})^{(r,m)}\right| \leqslant \omega_{\mathbf{h}}^{(r)}$ on $E_{\mathbf{h}}$. Then the condition (3.19) is satisfied with $\alpha(\mathbf{h}) = \omega_{\mathbf{h}}(a)$. This completes the proof.

4. Numerical examples

We apply the results presented in Section 3 to a differential equation with deviated variables and to a differential integral problem. Let $n = 2$ and

$$E = [0,0.5] \times [-0.5,0.5] \times [-0.5,0.5], \quad E_0 = \{0\} \times [-0.5,0.5] \times [-0.5,0.5].$$

Initial-boundary problems considered in the present section have solutions on E.

The following examples satisfy all the assumptions of Theorem 3.3.

EXAMPLE 4.1. Consider the differential equation with deviated variables

$$\begin{aligned} \partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) \\ + xy\partial_{xy} z(t,x,y) + z(t,0.5(x+y),0.5(x-y)) \\ + \left[x^2 - y^2 + 4t^2(x^2y^2 - x^2 - y^2) - e^{t(xy-x^2+y^2)}\right] z(t,x,y) \end{aligned} \tag{4.1}$$

and the initial-boundary conditions

$$z(0,x,y) = 1, \quad (x,y) \in [-0.5,0.5] \times [-0.5,0.5] \tag{4.2}$$

and

$$\begin{aligned} z(t,0.5,y) = z(t,-0.5,y) = e^{t(0.25-y^2)}, \ (t,y) \in [0,0.5] \times [-0.5,0.5], \\ z(t,x,0.5) = z(t,x,-0.5) = e^{t(x^2-0.25)}, \ (t,x) \in [0,0.5] \times [-0.5,0.5]. \end{aligned} \tag{4.3}$$

The solution of (4.1)- (4.3) is known, it is $v(t,x,y) = e^{t(x^2-y^2)}$. Let us denote by $u_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R}$ the solution of implicit difference problem corresponding to (4.1) -(4.3). Write

$$\varepsilon_{\mathbf{h}}^{(r)} = \frac{1}{(2N_1+1)(2N_2+1)} \sum_{m \in M} \left|u_{\mathbf{h}}^{(r,m)} - v_{\mathbf{h}}^{(r,m)}\right|, \quad 0 \leqslant r \leqslant N_0, \tag{4.4}$$

where $v_{\mathbf{h}}$ is the restriction of v to the set $E_{\mathbf{h}}$ and

$$M = \{m \in (m_1,m_2) : -N_1 \leqslant m_1 \leqslant N_1, \ -N_2 \leqslant m_2 \leqslant N_2\}$$

and $N_0 h_0 = 0.5$, $N_1 h_1 = 0.5$, $N_2 h_2 = 0.5$. The numbers $\varepsilon_{\mathbf{h}}^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$. We give experimental values of the above defined errors for $h_0 = \frac{1}{128}$, $h_1 = h_2 = \frac{1}{128}$.

Table I

$t^{(r)}$:	0.0625	0.1250	0.1875	0.2500	0.3125	0.3750	0.4375
$\varepsilon_h^{(r)}$:	0.000006	0.000017	0.000030	0.000044	0.000058	0.000072	0.000086

Note that condition (1.13) is not satisfied in our example and the explicit difference method is not convergent. In fact, the average errors of that method exceeded 10^8.

EXAMPLE 4.2. Consider the differential integral equation

$$\begin{aligned}\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + xy\partial_{xy} z(t,x,y) + \partial_{yy} z(t,x,y) \\ + xy^2 \int_0^x z(t,s,y)ds + yx^2 \int_0^y z(t,x,s)ds \\ - f(x,y)z(t,x,y) + g(t,x,y),\end{aligned} \tag{4.5}$$

with initial-boundary conditions

$$z(0,x) = 0 \quad \text{for } x \in [-0.5, 0.5], \tag{4.6}$$

and

$$\begin{aligned} z(t,-0.5,y) - \partial_x z(t,-0.5,y) = (1-y)\sin(\pi t)\exp(-0.5y), \\ z(t,0.5,y) + \partial_x z(t,0.5,y) = (1+y)\sin(\pi t)\exp(0.5y),\end{aligned} \tag{4.7}$$

where $(t,y) \in [0,0.5] \times [-0.5,0.5]$ and

$$\begin{aligned} z(t,x,-0.5) - \partial_x z(t,x,-0.5) = (1-x)\sin(\pi t)\exp(-0.5x), \\ z(t,x,0.5) + \partial_x z(t,x,0.5) = (1+x)\sin(\pi t)\exp(0.5x),\end{aligned}$$

where $(t,x) \in [0,0.5] \times [-0.5,0.5]$ and

$$\begin{aligned} f(x,y) = x^2y^2 + 3xy + x^2 + y^2, \\ g(t,x,y) = \pi\cos(\pi t)\exp(xy) + 2xy\sin(\pi t).\end{aligned}$$

The solution of (4.5)-(4.7) is known, it is $z(t,x,y) = \sin(\pi t)\exp(xy)$.

Let us denote by $u_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R}$ the solution of implicit difference problem corresponding to (4.5)-(4.7). Let $\varepsilon_{\mathbf{h}}^{(r)}$ be the arithmetical means of the errors defined by (4.4). In Table II we give experimental values of $\varepsilon_{\mathbf{h}}^{(r)}$ for $h_0 = \frac{1}{200}$, $h_1 = h_2 = \frac{1}{200}$.

Table II

$t^{(r)}$:	0.075	0.150	0.225	0.300	0.375	0.450	0.500
$\varepsilon_h^{(r)}$:	0.0003721	0.0010198	0.0018419	0.0027397	0.0036224	0.0044096	0.0048474

In the considered case condition (1.13) is not satisfied and the explicit difference method is not convergent. The average errors of that method exceeded 10^8.

The above examples show that there are implicit difference shcemes for parabolic functional differential equations which are convergent and the corresponding classical methods are not convergent. This is due to the fact that we need relation (1.13) for steps of the mesh in the classical case and we do not need this condition for our implicit difference schemes.

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(Received August 16, 2010)

(Revised August 7, 2010)

Milena Netka
Institute of Mathematics
University of Gdańsk
Wit Stwosz Street 57
80-952 Gdańsk, Poland e-mail: mnetka@wp.pl