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ASYMPTOTIC STABILITY AND STABILITY SWITCHES IN A LINEAR INTEGRO–DIFFERENTIAL SYSTEM

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Abstract. This paper is concerned with the stability problem of a linear integro-differential system with distributed delay in the diagonal terms. We establish some explicit conditions for the zero solution of the system to be asymptotically stable. In particular, as the delay parameter increases monotonously under certain conditions, the zero solution switches finite times from stability to instability to stability, and becomes unstable eventually.

1. Introduction

Recently, several authors have studied the asymptotic stability of linear differential equations with distributed delay as well as with discrete delay; for example, explicit stability conditions can be found in [1, 2, 4, 7, 8, 10, 11, 12, 14]. It is well known that for the linear and autonomous case, the zero solution being asymptotically stable is equivalent to all solutions having limit zero as $t \to \infty$ which in turn is true if and only if all roots of an associated characteristic equation have negative real parts; see, e.g., [5].

In this paper we consider a linear integro-differential system

$$\begin{cases} x'(t) = -a\displaystyle\int_{t-\tau}^{t} x(s)ds - by(t), \\ y'(t) = -cx(t) - a\displaystyle\int_{t-\tau}^{t} y(s)ds, \end{cases} \tag{1.1}$$

where a, b and c are real numbers and τ is a positive constant. The purpose of this paper is to establish necessary and sufficient conditions for the zero solution of (1.1) to be asymptotically stable.

System (1.1) can be rewritten as

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) - a\int_{t-\tau}^{t} \boldsymbol{x}(s)ds,$$

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where $\boldsymbol{x}(t) = \mathrm{col}\,(x(t), y(t))$ and A is the 2×2 matrix given by

$$A = \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix}.$$

So the stability problem of (1.1) may regard as *time-delayed feedback control*, that is, the stabilization of unstable steady states of the system

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) - \boldsymbol{F}(t), \tag{1.2}$$

where $\boldsymbol{F}(t)$ denotes the control force given by

$$\boldsymbol{F}(t) = a \int_{t-\tau}^{t} \boldsymbol{x}(s)ds.$$

Notice that system (1.2) with $\boldsymbol{F}(t) \equiv 0$ has periodic solutions (if $bc < 0$) or unbounded solutions (if $bc > 0$). For the general background of time-delayed feedback control, one can refer to a recent book [13, Chapter 4] and the references cited therein.

In case $\boldsymbol{F}(t) = a\boldsymbol{x}(t - \sigma)$, where σ is a positive constant, system (1.2) becomes the linear differential-difference system

$$\begin{cases} x'(t) = -ax(t-\sigma) - by(t), \\ y'(t) = -cx(t) - ay(t-\sigma). \end{cases} \tag{1.3}$$

Recently, the first author [9] has obtained the following stability condition of (1.3).

THEOREM A. *The zero solution of* (1.3) *is asymptotically stable if and only if any one of the following four conditions holds:*

(i) $\sqrt{bc} < a$ *and* $0 < \sigma < \dfrac{1}{\sqrt{a^2 - bc}} \arccos \dfrac{\sqrt{bc}}{a}$,

(ii) $\sqrt{-bc} \leqslant 4a$ *and* $0 < \sigma < \dfrac{\pi}{2(a + \sqrt{-bc})}$,

(iii) $0 < 4a < \sqrt{-bc}$ *and* $\sigma \in (0, \sigma_{1,0}) \cup (\sigma_{2,0}, \sigma_{1,1}) \cup \cdots \cup (\sigma_{2,k_*-1}, \sigma_{1,k_*})$,

(iv) $-\sqrt{-bc} < 2a < 0$ *and* $\sigma \in (\sigma_{1,0}, \sigma_{2,0}) \cup (\sigma_{1,1}, \sigma_{2,1}) \cup \cdots \cup (\sigma_{1,l_*}, \sigma_{2,l_*})$.

Here $\sigma_{1,n}$, $\sigma_{2,n}$, k_* *and* l_* *are defined by*

$$\sigma_{1,n} = \frac{(4n+1)\pi}{2(a + \sqrt{-bc})}, \quad \sigma_{2,n} = \frac{(4n+3)\pi}{2(-a + \sqrt{-bc})} \quad \text{for } n = 0, 1, 2, \ldots,$$

$$k_* = \left\lfloor \frac{\sqrt{-bc}}{4a} \right\rfloor, \quad l_* = \left\lfloor -\frac{\sqrt{-bc}}{4a} - \frac{1}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ *denotes the floor function, namely,* $\lfloor x \rfloor = \max\{n \in \mathbf{Z} \mid n \leqslant x\}$.

Theorem A provides some cases where the delay σ has a stabilizing effect on the solutions of (1.3). In particular, as σ increases monotonously from 0, the zero solution of (1.3) switches finite times from stability to instability to stability if $0 < 4a < \sqrt{-bc}$; from instability to stability to instability if $-\sqrt{-bc} < 2a < 0$; and it becomes unstable eventually. Such phenomena for increasing delay are often referred to as *stability switches*; see, e.g., [1, 3, 8]. Judging from this, we can expect that stability switches also appear in (1.1) as τ increases under certain conditions.

Our main result is stated as follows.

THEOREM 1.1. *Suppose that* $8a \neq bc$ *when* $bc < 0$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if any one of the following three conditions holds:*

$$0 \leqslant bc < 2a \quad \text{and} \quad \frac{\sqrt{bc}}{a} < \tau < \frac{1}{\sqrt{2a-bc}} \arccos \frac{bc-a}{a}, \tag{1.4}$$

$$bc < 0 < a \quad \text{and} \quad \tau \in (0, \tau_{2,0}) \cup (\tau_{1,1}, \tau_{2,1}) \cup \cdots \cup (\tau_{1,k}, \tau_{2,k}), \tag{1.5}$$

$$bc < 8a < 0 \quad \text{and} \quad \tau \in (\tau_{2,0}, \tau_{1,1}) \cup (\tau_{2,1}, \tau_{1,2}) \cup \cdots \cup (\tau_{2,l}, \tau_{1,l+1}). \tag{1.6}$$

Here $\tau_{1,n}$, $\tau_{2,n}$, k *and* l *are defined by*

$$\tau_{1,n} = \frac{2n\pi}{\sqrt{-bc}}, \quad \tau_{2,n} = \frac{2(2n+1)\pi}{\sqrt{-bc}+\sqrt{8a-bc}} \quad \text{for } n = 0, 1, 2, \ldots,$$

$$k = \left\lceil \frac{\sqrt{-bc}}{\sqrt{8a-bc}-\sqrt{-bc}} \right\rceil - 1, \quad l = \left\lceil \frac{\sqrt{8a-bc}}{\sqrt{-bc}-\sqrt{8a-bc}} \right\rceil - 1,$$

where $\lceil \cdot \rceil$ *denotes the ceiling function, namely,* $\lceil x \rceil = \min\{n \in \mathbf{Z} \mid x \leqslant n\}$.

REMARK 1.1. In the case $bc = 0$, system (1.1) is reduced to

$$x'(t) = -a \int_{t-\tau}^{t} x(s)ds. \tag{1.7}$$

Theorem 1.1 asserts that the zero solution of (1.7) is asymptotically stable if and only if $0 < \tau < \pi/\sqrt{2a}$, which coincides with the stability condition given by Theorem 4.1 in [2].

REMARK 1.2. Theorem 1.1 shows that as τ increases monotonously from 0, the zero solution of (1.1) switches finite times from stability to instability to stability if $0 < 8a < -3bc$; from instability to stability to instability if $0 \leqslant bc < 2a$ or $bc < 8a < 0$; and it becomes unstable eventually.

REMARK 1.3. The exact region of asymptotic stability of (1.1) with fixed b and c is presented in Figure 1. In the case $bc < 0$, the vertical boundaries and the boundary curves of the stability region are given by

$$\tau = \frac{2n\pi}{\sqrt{-bc}} \quad \text{and} \quad \begin{cases} a = \dfrac{(2n+1)\pi}{2\tau}\left(\dfrac{(2n+1)\pi}{\tau} - \sqrt{-bc}\right), \\ \dfrac{2n\pi}{\sqrt{-bc}} \leqslant \tau \leqslant \dfrac{2(n+1)\pi}{\sqrt{-bc}} \end{cases}$$

for $n = 0, 1, 2, \ldots$, respectively.

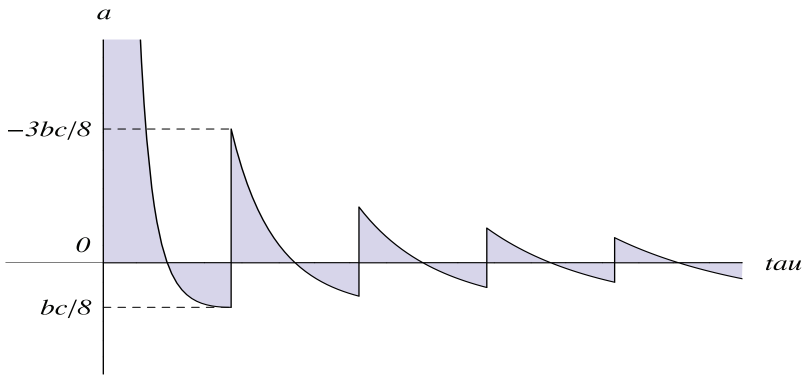

Figure 1. Stability region of (1.1) with $bc < 0$

2. Proof of Main Theorem

The characteristic equation associated with (1.1) is given by

$$F(\lambda) \equiv \det \begin{pmatrix} \lambda + a\int_{-\tau}^{0} e^{\lambda s} ds & b \\ c & \lambda + a\int_{-\tau}^{0} e^{\lambda s} ds \end{pmatrix} = 0. \tag{2.1}$$

Notice that

$$\begin{aligned} F(\lambda) &= \left(\lambda + a\int_{-\tau}^{0} e^{\lambda s} ds\right)^2 - bc \\ &= \begin{cases} \left(\lambda + a\int_{-\tau}^{0} e^{\lambda s} ds + \sqrt{bc}\right)\left(\lambda + a\int_{-\tau}^{0} e^{\lambda s} ds - \sqrt{bc}\right), & bc \geqslant 0, \\ \left(\lambda + a\int_{-\tau}^{0} e^{\lambda s} ds + i\sqrt{-bc}\right)\left(\lambda + a\int_{-\tau}^{0} e^{\lambda s} ds - i\sqrt{-bc}\right), & bc < 0. \end{cases} \end{aligned}$$

First, we consider the case of $bc > 0$. Then equation (2.1) is reduced to

$$\lambda + \sqrt{bc} + a\int_{-\tau}^{0} e^{\lambda s} ds = 0 \tag{2.2}$$

or

$$\lambda - \sqrt{bc} + a\int_{-\tau}^{0} e^{\lambda s} ds = 0. \tag{2.3}$$

The locations of roots of these equations have been studied by Funakubo et al. [4] and Hara and Sakata [6]. By virtue of their work, we have the following result.

LEMMA 2.1. *Let p and q be real numbers. Then all roots of the transcendental equation*

$$\lambda + p + q\int_{-\tau}^{0} e^{\lambda s}ds = 0$$

have negative real parts if and only if any one of the following four conditions holds:

(i) $p > 0, \quad q \geqslant 0 \quad \text{and} \quad 2q - p^2 \leqslant 0,$

(ii) $p \geqslant 0, \quad q > 0, \quad 2q - p^2 > 0 \quad \text{and} \quad \tau < \dfrac{1}{\sqrt{2q-p^2}}\left(2\pi - \arccos\dfrac{p^2-q}{q}\right),$

(iii) $p > 0, \quad q < 0 \quad \text{and} \quad \tau < \left|\dfrac{p}{q}\right|,$

(iv) $p < 0, \quad q > 0, \quad 2q - p^2 > 0 \quad \text{and} \quad \left|\dfrac{p}{q}\right| < \tau < \dfrac{1}{\sqrt{2q-p^2}}\arccos\dfrac{p^2-q}{q}.$

By Lemma 2.1, the necessary and sufficient condition for all roots of (2.2) to have negative real parts is given by

$$a \geqslant 0 \quad \text{and} \quad 2a - bc \leqslant 0,$$

or

$$a > 0, \quad 2a - bc > 0 \quad \text{and} \quad \tau < \frac{1}{\sqrt{2a-bc}}\left(2\pi - \arccos\frac{bc-a}{a}\right),$$

or

$$a < 0 \quad \text{and} \quad \tau < -\frac{\sqrt{bc}}{a}.$$

Also, by Lemma 2.1, the necessary and sufficient condition for all roots of (2.3) to have negative real parts is given by

$$a > 0, \quad 2a - bc > 0 \quad \text{and} \quad \frac{\sqrt{bc}}{a} < \tau < \frac{1}{\sqrt{2a-bc}}\arccos\frac{bc-a}{a}.$$

Taking into account that $0 \leqslant \arccos(bc/a - 1) \leqslant \pi$, one can immediately obtain the following result.

PROPOSITION 2.1. *Let $bc \geqslant 0$. Then all roots of* (2.1) *have negative real parts if and only if condition* (1.4) *holds.*

Next, we consider the case of $bc < 0$. Here and hereafter, let $\beta = \sqrt{-bc} > 0$. Then we can easily see that all roots of (2.1) have negative real parts if and only if all roots of

$$f(\lambda) \equiv \lambda + i\beta + a\int_{-\tau}^{0} e^{\lambda s}ds = 0 \tag{2.4}$$

have negative real parts since the function $F(\lambda)$ is written as

$$F(\lambda) = f(\lambda)\overline{f(\bar{\lambda})},$$

where $\bar{\lambda}$ denotes the complex conjugate of any complex λ. Notice that $\lambda = 0$ is not a root of (2.4) because of $f(0) = a\tau + i\beta \neq 0$. Hence, the function $f(\lambda)$ is expressed as

$$f(\lambda) = \lambda + i\beta + \frac{a}{\lambda}(1 - e^{-\lambda\tau}),$$

which implies that equation (2.4) is equivalent to

$$\hat{f}(\lambda) \equiv \lambda^2 + i\beta\lambda + a(1 - e^{-\lambda\tau}) = 0.$$

Since $\hat{f}(\lambda)$ is an analytic function of λ and τ for fixed a and β, one can regard the root $\lambda = \lambda(\tau)$ as a continuous function of τ. The next lemma established by Theorem 2.4 in [3] plays an essential role in our proofs (see also [8, Chapter 3]).

LEMMA 2.2. *As τ varies, the sum of the multiplicities of roots of* (2.4) *in the open right half-plane can change only if a root appears on or crosses the imaginary axis.*

Consequently, we will find the value of τ at which equation (2.4) has roots on the imaginary axis. For the sake of convenience, we define

$$\omega_1 = -\beta, \quad \omega_2 = -\frac{\beta + \sqrt{\beta^2 + 8a}}{2}, \quad \omega_3 = -\frac{\beta - \sqrt{\beta^2 + 8a}}{2}$$

and

$$\tau_{1,n} = -\frac{2n\pi}{\omega_1}, \quad \tau_{2,n} = -\frac{(2n+1)\pi}{\omega_2}, \quad \tau_{3,n} = \operatorname{sgn}(a)\frac{(2n+1)\pi}{\omega_3}$$

for $n = 0, 1, 2, \ldots$. Note that the equality $\omega_j^2 + \beta\omega_j - 2a = 0$ is satisfied for $j = 2, 3$.

LEMMA 2.3. *Let $a \neq 0$. Suppose that $i\omega$ is a root of* (2.4) *with $\tau \geqslant 0$, where ω is a nonzero real number. Then the values of ω and τ are expressed as*

$$\omega = \omega_1 \ \text{ and } \ \tau = \tau_{1,n} \quad \text{for } \ n = 0, 1, 2, \ldots$$

or

$$\omega = \omega_j \ \text{ and } \ \tau = \tau_{j,n} \quad \text{for } \ j = 2, 3 \ \text{ and } \ n = 0, 1, 2, \ldots \quad \text{if } \ \beta^2 + 8a \geqslant 0.$$

Conversely, if $\tau = \tau_{j,n}$ for $j = 1, 2, 3$ and $n = 0, 1, 2, \ldots$, then $i\omega_j$ is a root of (2.4).

Proof. Let $\hat{f}(i\omega) = 0$ with $\omega \neq 0$. Then $-\omega^2 - \beta\omega + a(1 - e^{-i\omega\tau}) = 0$, namely,

$$a\cos\omega\tau = -(\omega^2 + \beta\omega - a) \tag{2.5}$$

and

$$a\sin\omega\tau = 0. \tag{2.6}$$

By squaring both sides of (2.5) and (2.6), and adding them together, we have $a^2 = (\omega^2 + \beta\omega - a)^2$, that is, $(\omega + \beta)(\omega^2 + \beta\omega - 2a) = 0$. In case $\omega + \beta = 0$, equations

(2.5) and (2.6) become $\cos\omega\tau = 1$ and $\sin\omega\tau = 0$, which, together with $\omega = -\beta \equiv \omega_1 < 0$, yield

$$\tau = -\frac{2n\pi}{\omega_1} \equiv \tau_{1,n} \quad \text{for } n = 0,1,2,\ldots.$$

In case $\omega^2 + \beta\omega - 2a = 0$ and $\beta^2 + 8a \geqslant 0$, equations (2.5) and (2.6) become $\cos\omega\tau = -1$ and $\sin\omega\tau = 0$. If

$$\omega = -\frac{\beta + \sqrt{\beta^2 + 8a}}{2} \equiv \omega_2 < 0,$$

then

$$\tau = -\frac{(2n+1)\pi}{\omega_2} \equiv \tau_{2,n} \quad \text{for } n = 0,1,2,\ldots.$$

If

$$\omega = -\frac{\beta - \sqrt{\beta^2 + 8a}}{2} \equiv \omega_3,$$

then $\operatorname{sgn}(\omega_3) = \operatorname{sgn}(a)$, and hence,

$$\tau = \operatorname{sgn}(a)\frac{(2n+1)\pi}{\omega_3} \equiv \tau_{3,n} \quad \text{for } n = 0,1,2,\ldots.$$

Conversely, if $\tau = \tau_{1,n}$, we see that

$$\hat{f}(i\omega_1) = -\omega_1^2 - \beta\omega_1 + a(1 - e^{-i\omega_1\tau_{1,n}}) = -\omega_1^2 - \beta\omega_1 + a(1 - e^{-2n\pi i}) = 0,$$

which implies that $i\omega_1$ is a root of (2.4). Similarly, if $\tau = \tau_{j,n}$ $(j = 2,3)$, one can verify that $i\omega_j$ is a root of (2.4). The proof is complete. $\square$

REMARK 2.1. The root $i\omega_1$ of (2.4) is simple. In fact, since

$$\frac{d\hat{f}(\lambda)}{d\lambda} = 2\lambda + i\beta + a\tau e^{-\lambda\tau} = 2\lambda + i\beta + \tau(\lambda^2 + i\beta\lambda + a),$$

it follows that

$$\frac{d\hat{f}(i\omega)}{d\lambda} = i(2\omega + \beta) - \tau(\omega^2 + \beta\omega - a),$$

which implies $d\hat{f}(i\omega_1)/d\lambda = a\tau - i\beta \neq 0$, and the assertion is verified. Also, if $\beta^2 + 8a > 0$, then the roots $i\omega_2$ and $i\omega_3$ of (2.4) are simple because

$$\frac{d\hat{f}(i\omega_j)}{d\lambda} = (-1)^{j-1}\sqrt{\beta^2 + 8a} \neq 0 \quad \text{for } j = 2,3.$$

On the other hand, if $\beta^2 + 8a = 0$, then the root $i\omega_2$ $(= i\omega_3)$ of (2.4) is double because $d\hat{f}(i\omega_2)/d\lambda = 0$ and $d^2\hat{f}(i\omega_2)/d\lambda^2 = 2 \neq 0$.

Next, we will observe how the roots of (2.4) cross the imaginary axis as τ varies.

LEMMA 2.4. *As τ increases, the purely imaginary roots of* (2.4) *move as follows.*

1. *If $a>0$, then the root $i\omega_1$ crosses the imaginary axis from right to left, while the roots $i\omega_2$ and $i\omega_3$ cross the imaginary axis from left to right.*

2. *If $a<0$, then the root $i\omega_1$ crosses the imaginary axis from left to right. Moreover, if $-\beta^2/8<a<0$, then the root $i\omega_2$ crosses the imaginary axis from right to left, while the root $i\omega_3$ crosses the imaginary axis from left to right.*

Proof. By taking the derivative of λ with respect to τ on (2.4), we have

$$\frac{d\lambda}{d\tau}+\frac{a}{\lambda^2}\left\{\left(\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau}+\lambda e^{-\lambda\tau}\right)\lambda-(1-e^{-\lambda\tau})\frac{d\lambda}{d\tau}\right\}=0,$$

that is,

$$\frac{d\lambda}{d\tau}=-\frac{a\lambda^2e^{-\lambda\tau}}{\lambda^2-a+a(1+\lambda\tau)e^{-\lambda\tau}}=-\frac{a\lambda^2}{(\lambda^2-a)e^{\lambda\tau}+a(1+\lambda\tau)}.$$

This yields that

$$\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega}=-\frac{a\omega^2}{(\omega^2+a)e^{i\omega\tau}-a-ia\omega\tau}. \tag{2.7}$$

In case $\omega=\omega_1$, equality (2.7) with $\omega_1\tau_{1,n}=2n\pi$ leads to

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega_1}=\operatorname{Re}\left(-\frac{a\omega_1^2}{\omega_1^2-ia\omega_1\tau_{1,n}}\right)=-\frac{a\omega_1^4}{\omega_1^4+(a\omega_1\tau_{1,n})^2}.$$

If $a>0$, then $\operatorname{Re}(d\lambda/d\tau)\big|_{\lambda=i\omega_1}<0$, which implies that the root $i\omega_1$ crosses the imaginary axis from right to left as τ increases. On the other hand, if $a<0$, then $\operatorname{Re}(d\lambda/d\tau)\big|_{\lambda=i\omega_1}>0$, which yields that the root $i\omega_1$ crosses the imaginary axis from left to right as τ increases.

In case $\beta^2+8a>0$ and $\omega=\omega_j$ $(j=2,3)$, equality (2.7) with $\omega_j\tau_{j,n}=|(2j+1)\pi|$ leads to

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega_j}=\operatorname{Re}\left(\frac{a\omega_j^2}{\omega_j^2+2a+ia\omega_j\tau_{j,n}}\right)=\frac{a\omega_j^2(\omega_j^2+2a)}{(\omega_j^2+2a)^2+(a\omega_j\tau_{j,n})^2}.$$

If $a>0$, then $\operatorname{Re}(d\lambda/d\tau)\big|_{\lambda=i\omega_j}>0$ for $j=2,3$, which implies that the roots $i\omega_2$ and $i\omega_3$ cross the imaginary axis from left to right as τ increases. Also, if $-\beta^2/8<a<0$, then

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega_2}=\frac{a\omega_2^2\left(\beta^2+8a+\beta\sqrt{\beta^2+8a}\right)}{2\{(\omega_2^2+2a)^2+(a\omega_2\tau_{2,n})^2\}}<0,$$

$$\operatorname{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega_3}=\frac{a\omega_3^2\left(\beta^2+8a-\beta\sqrt{\beta^2+8a}\right)}{2\{(\omega_3^2+2a)^2+(a\omega_3\tau_{3,n})^2\}}>0.$$

This yields that as τ increases under $-\beta^2/8 < a < 0$, the root $i\omega_2$ crosses the imaginary axis from right to left, while the root $i\omega_3$ crosses the imaginary axis from left to right. This completes the proof. $\square$

We are now ready to present necessary and sufficient conditions for all roots of (2.1) to have negative real parts when $a \neq 0$, $bc < 0$ and $8a \neq bc$ (Propositions 2.2 and 2.3).

PROPOSITION 2.2. *Let* $a > 0$ *and* $bc < 0$. *Then all roots of* (2.1) *have negative real parts if and only if condition* (1.5) *holds.*

Proof. It suffices to verify that under $a > 0$ and $\beta = \sqrt{-bc} > 0$, all roots of (2.4) lie in the left half-plane if and only if condition (1.5) holds. To this end, we will examine the locations of roots of (2.4) by taking τ as a parameter.

For the sake of brevity, we denote by $\nu(\tau)$ the number of roots of (2.4) including multiplicity whose real parts are positive at τ. Clearly, for $\tau = 0$, equation (2.4) has the only root $-i\beta$ $(= i\omega_1)$, and hence, $\nu(0) = 0$. Lemma 2.4 (i) then asserts that $i\omega_1$ moves in the left half-plane as τ increases from 0, which yields $\nu(\tau) = 0$ for all sufficiently small $\tau > 0$ by the continuity of the roots with respect to τ. By virtue of Lemma 2.2, we will focus on purely imaginary roots of (2.4) to determine the value of $\nu(\tau)$.

By Lemma 2.3, if equation (2.4) has a purely imaginary root $i\omega$, then $\omega = \omega_j$ for some $j = 1,2,3$. Let $\lambda_{j,n}(\tau)$ be the branch of the root of (2.4) satisfying $\lambda_{j,n}(\tau_{j,n}) = i\omega_j$ for $j = 1,2,3$ and $n = 0,1,2,\ldots$. Then, as shown in Lemma 2.4 (i), $\lambda_{1,n}(\tau)$ moves in the left half-plane as τ increases from $\tau_{1,n}$, while for $j = 2,3$, $\lambda_{j,n}(\tau)$ moves in the right half-plane as τ increases from $\tau_{j,n}$. This implies that $\lambda_{2,n}(\tau)$ or $\lambda_{3,n}(\tau)$ may cross the imaginary axis at $\lambda = i\omega_1$ from right to left as τ increases. Consequently, we will observe the order relation between $\tau_{1,n}$, $\tau_{2,n}$, and $\tau_{3,n}$ in detail. An easy calculation yields that for $n = 0,1,2,\ldots$,

$$\tau_{3,n} - \tau_{2,n} = \frac{(2n+1)\pi\beta}{2a} > 0,$$

$$\tau_{2,n} - \tau_{1,n} = \frac{2\pi(\sqrt{\beta^2+8a}-\beta)}{\beta(\sqrt{\beta^2+8a}+\beta)}\left(\frac{\beta}{\sqrt{\beta^2+8a}-\beta} - n\right), \tag{2.8}$$

$$\tau_{3,0} - \tau_{2,n} = \frac{\pi(\sqrt{\beta^2+8a}-\beta)}{2a}\left(\frac{\beta}{\sqrt{\beta^2+8a}-\beta} - n\right). \tag{2.9}$$

Also, since

$$\tau_{2,n+1} - \tau_{2,n} = \frac{2\pi}{|\omega_2|} < \frac{2\pi}{|\omega_1|} = \tau_{1,n+1} - \tau_{1,n} \quad \text{for } n = 0,1,2,\ldots,$$

there exists a positive integer $N_1(n)$ such that

$$\tau_{1,n} < \tau_{2,N_1(n)} < \tau_{1,n+1}. \tag{2.10}$$

We consider the following two cases. For simplicity, let k be a nonnegative integer defined by

$$k = \left\lceil \frac{\beta}{\sqrt{\beta^2+8a}-\beta} \right\rceil - 1,$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

Case (i): $a \neq (2m+1)\beta^2/(8m^2)$ for any $m = 1,2,3,\ldots$. Since $\beta/(\sqrt{\beta^2+8a}-\beta)$ is not an integer, equalities (2.8) and (2.9) imply that

$$\begin{aligned} &\tau_{2,n} - \tau_{1,n} > 0 \quad \text{and} \quad \tau_{3,0} - \tau_{2,n} > 0 \quad \text{for } n = 0,1,\ldots,k, \\ &\tau_{2,n} - \tau_{1,n} < 0 \quad \text{and} \quad \tau_{3,0} - \tau_{2,n} < 0 \quad \text{for } n = k+1, k+2, \ldots, \end{aligned}$$

namely,

$$0 = \tau_{1,0} < \tau_{2,0} < \cdots < \tau_{1,k} < \tau_{2,k} < \tau_{3,0} < \tau_{2,k+1} < \tau_{1,k+1} < \cdots.$$

From this, together with (2.10) and the crossing of the imaginary axis, we thus obtain

$$\begin{cases} \nu(\tau) = 0 & \text{if } \tau \in (0,\tau_{2,0}) \cup (\tau_{1,1},\tau_{2,1}) \cup \cdots \cup (\tau_{1,k},\tau_{2,k}), \\ \nu(\tau) = 1 & \text{if } \tau \in (\tau_{2,0},\tau_{1,1}) \cup (\tau_{2,1},\tau_{1,2}) \cup \cdots \cup (\tau_{2,k-1},\tau_{1,k}), \\ \nu(\tau) \geqslant 1 & \text{if } \tau \in (\tau_{2,k},\infty). \end{cases} \tag{2.11}$$

Case (ii): $a = (2m+1)\beta^2/(8m^2)$ for some $m = 1,2,3,\ldots$. By $\beta/(\sqrt{\beta^2+8a}-\beta) = m$, equalities (2.8) and (2.9) yield that

$$0 = \tau_{1,0} < \tau_{2,0} < \cdots < \tau_{1,k} < \tau_{2,k} < \tau_{3,0} = \tau_{2,k+1} = \tau_{1,k+1} < \tau_{2,k+2} < \cdots.$$

From this, together with (2.10) and the crossing of the imaginary axis, we obtain (2.11).

By virtue of the preceding argument and Lemma 2.3, we therefore conclude that under $a > 0$ and $\beta > 0$, all roots of (2.4) lie in the left half-plane if and only if condition (1.5) holds. $\square$

PROPOSITION 2.3. *Let $a < 0$, $bc < 0$ and $8a \neq bc$. Then all roots of* (2.1) *have negative real parts if and only if condition* (1.6) *holds.*

Proof. It suffices to verify that under $a < 0$, $\beta = \sqrt{-bc} > 0$ and $\beta^2 + 8a \neq 0$, all roots of (2.4) lie in the left half-plane if and only if condition (1.6) holds.

For the sake of convenience, we denote by $\nu(\tau)$ the number of roots of (2.4) including multiplicity whose real parts are positive at τ. Recall that for $\tau = 0$, equation (2.4) has the only root $i\omega_1$, and hence, $\nu(0) = 0$. Lemma 2.4 (ii) then asserts that $i\omega_1$ moves in the right half-plane as τ increases from 0, which implies $\nu(\tau) = 1$ for all sufficiently small $\tau > 0$ by the continuity of the roots with respect to τ. By virtue of Lemma 2.2, we will focus on purely imaginary roots of (2.4) to determine the value of $\nu(\tau)$. Our argument is divided into two cases.

Case (I): $\beta^2 + 8a < 0$. By Lemma 2.3, if equation (2.4) has a purely imaginary root $i\omega$, then $\omega = \omega_1$. Let $\lambda_{1,n}(\tau)$ be the branch of the root of (2.4) satisfying $\lambda_{1,n}(\tau_{1,n}) =$

$i\omega_1$. Lemma 2.4 (ii) shows that $\lambda_{1,n}(\tau)$ moves in the right half-plane and cannot move in the left half-plane crossing on the imaginary axis as τ increases from $\tau_{1,n}$. This yields $\nu(\tau) \geqslant 1$ for all $\tau > 0$.

Case (II): $\beta^2 + 8a > 0$. By Lemma 2.3, if equation (2.4) has a purely imaginary root $i\omega$, then $\omega = \omega_j$ for some $j = 1,2,3$. Let $\lambda_{j,n}(\tau)$ be the branch of the root of (2.4) satisfying $\lambda_{j,n}(\tau_{j,n}) = i\omega_j$ for $j = 1,2,3$ and $n = 0,1,2,\dots$. Then, as shown in Lemma 2.4 (ii), $\lambda_{2,n}(\tau)$ moves in the left half-plane as τ increases from $\tau_{2,n}$, while for $j = 1,3$, $\lambda_{j,n}(\tau)$ moves in the right half-plane as τ increases from $\tau_{j,n}$. This implies that $\lambda_{1,n}(\tau)$ or $\lambda_{3,n}(\tau)$ may cross the imaginary axis at $\lambda = i\omega_2$ from right to left as τ increases. Consequently, we will observe the order relation between $\tau_{1,n}$, $\tau_{2,n}$, and $\tau_{3,n}$ in detail. An easy calculation yields that for $n = 0,1,2,\dots$,

$$\tau_{3,n} - \tau_{2,n} = -\frac{(2n+1)\pi\sqrt{\beta^2+8a}}{2a} > 0,$$

$$\tau_{2,n} - \tau_{1,n} = \frac{2\pi\{\beta + n(\beta - \sqrt{\beta^2+8a})\}}{\beta(\beta+\sqrt{\beta^2+8a})} > 0,$$

$$\tau_{1,n+1} - \tau_{2,n} = \frac{2\pi(\beta - \sqrt{\beta^2+8a})}{\beta(\beta+\sqrt{\beta^2+8a})}\left(\frac{\sqrt{\beta^2+8a}}{\beta-\sqrt{\beta^2+8a}} - n\right), \tag{2.12}$$

$$\tau_{3,0} - \tau_{1,n+1} = \frac{2\pi}{\beta}\left(\frac{\sqrt{\beta^2+8a}}{\beta-\sqrt{\beta^2+8a}} - n\right). \tag{2.13}$$

Also, since

$$\tau_{1,n+1} - \tau_{1,n} = \frac{2\pi}{|\omega_1|} < \frac{2\pi}{|\omega_2|} = \tau_{2,n+1} - \tau_{2,n} \quad \text{for } n = 0,1,2,\dots,$$

there exists a positive integer $N_2(n)$ such that

$$\tau_{2,n} < \tau_{1,N_2(n)} < \tau_{2,n+1}. \tag{2.14}$$

We consider the following two subcases. For brevity, let l be a nonnegative integer defined by

$$l = \left\lceil \frac{\sqrt{\beta^2+8a}}{\beta-\sqrt{\beta^2+8a}} \right\rceil - 1.$$

Subcase (II-i): $a \neq -(2m+1)\beta^2/(8(m+1)^2)$ for any $m = 1,2,3,\dots$. Since $\sqrt{\beta^2+8a}/(\beta - \sqrt{\beta^2+8a})$ is not an integer, equalities (2.12) and (2.13) imply that

$$\begin{aligned} &\tau_{1,n+1} - \tau_{2,n} > 0 \quad \text{and} \quad \tau_{3,0} - \tau_{1,n+1} > 0 \quad \text{for } n = 0,1,\dots,l, \\ &\tau_{1,n+1} - \tau_{2,n} < 0 \quad \text{and} \quad \tau_{3,0} - \tau_{1,n+1} < 0 \quad \text{for } n = l+1, l+2,\dots, \end{aligned}$$

namely,

$$0 < \tau_{2,0} < \tau_{1,1} < \cdots < \tau_{2,l} < \tau_{1,l+1} < \tau_{3,0} < \tau_{1,l+2} < \tau_{2,l+1} < \cdots.$$

From this, together with (2.14) and the crossing of the imaginary axis, we thus obtain

$$\begin{cases} \nu(\tau)=1 & \text{if } \tau\in(0,\tau_{2,0})\cup(\tau_{1,1},\tau_{2,1})\cup\cdots\cup(\tau_{1,l},\tau_{2,l}),\\ \nu(\tau)=0 & \text{if } \tau\in(\tau_{2,0},\tau_{1,1})\cup(\tau_{2,1},\tau_{1,2})\cup\cdots\cup(\tau_{2,l},\tau_{1,l+1}),\\ \nu(\tau)\geqslant 1 & \text{if } \tau\in(\tau_{1,l+1},\infty). \end{cases} \tag{2.15}$$

Subcase (II-ii): $a=-(2m+1)\beta^2/(8(m+1)^2)$ for some $m=1,2,3,\ldots$. By $\sqrt{\beta^2+8a}/(\beta-\sqrt{\beta^2+8a})=m$, equalities (2.12) and (2.13) yield that

$$0<\tau_{2,0}<\tau_{1,1}<\cdots<\tau_{2,l}<\tau_{1,l+1}<\tau_{3,0}=\tau_{1,l+2}=\tau_{2,l+1}<\cdots.$$

From this, together with (2.14) and the crossing of the imaginary axis, we obtain (2.15).

By virtue of the preceding argument and Lemma 2.3, we therefore conclude that under $a<0$, $\beta>0$ and $\beta^2+8a\neq 0$, all roots of (2.4) lie in the left half-plane if and only if condition (1.6) holds. $\square$

Finally, we will prove our main theorem.

PROOF OF THEOREM 1.1. Suppose that $8a\neq bc$ when $bc<0$. In case $a=0$ and $bc<0$, equation (2.1) has the roots $\pm i\sqrt{-bc}$ whose real parts are zero. This fact and Propositions 2.1–2.3 show that all roots of (2.1) have negative real parts if and only if any one of conditions (1.4)–(1.6) holds. $\square$

REMARK 2.2. As mentioned in Remark 2.1, in case $8a=bc<0$, the root $i\omega_2$ $(=i\omega_3)$ of (2.4) is double, and hence, we cannot analyze the behavior of the root $i\omega_2$ by using the derivative $\mathrm{Re}\,(d\lambda/d\tau)\big|_{\lambda=i\omega_2}$. In the critical case, we believe that equation (2.4) has a root with nonnegative real part for all $\tau>0$ and as a result, the assumption that $8a\neq bc$ when $bc<0$ in Theorem 1.1 might be dropped.

REFERENCES

[1] K. L. COOKE AND Z. GROSSMAN, *Discrete delay, distributed delay and stability switches*, J. Math. Anal. Appl., **86** (1982), 592–627.

[2] J. C. F. DE OLIVEIRA AND L. A. V. CARVALHO, *A Lyapunov functional for a retarded differential equation*, SIAM. J. Math. Anal., **16** (1985), 1925–1305.

[3] H. I. FREEDMAN AND Y. KUANG, *Stability switches in linear scalar neutral delay equations*, Funkcial. Ekvac., **34** (1991), 187–209.

[4] M. FUNAKUBO, T. HARA AND S. SAKATA, *On the uniform asymptotic stability for a linear integro-differential equation of Volterra type*, J. Math. Anal. Appl., **324** (2006), 1036–1049.

[5] J. K. HALE AND S. M. V. LUNEL, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[6] T. HARA AND S. SAKATA, *An application of the Hurwitz theorem to the root analysis of the characteristic equation*, Appl. Math. Lett., **24** (2011), 12–15.

[7] C.-H. HSU, S.-Y. YANG, T.-H. YANG AND T.-S. YANG, *Stability and bifurcation of a two-neuron network with distributed time delays*, Nonlinear Anal. Real World Appl., **11** (2010), 1472–1490.

[8] Y. KUANG, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.

[9] H. MATSUNAGA, *Stability switches in a system of linear differential equations with diagonal delay*, Appl. Math. Comput., **212** (2009), 145–152.

[10] R. MIYAZAKI, *Characteristic equation and asymptotic behavior of delay-differential equation*, Funkcial. Ekvac., **40** (1997), 471–481.

[11] R. MIYAZAKI, *Analysis of characteristic roots of delay-differential systems*, Dynam. Contin. Discrete Impuls. Systems, **5** (1999), 195–207.

[12] S. SAKATA AND T. HARA, *Stability regions for linear differential equations with two kinds of time lags*, Funkcial. Ekvac., **47** (2004), 129–144.

[13] E. SCHÖLL, P. HÖVEL, V. FLUNKERT AND M. A. DAHLEM, *Time-delayed feedback control: from simple models to lasers and neural systems*, Complex Time-Delay Systems, F. M. Atay (ed.) pp. 85–150, Underst. Complex Syst., Springer, Berlin, 2010.

[14] G. STÉPÁN, *Retarded Dynamical Systems: Stability and Characteristic Functions*, Longman Scientific and Technical, U.K., 1989.

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