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EXISTENCE OF POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS WITH SOBOLEV CRITICAL EXPONENTS

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Abstract. In this paper, we consider the existence of positive solutions to the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \frac{\partial F}{\partial u}(u,v) + \varepsilon^{p-1}g(x) & \text{in } \Omega, \\ -\mathrm{div}(|\nabla v|^{q-2}\nabla v) = \frac{\partial F}{\partial v}(u,v) + \varepsilon^{q-1}h(x) & \text{in } \Omega, \\ u,v > 0 \ \text{ in } \Omega, \\ u = v = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$; $F \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ is positively homogeneous of degree μ; $g,h \in C^1(\overline{\Omega})\backslash\{0\}$; and ε is a positive parameter. Using sub-supersolution method and comparison principle, we prove the existence of positive solutions for the above problem.

1. Introduction and main results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We are concerned with the following problem:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \frac{\partial F}{\partial u}(u,v) + \varepsilon^{p-1}g(x) & \text{in } \Omega, \\ -\mathrm{div}(|\nabla v|^{q-2}\nabla v) = \frac{\partial F}{\partial v}(u,v) + \varepsilon^{q-1}h(x) & \text{in } \Omega, \\ u,v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $g,h \in C^1(\overline{\Omega})\backslash\{0\}$, $p,q \in \mathbb{R}$ such that $p > 1$ and $q > 1$, the parameter ε is positive, and $F \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ is positively homogeneous of degree μ, that is, $F(tz) = t^\mu F(z)$ holds for all $z \in (\mathbb{R}^+)^2$ and $t > 0$, here, $\mathbb{R}^+ = [0, +\infty)$.

Systems of the above form are mathematical models occurring in studies of the (p,q)-Laplace system, generalized reaction-diffusion theory, non-Newtonian fluid theory ([4],[34]), non-Newtonian filtration ([29]) and the turbulent flow of a gas in porous medium ([19]). In the non-Newtonian fluid theory, the quantity (p,q) is characteristic of the medium. Media with $(p,q) > (2,2)$ are called dilatant fluids and those with $(p,q) < (2,2)$ are called pseudoplastics. If $(p,q) = (2,2)$, they are Newtonian fluids.

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In recent years, the existence and uniqueness of the positive solutions for the single quasilinear elliptic equation with eigenvalue problems

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 \ \text{ in } \Omega, \\ u(x) = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{2}$$

with $\lambda > 0, p > 1, \Omega \subset \mathbb{R}^N, N \geqslant 2$, have been studied by many authors, see [23]-[27], [33], [35], [48]-[55] and the references therein. When f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leqslant \alpha_1 + \alpha_2 s^\mu, 0 < \mu < p-1, \alpha_1, \alpha_2 > 0$, it was shown in [25] that there exist at least two positive solutions for the problem (2) when λ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [26] that the problem (2) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [24] that problem (2) has a unique positive large solution and at least one positive small solution when λ is large if f is nondecreasing; there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leqslant \alpha_1 + \alpha_2 s^\beta, 0 < \beta < p-1; \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$, and there exist $T, Y > 0$ with $Y \geqslant T$ such that

$$(f(s)/s^{p-1})' > 0 \ \text{ for } \ s \in (0,T)$$

and

$$(f(s)/s^{p-1})' < 0 \ \text{ for } \ s > Y.$$

Recently, Hai [27] considered the case when Ω is an annular domain, and obtained the existence of positive large solutions for the problem (2) when λ is sufficiently small. Xuan & Chen proved in [47] that the singular problem (2) has a unique positive radial solution if f is a continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball). The existence of entire solutions have been obtained for singular and non-singular problem (2), see [26], [53], [55]. For $p = 2$, the related results to a singular semilinear elliptic boundary value problem

$$\begin{cases} \Delta u + \lambda m(x) u^\gamma = 0, \ \ x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

have been extensively studied when $\Omega \subset \mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, see [16], [18], [30]. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be founded in [24]-[26].

Since 1980s, many important results have been obtained for quasilinear elliptic systems. We will introduce some results in the following. Existence and non-existence of solutions of the quasilinear elliptic system

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u,v) = 0, \ \ x \in \mathbb{R}^N, \\ \operatorname{div}(|\nabla v|^{q-2}\nabla v) + g(u,v) = 0, \ \ x \in \mathbb{R}^N, \end{cases} \tag{3}$$

have gained much attention recently. See, for example, [10], [20], [23], [38], [51], [54].

When $p = q = 2$, system (3) becomes

$$\begin{cases} \Delta u + f(u,v) = 0, & x \in \mathbb{R}^N, \\ \Delta v + g(u,v) = 0, & x \in \mathbb{R}^N, \end{cases}$$

for which the existence and the non-existence of positive solutions and positive boundary blow-up solutions have been investigated extensively. We list here, for example, [9], [11], [31], [36], [37], [42], and refer to the references therein.

When $p = q = 2, f = -a(|x|)v^\alpha, g = -b(|x|)u^\beta$, system (3) becomes

$$\begin{cases} \Delta u = a(|x|)v^\alpha, & x \in \mathbb{R}^N, \\ \Delta v = b(|x|)u^\beta, & x \in \mathbb{R}^N, \end{cases} \tag{4}$$

for which existence results for positive boundary blow-up solutions can be found in a recent paper by Lair and Wood [31]. Lair and Wood established that all positive entire radial solutions of (4) are boundary blow-up provided that

$$\int_0^\infty ta(t)dt = \infty, \quad \int_0^\infty tb(t)dt = \infty.$$

On the other hand, if

$$\int_0^\infty ta(t)dt < \infty \ \text{ and } \ \int_0^\infty tb(t)dt < \infty,$$

then all positive entire radial solutions of (4) are bounded.

F. Cirstea and V. D. Radulescu [11], extended the above results to a larger class of systems

$$\begin{cases} \Delta u = a(|x|)g(v), & x \in \mathbb{R}^N, \\ \Delta v = b(|x|)f(u), & x \in \mathbb{R}^N. \end{cases}$$

Z. D. Yang [51], extended the above results to a class of systems

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(|x|)g(v), & x \in \mathbb{R}^N, \\ \operatorname{div}(|\nabla v|^{q-2}\nabla v) = b(|x|)f(u), & x \in \mathbb{R}^N. \end{cases}$$

For $p = q = 2$, system (3) becomes

$$\begin{cases} -\Delta u = \frac{\partial F}{\partial u}(u,v) + \varepsilon g(x) & \text{in } \Omega, \\ -\Delta v = \frac{\partial F}{\partial v}(u,v) + \varepsilon g(x) & \text{in } \Omega, \\ u,v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

for which the existence and multiplicity of positive solutions for semilinear elliptic problems have been investigated extensively. Results relating to these problems can be find in [1], [5], [7]-[8], [12], [14]-[15], [21], [28], [45], and the references therein.

In a recent paper, Chu and Tang [8] have studied the following systems

$$\begin{cases} -\Delta u = \frac{\partial F}{\partial u}(u,v) + \varepsilon g(x) & \text{in } \Omega, \\ -\Delta v = \frac{\partial F}{\partial v}(u,v) + \varepsilon g(x) & \text{in } \Omega, \\ u,v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $g,h \in C^1(\overline{\Omega})\backslash\{0\}$; ε is a positive parameter, and $F \in C^1((\mathbb{R}^+)^2,\mathbb{R}^+)$ is positively homogeneous of degree μ, that is $F(tz) = t^\mu F(z)$ holds for all $z \in (\mathbb{R}^+)^2$ and $t > 0$, here, $\mathbb{R}^+ = [0,+\infty)$. By means of sub-supersolution method, they have proved that if $\mu \in (1,2)$, (f_1) and (f_2) (the same as the following assumptions) hold, then the problem (5) has at least one positive solution for all $\varepsilon > 0$ if and only if both problems

$$\begin{cases} -\Delta u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

and

$$\begin{cases} -\Delta v = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

have nonnegative solutions.

Motivated by the results of papers [1], [5], [7]-[8], [12], [14]-[15], [21], [28], [45], in this paper, we consider the quasilinear elliptic system (1). We modify the method developed by Chu and Tang [8], Han [3] and extend the results of [8] to a quasilinear elliptic system (1).

The outline of this paper is as following. In section 2, we investigate the existence of solution for single equation of singular quasilinear elliptic system (1). Section 3 is devoted to the existence of solutions to system (1).

Before stating our results, we need to give some assumptions. Let $E = H_0^1(\Omega) \times H_0^1(\Omega)$, $z = (u,v), |z| = \sqrt{u^2+v^2}, \|z\|_s = (\int_\Omega (u^2+v^2)^{\frac{s}{2}} dx)^{\frac{1}{s}}$. Let

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

and

$$\begin{cases} -\operatorname{div}(|\nabla v|^{q-2}\nabla v) = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

(f_1) $g,h \in C(\overline{\Omega})\backslash\{0\}$, and there exists $x_0 \in \Omega$ such that $g(x_0) > 0, h(x_0) > 0$.

(f_2) $\frac{\partial F}{\partial u}(u,v), \frac{\partial F}{\partial v}(u,v)$ are strictly increasing functions about u and v for all $u,v > 0$.

In addition, we denote positive constants by $C, C_1, C_2, \ldots$.

The main results of the paper are the following theorems.

THEOREM 1. *Suppose that $F \in C^1((\mathbb{R}^+)^2,\mathbb{R}^+)$ is positively homogeneous of degree $\mu > 1$. Assume that (f_1), (f_2) hold, and $1 < \mu < \min\{p,q\}$, then problem* (1) *has at least one positive solution for all $\varepsilon > 0$ if both problems* (8) *and* (9) *have nonnegative solutions.*

For $p = q$, we give the following theorems.

THEOREM 2. *Suppose that $F \in C^1((\mathbb{R}^+)^2,\mathbb{R}^+)$ is positively homogeneous of degree $\mu > 1$. Assume that (f_1), (f_2) hold, and $1 < \mu < p$, if problem* (1) *has at least one positive solution for all $\varepsilon > 1$, then both problems* (8) *and* (9) *have nonnegative solutions.*

In particular, for the supercritical case, the condition that problem (8) and problem (9) have nonnegative solutions are also a necessary condition that guarantees the existence of positive solutions for problem (1). In fact,we have

THEOREM 3. *Suppose that* $F \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ *is positively homogeneous of degree* $\mu > 2^*, (2^* = \frac{pN}{N-p})$. *Let* $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ *be a star-shaped domain,* (f_1), (f_2) *hold, and* $1 < p < \mu$, *if problem* (1) *has at least one positive solution for all* $\varepsilon > 1$, *then both problems* (8) *and* (9) *have nonnegative solutions.*

REMARK 1. If $p \geqslant N$, we do not know whether or not Theorem 3 holds. From [56], we guess problem (1) has no positive solution for $p \geqslant N$. This is an open problem.

2. The existence of positive solutions for problem (1)

In this section, we will prove Theorems 1-3. It is well known that the following lemma holds. From [8], we give the following lemma.

LEMMA 1. *Suppose that* (f_2) *holds. Assume that* $F \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ *is positively homogeneous of degree* μ *with* $\mu > 1$, *then* $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \in C((\mathbb{R}^+)^2, \mathbb{R}^+)$ *are positively homogeneous of degree* $\mu - 1$.

From [27] and [42], we give the following lemma.

LEMMA 2. (Weak comparison principle) *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ $(N \geqslant 2)$ *with smooth boundary* $\partial\Omega$ *and* $\theta : (0,\infty) \to (0,\infty)$ *be continuous and nondecreasing function. Let* u_1, $u_2 \in W^{1,p}(\Omega)$ *be such that*

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla \psi dx + \int_\Omega \theta(u_1)\psi dx \leqslant \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla \psi dx + \int_\Omega \theta(u_2)\psi dx$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$. *If*

$$u_1 \leqslant u_2 \quad on \quad \partial\Omega,$$

then

$$u_1 \leqslant u_2 \quad in \quad \Omega.$$

PROOF OF THEOREM 1. First, let u_0, v_0 be nonnegative solutions of problem (8) and (9) , respectively. It implies from Lemma 1 that $(\varepsilon u_0, \varepsilon v_0)$ satisfies:

$$\begin{aligned}
-\mathrm{div}(|\nabla(\varepsilon u_0)|^{p-2}\nabla(\varepsilon u_0)) &= -\varepsilon^{p-1}\mathrm{div}(|\nabla u_0|^{p-2}\nabla u_0) \\
&= \varepsilon^{p-1} g(x) \\
&\leqslant \frac{\partial F}{\partial u}(\varepsilon u_0, \varepsilon v_0) + \varepsilon^{p-1} g(x),
\end{aligned}$$

and

$$\begin{aligned}
-\operatorname{div}(|\nabla(\varepsilon v_0)|^{q-2}\nabla(\varepsilon v_0)) &= -\varepsilon^{q-1}\operatorname{div}(|\nabla v_0|^{q-2}\nabla v_0) \\
&= \varepsilon^{q-1}h(x) \\
&\leqslant \frac{\partial F}{\partial v}(\varepsilon u_0, \varepsilon v_0) + \varepsilon^{q-1}h(x)
\end{aligned}$$

for all $\varepsilon > 0$. Hence, $(\varepsilon u_0, \varepsilon v_0)$ is a subsolution of problem (1). In addition, let d and e denote the solutions of the following problem, and (10) respectively:

$$\begin{cases} -\operatorname{div}(|\nabla d|^{p-2}\nabla d) = 1 & \text{in } \Omega, \\ d = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\operatorname{div}(|\nabla e|^{q-2}\nabla e) = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

It follows from the strong maximum principle (see Serrin and Zou [43]) that $d(x) > 0$, $e(x) > 0$ in Ω. Choose R_0 so large that

$$\begin{aligned}
R_0^{p-1} &\geqslant R_0^{\mu-1} \max_{x\in\overline{\Omega}} \frac{\partial F}{\partial u}(d,e) + \varepsilon^{p-1} \max_{x\in\overline{\Omega}} |g|, \\
R_0^{q-1} &\geqslant R_0^{\mu-1} \max_{x\in\overline{\Omega}} \frac{\partial F}{\partial v}(d,e) + \varepsilon^{q-1} \max_{x\in\overline{\Omega}} |h|.
\end{aligned}$$

This is possible, since $1 < \mu < p$, $1 < \mu < q$ and

$$\begin{aligned}
R^{1-p}\Big(R^{\mu-1} \max_{x\in\overline{\Omega}} \frac{\partial F}{\partial u}(d,e) + \varepsilon^{p-1} \max_{x\in\overline{\Omega}} |g|\Big) &\to 0, \\
R^{1-q}\Big(R^{\mu-1} \max_{x\in\overline{\Omega}} \frac{\partial F}{\partial v}(d,e) + \varepsilon^{q-1} \max_{x\in\overline{\Omega}} |h|\Big) &\to 0,
\end{aligned}$$

as $R \to +\infty$. Set $\overline{\omega}_1 = R_0 d$, $\overline{\omega}_2 = R_0 e$, we have

$$\begin{aligned}
&-\operatorname{div}(|\nabla\overline{\omega}_1)|^{p-2}\nabla\overline{\omega}_1)) - \frac{\partial F}{\partial u}(\overline{\omega}_1, \overline{\omega}_2) - \varepsilon^{p-1}g(x) \\
&\quad = -\operatorname{div}(|\nabla(R_0 d))|^{p-2}\nabla(R_0 d)) - R_0^{\mu-1}\frac{\partial F}{\partial u}(d,e) - \varepsilon^{p-1}g(x) \\
&\quad = R_0^{p-1} - R_0^{\mu-1}\frac{\partial F}{\partial u}(d,e) - \varepsilon^{p-1}g(x) \geqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
&-\operatorname{div}(|\nabla\overline{\omega}_2)|^{q-2}\nabla\overline{\omega}_2)) - \frac{\partial F}{\partial v}(\overline{\omega}_1, \overline{\omega}_2) - \varepsilon^{q-1}h(x) \\
&\quad = -\operatorname{div}(|\nabla(R_0 e))|^{q-2}\nabla(R_0 e)) - R_0^{\mu-1}\frac{\partial F}{\partial v}(d,e) - \varepsilon^{q-1}h(x) \\
&\quad = R_0^{q-1} - R_0^{\mu-1}\frac{\partial F}{\partial v}(e,e) - \varepsilon^{q-1}h(x) \geqslant 0.
\end{aligned}$$

These imply that $(\overline{\omega}_1,\overline{\omega}_2)$ is a supersolution of problem (1). According to $\overline{\omega}_1 \geqslant 0$, $\overline{\omega}_2 \geqslant 0$ and Lemma 1, we have

$$\begin{cases} -\mathrm{div}(|\nabla\overline{\omega}_1)|^{p-2}\nabla\overline{\omega}_1)) - (-\mathrm{div}(|\nabla(\varepsilon u_0)|^{p-2}\nabla(\varepsilon u_0)) \\ \qquad\qquad \geqslant \frac{\partial F}{\partial u}(\overline{\omega}_1,\overline{\omega}_2) + \varepsilon^{p-1}g(x) - \varepsilon^{p-1}g(x) \geqslant 0 \quad \text{in } \Omega, \\ -\mathrm{div}(|\nabla\overline{\omega}_2)|^{q-2}\nabla\overline{\omega}_2)) - (-\mathrm{div}(|\nabla(\varepsilon v_0)|^{q-2}\nabla(\varepsilon v_0)) \\ \qquad\qquad \geqslant \frac{\partial F}{\partial v}(\overline{\omega}_1,\overline{\omega}_2) + \varepsilon^{q-1}h(x) - \varepsilon^{q-1}h(x) \geqslant 0 \quad \text{in } \Omega, \\ \overline{\omega}_1 = \overline{\omega}_2 = \varepsilon u_0 = \varepsilon v_0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By Lemma 2, we have $0 \leqslant \varepsilon u_0 \leqslant \overline{\omega}_1$, $0 \leqslant \varepsilon v_0 \leqslant \overline{\omega}_2$. By $C^{1,\alpha}(\overline{\Omega})$ estimates in [32] and monotonic iteration in [2] or [41], we conclude that problem (1) has at least one positive solution $(u_\varepsilon^*, v_\varepsilon^*)$ which satisfies $0 \leqslant \varepsilon u_0 \leqslant u_\varepsilon^* \leqslant \overline{\omega}_1$, $0 \leqslant \varepsilon v_0 \leqslant v_\varepsilon^* \leqslant \overline{\omega}_2$.

It implies from (f_1) that there exists $\Omega_0 = B(x_0) \subset\subset \Omega$ such that $g(x) > 0$, $h(x) > 0$ in Ω_0. Moreover, u_0 and v_0 satisfy

$$\begin{cases} -\mathrm{div}(|\nabla u_0|^{p-2}\nabla u_0) = g(x) \quad \text{in } \Omega_0, \\ u_0 \geqslant 0 \quad \text{on } \partial\Omega_0, \end{cases}$$

and

$$\begin{cases} -\mathrm{div}(|\nabla v_0|^{q-2}\nabla v_0) = h(x) \quad \text{in } \Omega_0, \\ v_0 \geqslant 0 \quad \text{on } \partial\Omega_0. \end{cases}$$

By the strong maximum principle (see Serrin and Zou [43] or Vazquez's [46]), we have $u_0 > 0, v_0 > 0$ in Ω_0. We infer from (f_2) and Lemma 1 that

$$\frac{\partial F}{\partial u}(\varepsilon u_0, \varepsilon v_0) > 0, \quad \frac{\partial F}{\partial v}(\varepsilon u_0, \varepsilon v_0) > 0$$

in Ω_0. Hence we conclude that $(\varepsilon u_0, \varepsilon v_0)$ is not the solution of the problem (1). So, we have $u_\varepsilon^* > \varepsilon u_0 \geqslant 0, v_\varepsilon^* > \varepsilon v_0 \geqslant 0$ in Ω, that is, $(u_\varepsilon^*, v_\varepsilon^*)$ is a positive solution of problem (1). The proof of Theorem 1 is completed. □

PROOF OF THEOREM 2. Assume that problem (1) has at least one positive solution for all $\varepsilon > 1$, we shall prove that problems (8) and (9) have nonnegative solutions. To this end, let $(\overline{u}_\varepsilon, \overline{v}_\varepsilon)$ be any positive solution of problem (1) with respect to the parameter ε. Set $\omega_{1\varepsilon} = \varepsilon^{-1}\overline{u}_\varepsilon$, $\omega_{2\varepsilon} = \varepsilon^{-1}\overline{v}_\varepsilon$, then $(\omega_{1\varepsilon}, \omega_{2\varepsilon})$ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla\omega_{1\varepsilon}|^{p-2}\nabla\omega_{1\varepsilon}) = \varepsilon^{\mu-p}\frac{\partial F}{\partial u}(\omega_{1\varepsilon},\omega_{2\varepsilon}) + g(x) \quad \text{in } \Omega, \\ -\mathrm{div}(|\nabla\omega_{2\varepsilon}|^{p-2}\nabla\omega_{2\varepsilon}) = \varepsilon^{\mu-p}\frac{\partial F}{\partial v}(\omega_{1\varepsilon},\omega_{2\varepsilon}) + h(x) \quad \text{in } \Omega, \\ \omega_{1\varepsilon}, \omega_{2\varepsilon} > 0 \quad \text{in } \Omega, \\ \omega_{1\varepsilon} = \omega_{2\varepsilon} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{11}$$

Since $F \in ((R^+)^2, R^+)$ is positively homogeneous of degree μ, we have

$$m|z|^\mu \leqslant F(z) \leqslant M|z|^\mu,$$

where

$$m = \min_{\{z\in(R^+)^2:|z|=1\}} F(z) \quad \text{and} \quad M = \max_{\{z\in(R^+)^2:|z|=1\}} F(z).$$

Let $|\Omega|$ denote the Lebesgue measure of Ω. Multiplying the two differential equations in problem (11) by $\omega_{1\varepsilon},\omega_{2\varepsilon}$, and integrating over Ω, respectively, according to the Schwartz inequality, the Hölder inequality, the Poincaré ineuality and $1<\mu<p$, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla\omega_{1\varepsilon}|^p+|\nabla\omega_{2\varepsilon}|^p dx\\
&\quad=\varepsilon^{\mu-p}\int_\Omega \omega_{1\varepsilon}\frac{\partial F}{\partial u}(\omega_{1\varepsilon},\omega_{2\varepsilon})+\omega_{1\varepsilon}\frac{\partial F}{\partial v}(\omega_{v\varepsilon},\omega_{2\varepsilon})dx+\int_\Omega \omega_{1\varepsilon}g dx+\int_\Omega \omega_{2\varepsilon}h dx\\
&\quad=\mu\varepsilon^{\mu-p}\int_\Omega F(\omega_{1\varepsilon},\omega_{2\varepsilon})dx+\int_\Omega \omega_{1\varepsilon}g dx+\int_\Omega \omega_{2\varepsilon}h dx\\
&\quad\leqslant \mu M\varepsilon^{\mu-p}\int_\Omega (\omega_{1\varepsilon}^2+\omega_{1\varepsilon}^2)^{\frac{\mu}{2}}dx+\int_\Omega \omega_{1\varepsilon}g dx+\int_\Omega \omega_{2\varepsilon}h dx\\
&\quad\leqslant \mu M\varepsilon^{\mu-p}2^{\frac{\mu}{2}}\int_\Omega ((\omega_{1\varepsilon}^\mu+\omega_{1\varepsilon}^\mu)dx+\|\omega_{1\varepsilon}\|_2\|g\|_2+\|\omega_{1\varepsilon}\|_2\|h\|_2\\
&\quad\leqslant \mu M\varepsilon^{\mu-p}2^{\frac{\mu}{2}}|\Omega|^{\frac{2-\mu}{2}}\left(\|\omega_{1\varepsilon}\|_2^{\frac{\mu}{2}}+\|\omega_{2\varepsilon}\|_2^{\frac{\mu}{2}}\right)+\|\omega_{1\varepsilon}\|_2\|g\|_2+\|\omega_{1\varepsilon}\|_2\|h\|_2\\
&\quad\leqslant C(\Omega,g,h)\left(\|\omega_{1\varepsilon}\|^{\frac{\mu}{2}}+\|\omega_{2\varepsilon}\|^{\frac{\mu}{2}}+\|\omega_{1\varepsilon}\|+\|\omega_{2\varepsilon}\|\right), \qquad (12)
\end{aligned}$$

for all $\varepsilon\geqslant 1$, where $C(\Omega,g,h)$ depends only on Ω, g and h. Therefore, there exist a constant $C>0$ independent of $\varepsilon\geqslant 1$ such that

$$\int_\Omega |\nabla\omega_{1\varepsilon}|^p+|\nabla\omega_{2\varepsilon}|^p dx\leqslant C_1. \qquad (13)$$

By the Sobolev embedding theorem, we have

$$\int_\Omega |\omega_{1\varepsilon}|^{p^*}dx\leqslant C_1,\int_\Omega |\omega_{2\varepsilon}|^{p^*}dx\leqslant C_1.$$

By choosing a subsequence if necessary, we may assume that as $\varepsilon\to\infty$,

$$(\omega_{1\varepsilon},\omega_{2\varepsilon})\rightharpoonup(\omega_1,\omega_2)\quad \text{weakly in } E. \qquad (14)$$

Combining (12) and (13), we deduce that there exists a constant C_2 independent of ε such that

$$\varepsilon^{\mu-p}\int_\Omega F(\omega_{1\varepsilon},\omega_{2\varepsilon})dx\leqslant C_2.$$

Set $z_\varepsilon=(\omega_{1\varepsilon},\omega_{2\varepsilon})$, according to $F(z)\geqslant m|z|^\mu$ and $m>0$, we have

$$\varepsilon^{\mu-p}\int_\Omega |z_\varepsilon|^\mu dx\leqslant \frac{1}{m}C_2. \qquad (15)$$

Since $F\in C^1((R^+)^2,R^+)$ is positively homogeneous of degree $\mu>1$, $\frac{\partial F}{\partial u}(z)$, $\frac{\partial F}{\partial v}(z)$ is positively homogeneous of degree $\mu-1$. Moreover, we have

$$\frac{\partial F}{\partial u}(z)\leqslant M_1|z|^{\mu-1},$$

where

$$M_1 = \max_{\{z \in (R^+)^2 : |z|=1\}} \frac{\partial F}{\partial u}(z).$$

By (15) and Hölder inequality, for any $\varphi, \psi \in C_0^\infty$, we infer that

$$\begin{aligned}
\Big| \int_\Omega \frac{\partial F}{\partial u}(\omega_{1\varepsilon}, \omega_{2\varepsilon}) \varphi dx \Big| &\leqslant M_1 \int_\Omega |z_\varepsilon|^{\mu-1} |\varphi| dx \\
&\leqslant M_1 \|\varphi\|_\infty |\Omega|^{\frac{1}{\mu}} \Big(\int_\Omega |z_\varepsilon|^\mu dx \Big)^{\frac{\mu-1}{\mu}} \\
&\leqslant M_1 m^{\frac{1-\mu}{\mu}} C_2^{\frac{\mu-1}{\mu}} |\Omega|^{\frac{1}{\mu}} \|\varphi\|_\infty \varepsilon^{\frac{(\mu-p)(1-\mu)}{\mu}}.
\end{aligned}$$

Since $1 < \mu < p$, we have

$$\varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial u}(\omega_{1\varepsilon}, \omega_{2\varepsilon}) \varphi dx \to 0 \text{ as } \varepsilon \to \infty. \tag{16}$$

Similarly, one has

$$\varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial v}(\omega_{1\varepsilon}, \omega_{2\varepsilon}) \psi dx \to 0 \text{ as } \varepsilon \to \infty. \tag{17}$$

Multiplying the two differential equations in problem (11) by φ, ψ, and integrating over Ω, respectively, we obtain

$$\int_\Omega |\nabla \omega_{1\varepsilon}|^{p-2} \nabla \omega_{1\varepsilon} \nabla \varphi dx = \varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial u}(\omega_{1\varepsilon}, \omega_{2\varepsilon}) \varphi dx + \int_\Omega g \varphi dx, \tag{18}$$

$$\int_\Omega |\nabla \omega_{2\varepsilon}|^{p-2} \nabla \omega_{2\varepsilon} \nabla \psi dx = \varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial v}(\omega_{1\varepsilon}, \omega_{2\varepsilon}) \psi dx + \int_\Omega g \psi dx. \tag{19}$$

Taking $\varepsilon \to \infty$ on both sides of the equalities in (18) and (19), and taking (16) and (17) into account, we have

$$\int_\Omega |\nabla \omega_1|^{p-2} \nabla \omega_1 \nabla \varphi = \int_\Omega g \varphi dx \text{ and } \int_\Omega |\nabla \omega_2|^{p-2} \nabla \omega_2 \nabla \psi = \int_\Omega g \psi dx,$$

which imply that ω_1, ω_2 are weakly solutions of problem (8) and (9), respectively. Since $\omega_{1\varepsilon}, \omega_{2\varepsilon} > 0$ in Ω, we infer that $\omega_1, \omega_2 \geqslant 0$ in Ω. By the regularity theory (see [31]), we know that ω_1, ω_2 are classical nonnegative solutions of problems (8) and (9), respectively. The proof of Theorem 2 is completed. □

Before giving the proof of Theorem 3, we introduce the Pohozaev identity, which is established in [39].

LEMMA 3. *Let u be in $C^1(\Omega) \cap C(\overline{\Omega})$, where Ω is a C^1 domain in $\mathbb{R}^N$ $(N \geqslant 3)$ satisfying $u = 0$ on $\partial\Omega$. Then we have*

$$\int_\Omega div(|\nabla u|^{p-2} \nabla u)(x, \nabla u) dx = \frac{N-2}{2} \int_\Omega |\nabla u|^p dx + \frac{1}{2} \int_\Omega (x, n) \Big| \frac{\partial u}{\partial n} \Big|^p d\sigma, \tag{20}$$

where n denotes the unit outward normal of $\partial\Omega$.

PROOF OF THEOREM 3. Let $(\widetilde{u}_\varepsilon,\widetilde{v}_\varepsilon)$ be any positive solution of problem (1) with respect to the parameter ε. Set $\widetilde{\omega}_{1\varepsilon}=\varepsilon^{-1}\widetilde{u}_\varepsilon$, $\omega_{2\varepsilon}=\varepsilon^{-1}\widetilde{v}_\varepsilon$, then $(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})$ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla\widetilde{\omega}_{1\varepsilon}|^{p-2}\nabla\widetilde{\omega}_{1\varepsilon})=\varepsilon^{\mu-p}\frac{\partial F}{\partial u}(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})+g(x) \quad \text{in } \Omega,\\ -\mathrm{div}(|\nabla\widetilde{\omega}_{2\varepsilon}|^{p-2}\nabla\widetilde{\omega}_{2\varepsilon})=\varepsilon^{\mu-p}\frac{\partial F}{\partial v}(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})+h(x) \quad \text{in } \Omega,\\ \widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon}>0 \quad \text{in } \Omega,\\ \widetilde{\omega}_{1\varepsilon}=\widetilde{\omega}_{2\varepsilon}=0 \quad \text{on } \partial\Omega. \end{cases} \tag{21}$$

Multiplying the two differential equations in problem (21) by $\widetilde{\omega}_{1\varepsilon}$, $\widetilde{\omega}_{2\varepsilon}$, and integrating over Ω, respectively, we obtain

$$\int_\Omega |\nabla\widetilde{\omega}_{1\varepsilon}|^p+|\nabla\widetilde{\omega}_{2\varepsilon}|^p dx=\mu\varepsilon^{\mu-p}\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx+\int_\Omega \widetilde{\omega}_{1\varepsilon}g dx+\int_\Omega \widetilde{\omega}_{2\varepsilon}h dx. \tag{22}$$

On the one hand, from Lemma 3 and (22), we obtain

$$\begin{aligned} &\int_\Omega \Delta\widetilde{\omega}_{1\varepsilon}(x,\nabla\widetilde{\omega}_{1\varepsilon})dx+\int_\Omega \Delta\widetilde{\omega}_{2\varepsilon}(x,\nabla\widetilde{\omega}_{2\varepsilon})dx\\ &\quad=\frac{N-2}{2}\int_\Omega(|\nabla\widetilde{\omega}_{1\varepsilon}|^p+|\nabla\widetilde{\omega}_{2\varepsilon}|^p)dx+\frac{1}{2}\int_{\partial\Omega}(x,n)(|\frac{\partial\widetilde{\omega}_{1\varepsilon}}{\partial n}|^p+|\frac{\partial\widetilde{\omega}_{2\varepsilon}}{\partial n}|^p)d\sigma\\ &\quad=\frac{N-2}{2}\mu\varepsilon^{\mu-p}\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx+\frac{N-2}{2}\int_\Omega(\widetilde{\omega}_{1\varepsilon}g+\widetilde{\omega}_{2\varepsilon}h)dx\\ &\qquad\quad+\frac{1}{2}\int_{\partial\Omega}(x,n)\Big(|\frac{\partial\widetilde{\omega}_{1\varepsilon}}{\partial n}|^p+|\frac{\partial\widetilde{\omega}_{2\varepsilon}}{\partial n}|^p\Big)d\sigma. \end{aligned} \tag{23}$$

On the other hand, noticing that $\widetilde{\omega}_{1\varepsilon}=\widetilde{\omega}_{2\varepsilon}=0,\ x\in\partial\Omega$ and the homogeneity of F, we have $F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})=0, x\in\partial\Omega$. Therefore, one has:

$$\begin{aligned} &\int_\Omega(\varepsilon^{\mu-p}\frac{\partial F}{\partial u}(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})+g(x))\cdot(x,\nabla\widetilde{\omega}_{1\varepsilon})dx\\ &\qquad+\int_\Omega(\varepsilon^{\mu-p}\frac{\partial F}{\partial v}(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})+h(x))\cdot(x,\nabla\widetilde{\omega}_{2\varepsilon})dx\\ &\quad=\varepsilon^{\mu-p}\int_\Omega(x,\nabla F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon}))dx+\int_\Omega g(x,\nabla\widetilde{\omega}_{1\varepsilon})dx+\int_\Omega h(x,\nabla\widetilde{\omega}_{2\varepsilon})dx\\ &\quad=-\varepsilon^{\mu-p}N\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx-N\int_\Omega(\widetilde{\omega}_{1\varepsilon}g+\widetilde{\omega}_{2\varepsilon}h)dx\\ &\qquad-\int_\Omega \widetilde{\omega}_{1\varepsilon}(x,\nabla g)dx-\int_\Omega \widetilde{\omega}_{2\varepsilon}(x,\nabla h)dx. \end{aligned} \tag{24}$$

According to (21), (23), (24), we obtain

$$\begin{aligned} &(\frac{N-2}{2}-\frac{N}{\mu})\mu\varepsilon^{\mu-p}\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx+\frac{1}{2}\int_{\partial\Omega}(x,n)\Big(|\frac{\partial\widetilde{\omega}_{1\varepsilon}}{\partial n}|^p+|\frac{\partial\widetilde{\omega}_{2\varepsilon}}{\partial n}|^p\Big)d\sigma\\ &\quad=\frac{N+2}{2}\int_\Omega(\widetilde{\omega}_{1\varepsilon}g+\widetilde{\omega}_{2\varepsilon}h)dx+\int_\Omega \widetilde{\omega}_{1\varepsilon}(x,\nabla g)dx+\int_\Omega \widetilde{\omega}_{2\varepsilon}(x,\nabla h)dx. \end{aligned} \tag{25}$$

As $\mu > 2^*$, we have $\frac{N-2}{2} - \frac{N}{\mu} > 0$. Since Ω is star-shaped, we have $(x,n) > 0$ on $\partial\Omega$. Therefore, one has

$$\int_{\partial\Omega}(x,n)\Big(|\frac{\partial\widetilde{\omega}_{1\varepsilon}}{\partial n}|^p + |\frac{\partial\widetilde{\omega}_{2\varepsilon}}{\partial n}|^p\Big)d\sigma \geqslant 0.$$

Consequently, using (25), we conclude that

$$\begin{aligned}&\varepsilon^{\mu-p}\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx\\ &\qquad\leqslant C_3\Big(\int_\Omega \widetilde{\omega}_{1\varepsilon} g dx + \int_\Omega \widetilde{\omega}_{2\varepsilon} h dx + \int_\Omega \widetilde{\omega}_{1\varepsilon}(x,\nabla g)dx + \int_\Omega \widetilde{\omega}_{2\varepsilon}(x,\nabla h)dx\Big).\end{aligned}$$

From (22), we obtain

$$\begin{aligned}&\int_\Omega(|\nabla\widetilde{\omega}_{1\varepsilon}|^p + |\nabla\widetilde{\omega}_{2\varepsilon}|^p)dx\\ &\quad= \mu\varepsilon^{\mu-p}\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx + \int_\Omega \widetilde{\omega}_{1\varepsilon} g dx + \int_\Omega \widetilde{\omega}_{2\varepsilon} h dx\\ &\quad\leqslant C_4\Big(\int_\Omega \widetilde{\omega}_{1\varepsilon} g dx + \int_\Omega \widetilde{\omega}_{2\varepsilon} h dx + \int_\Omega \widetilde{\omega}_{1\varepsilon}(x,\nabla g)dx + \int_\Omega \widetilde{\omega}_{2\varepsilon}(x,\nabla h)dx\Big) \qquad (26)\end{aligned}$$

for all ε small enough, where $C(\Omega,g,h)$ depends only on Ω, g and h. Therefore, there exist a constant $C_5 > 0$ independent of ε such that

$$\int_\Omega |\nabla\widetilde{\omega}_{1\varepsilon}|^p + |\nabla\widetilde{\omega}_{2\varepsilon}|^p dx < C_5. \qquad (27)$$

By the Sobolev embedding theorem, we have

$$\int_\Omega |\widetilde{\omega}_{1\varepsilon}|^{p^*} dx \leqslant C_5, \int_\Omega |\widetilde{\omega}_{2\varepsilon}|^{p^*} dx \leqslant C_5.$$

By choosing a subsequence if necessary, we may assume that as $\varepsilon \to 0$,

$$(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon}) \rightharpoonup (\widetilde{\omega}_1,\widetilde{\omega}_2) \quad \text{weakly in } E. \qquad (28)$$

Since F is positively homogeneous of degree μ, we have

$$m|z|^\mu \leqslant F(z) \leqslant M|z|^\mu,$$

where

$$m = \min_{\{z\in(R^+)^2:|z|=1\}} F(z) \quad \text{and} \quad M = \max_{\{z\in(R^+)^2:|z|=1\}} F(z).$$

Combining (22) and (26), we deduce that there exists a constant C_6 independent of ε such that

$$\varepsilon^{\mu-p}\int_\Omega F(\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})dx < C_6.$$

Set $\widetilde{z}_\varepsilon = (\widetilde{\omega}_{1\varepsilon},\widetilde{\omega}_{2\varepsilon})$, according to $F(z) \geqslant m|z|^\mu$ and $m > 0$, we have

$$\varepsilon^{\mu-p}\int_\Omega |\widetilde{z}_{\varepsilon|^\mu} dx \leqslant \frac{1}{m}C_6. \qquad (29)$$

Since $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree $\mu > 2^*$, $\frac{\partial F}{\partial u}(z)$, $\frac{\partial F}{\partial v}(z)$ is positively homogeneous of degree $\mu - 1$, and $1 < p < \mu$. Moreover, we have

$$\frac{\partial F}{\partial u}(z) \leqslant M_1 |z|^{\mu-1},$$

where

$$M_1 = \max_{\{z \in (R^+)^2 : |z|=1\}} \frac{\partial F}{\partial u}(z).$$

By (29) and Hölder inequality, for any $\varphi, \psi \in C_0^\infty$, we infer that

$$\begin{aligned}\Big| \int_\Omega \frac{\partial F}{\partial u}(\widetilde{\omega}_{1\varepsilon}, \widetilde{\omega}_{2\varepsilon})\varphi dx \Big| &\leqslant M_1 \int_\Omega |\widetilde{z}_\varepsilon|^{\mu-1} |\varphi| dx \\ &\leqslant M_1 \|\varphi\|_\infty |\Omega|^{\frac{1}{\mu}} \Big(\int_\Omega |\widetilde{z}_\varepsilon|^\mu dx \Big)^{\frac{\mu-1}{\mu}} \\ &\leqslant M_1 m^{\frac{1-\mu}{\mu}} C_2^{\frac{\mu-1}{\mu}} |\Omega|^{\frac{1}{\mu}} \|\varphi\|_\infty \varepsilon^{\frac{(\mu-p)(1-\mu)}{\mu}}.\end{aligned}$$

Since $1 < p < \mu$, we have

$$\varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial u}(\widetilde{\omega}_{1\varepsilon}, \widetilde{\omega}_{2\varepsilon})\varphi dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{30}$$

Similarly, one has

$$\varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial v}(\widetilde{\omega}_{1\varepsilon}, \widetilde{\omega}_{2\varepsilon})\psi dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{31}$$

Multiplying the two differential equations in problem (21) by φ, ψ, and integrating over Ω, respectively, we obtain:

$$\int_\Omega |\nabla \omega_{1\varepsilon}|^{p-2} \nabla \omega_{1\varepsilon} \nabla \varphi dx = \varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial u}(\widetilde{\omega}_{1\varepsilon}, \widetilde{\omega}_{2\varepsilon})\varphi dx + \int_\Omega g\varphi dx, \tag{32}$$

$$\int_\Omega |\nabla \widetilde{\omega}_{2\varepsilon}|^{p-2} \nabla \widetilde{\omega}_{2\varepsilon} \nabla \psi dx = \varepsilon^{\mu-p} \int_\Omega \frac{\partial F}{\partial v}(\widetilde{\omega}_{1\varepsilon}, \widetilde{\omega}_{2\varepsilon})\psi dx + \int_\Omega g\psi dx. \tag{33}$$

Taking $\varepsilon \to 0$ on both sides of the equalities in (32), (33) and tanking (28), (30), (31) into account, we have

$$\int_\Omega |\nabla \widetilde{\omega}_1|^{p-2} \nabla \widetilde{\omega}_1 \nabla \varphi = \int_\Omega g\varphi dx \quad \text{and} \quad \int_\Omega |\nabla \widetilde{\omega}_2|^{p-2} \nabla \widetilde{\omega}_2 \nabla \psi = \int_\Omega g\psi dx,$$

which imply that $\widetilde{\omega}_1, \widetilde{\omega}_2$ are weakly solutions of problem (8) and (9), respectively. Since $\widetilde{\omega}_{1\varepsilon}, \widetilde{\omega}_{2\varepsilon} > 0$ in Ω, we infer that $\widetilde{\omega}_1, \widetilde{\omega}_2 \geqslant 0$ in Ω. By the regularity theory (see [17]), we know that $\widetilde{\omega}_1, \widetilde{\omega}_2$ are classical nonnegative solutions of problems (8) and (9), respectively. The proof of Theorem 3 is completed. $\square$

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