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SUBHARMONIC SOLUTIONS FOR NONAUTONOMOUS SUBLINEAR p–HAMILTONIAN SYSTEMS

KUN LIAO AND CHUN-LEI TANG

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Abstract. Some existence theorems are obtained for subharmonic solutions of nonautonomous p-Hamiltonian systems by the minimax methods in critical point theory.

1. Introduction and main results

Consider the p-Hamiltonian systems

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = \nabla F(t,u(t)), \quad \text{a.e. } t \in \mathbb{R}, \tag{1}$$

where $p > 2$ and $F : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$. In this paper, we always assume:

(A) $F(t,x)$ is T-periodic $(T > 0)$ and measurable in t for each $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leqslant a(|x|)b(t) \quad \text{and} \quad |\nabla F(t,x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

When $p = 2$, (1) reduces to the following second-order Hamiltonian systems

$$\ddot{u}(t) = \nabla F(t,u(t)), \ \text{ a.e. } t \in \mathbb{R}. \tag{2}$$

It has been proved that problem (2) has infinitely subharmonic solutions under suitable conditions (see [2, 3, 8, 11, 12]). In recent years, authors have devoted to the existence of T-periodic solutions for problem (1) (see [1, 6, 9, 13, 14, 15]). But there were a few papers about the infinitely subharmonic solutions for problem (1). Ma and Zhang in [4] studied the existence of subharmonic solutions for problem (1) by using the Generalized Mountain Pass Theorem, the subharmonic solutions for problem (1) under nonsmooth potential are considered in [7] and [13].

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Recently, Ye and Tang in [16] consider the existence of T-periodic solutions for problem (2) under conditions that

$$F(t,x)=G(x)+H(t,x),$$

where ∇H is sublinear, that is, there exist $f,\ g\in L^1(0,T;\mathbb{R}^+)$ and $\alpha\in[0,1)$ such that

$$|\nabla H(t,x)|\leqslant f(t)|x|^{\alpha}+g(t)$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, and there exists $r<\frac{4\pi^2}{T^2}$ such that

$$(\nabla G(x)-\nabla G(y),x-y)\geqslant -r|x-y|^2$$

for all $x,\ y\in\mathbb{R}^N$ and a.e. $t\in[0,T]$. Motivated by the results in [16], in the present paper, we obtain some new results of infinitely many subharmonic solutions for p-Hamiltonian systems (1) with $\alpha\in[0,p-1)$ by using the Saddle Point Theorem.

Our main results are the following theorems.

THEOREM 1. *Suppose that* $F(t,x)$ *satisfies* (A) *and the following conditions:*
(l_1) *there exists* $g\in L^1(0,T;\mathbb{R}^+)$ *such that*

$$|\nabla H(t,x)|\leqslant g(t)\quad \textit{for all } x\in\mathbb{R}^N \textit{ and a.e. } t\in[0,T],$$

(l_2) *there exists* $r\in\mathbb{R}$ *such that*

$$(\nabla G(x)-\nabla G(y),x-y)\geqslant -r|x-y|^2$$

for all $x,\ y\in\mathbb{R}^N$ *and a.e.* $t\in[0,T]$*,*
(l_3) *there exists* $\gamma\in L^1(0,T)$ *such that*

$$F(t,x)\leqslant\gamma(t)\quad \textit{for all } x\in\mathbb{R}^N \textit{ and a.e. } t\in[0,T],$$

(l_4) *there exists a subset* E *of* $[0,T]$ *with* meas$(E)>0$ *such that*

$$F(t,x)\to-\infty\quad \textit{as}\quad |x|\to\infty\quad \textit{for a.e. } t\in E.$$

Then problem (1) *has a* kT*-periodic solution* $u_k\in W_{kT}^{1,p}$ *for every positive integer* k *such that* $\|u_k\|_\infty\to\infty$ *as* $k\to\infty$*, where* $\|u_k\|_\infty=\max\limits_{0\leqslant t\leqslant kT}|u_k(t)|$*, and*

$$W_{kT}^{1,p}=\{u:[0,kT]\to\mathbb{R}^N\mid u \textit{ is absolutely continuous, } u(0)=u(kT),\ \textit{and } \dot u\in L^p(0,kT;\mathbb{R}^N)\}$$

is a Banach space with the norm

$$\|u\|=\left(\int_0^{kT}|u(t)|^p dt+\int_0^{kT}|\dot u(t)|^p dt\right)^{\frac{1}{p}}.$$

REMARK 1. Let $F(t,x) = G(x) + H(t,x)$ with $G(x) = -r\cos x_1$, where x_1 is the first coordinate of x, and

$$H(t,x) = -|\sin\omega t|\ln(1+|x|^p)$$

for all $x \in \mathbb{R}^N$ and $t \in [0,T]$. Then F satisfies our Theorem 1 but does not satisfy the assumptions in [4] and [13].

THEOREM 2. *Suppose that $F(t,x)$ satisfies (A), (l_2) and the following conditions:*

(l_5) *there exist $f,\ g \in L^1(0,T;\mathbb{R}^+)$ and $\alpha \in [0,p-1)$ such that*

$$|\nabla H(t,x)| \leqslant f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$,

(l_6) *there holds true*

$$|x|^{-q\alpha}F(t,x) \to -\infty \ \text{ as } |x| \to \infty \text{ uniformly for a.e. } t \in [0,T],$$

where α is the same as in (l_5) and $q = \frac{p}{p-1}$. Then problem (1) has a kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

REMARK 2. Let $F(t,x) = G(x) + H(t,x)$ with $G(x) = -\frac{r}{2}|x_1|^2$, where x_1 is the first coordinate of x, and $H(t,x) = -|x|^{1+\alpha}$, where $0 < \alpha < p-1$. Then F satisfies our Theorem 2 but does not satisfy the assumptions of [4] and [13].

We shall prove a more general result than Theorems 1 and 2.

THEOREM 3. *Suppose that $F(t,x)$ satisfies (A), (l_2), (l_3), (l_5) and the following condition:*

(l_7) *there exists a subset E of $[0,T]$ with* meas$(E) > 0$ *such that*

$$|x|^{-q\alpha}F(t,x) \to -\infty \ \text{ as } |x| \to \infty \text{ for a.e. } t \in E.$$

Then problem (1) has a kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$.

REMARK 3. When $p = 2$, the conditions of Theorem 3 are similar to the ones of Theorem 3 in [16]. But in [16], the authors studied the existence of T-periodic solutions for problem (2) while the infinitely many subharmonic solutions for problem (1) are considered in our paper. Since $\|u_k\|_\infty \to \infty$ as $k \to \infty$, then $\{\|u_k\|_\infty\}$ has a subsequence with infinitely distinct elements, so the infinitely many solutions for problem (1) are indeed distinct.

2. Proof of theorems

For $u \in W_{kT}^{1,p}$, let

$$\overline{u} = \frac{1}{kT}\int_0^{kT} u(t)dt \quad \text{and} \quad \widetilde{u}(t) = u(t) - \overline{u}.$$

Then due to Proposition 1.1 in [5], there exists a constant c_k such that

$$\|\widetilde{u}\|_\infty^p \leqslant c_k \int_0^{kT} |\dot{u}(t)|^p dt, \tag{3}$$

and

$$\int_0^{kT} |\widetilde{u}(t)|^p dt \leqslant c_k \int_0^{kT} |\dot{u}(t)|^p dt. \tag{4}$$

It follows from (A) that the functional φ_k on $W_{kT}^{1,p}$ given by

$$\varphi_k(u) = \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt + \int_0^{kT} F\left(t, u(t)\right) dt$$

is continuously differentiable on $W_{kT}^{1,p}$ (see Theorem 1.4 in [5]). Moreover, one has

$$\langle \varphi_k'(u), v\rangle = \int_0^{kT} |\dot{u}(t)|^{p-2}\left(\dot{u}(t), \dot{v}(t)\right) dt + \int_0^{kT} \left(\nabla F\left(t, u(t)\right), v(t)\right) dt$$

for all $u,\ v \in W_{kT}^{1,p}$. It is well known that the kT-periodic solutions of problem (1) correspond to the critical points of the functional φ_k.

For convenience to quote we state an analog of Egorov's theorem (see Lemma 2 in [10]), in which we replace F by $-F$.

LEMMA 1. (see [10]) *Suppose that F satisfies (A) and E is a measurable subset of $[0,T]$. Assume that*

$$F(t,x) \to -\infty \quad as \quad |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$ there exists a subset E_δ of E with $\operatorname{meas}(E \backslash E_\delta) < \delta$ *such that*

$$F(t,x) \to -\infty \quad as \quad |x| \to \infty$$

uniformly for all $t \in E_\delta$.

LEMMA 2. *Assume that $F(t,x)$ satisfies (A), (l_2), (l_3), (l_5) and (l_7). Then φ_k satisfies the $(P.S.)$ condition, that is, $\{u_n\}$ has a convergent subsequence whenever it satisfies $\varphi_k'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi_k(u_n)\}$ is bounded.*

Proof. By (4), we have

$$\left(\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p}} \leqslant \|\widetilde{u}_n\| \leqslant (1+c_k)^{\frac{1}{p}}\left(\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p}} \tag{5}$$

for all n. For any $\varepsilon>0$, it follows from (l_5) and Young's inequality that

$$\begin{aligned}\left|\int_0^{kT}(\nabla H(t,u(t)),\widetilde{u}(t))\,dt\right| &\leqslant \int_0^{kT} f(t)|\overline{u}+\widetilde{u}(t)|^{\alpha}|\widetilde{u}(t)|dt+\int_0^{kT} g(t)|\widetilde{u}(t)|dt\\ &\leqslant \int_0^{kT} 2^{\alpha} f(t)(|\overline{u}|^{\alpha}+|\widetilde{u}(t)|^{\alpha})|\widetilde{u}(t)|dt+\int_0^{kT} g(t)|\widetilde{u}(t)|dt\\ &\leqslant 2^{\alpha}(|\overline{u}|^{\alpha}+\|\tilde{u}\|_{\infty}^{\alpha})\|\tilde{u}\|_{\infty}\int_0^{kT} f(t)dt+\|\tilde{u}\|_{\infty}\int_0^{kT} g(t)dt\\ &\leqslant 2^{\alpha}\varepsilon\|\tilde{u}\|_{\infty}^{p}+2^{\alpha}\varepsilon^{-\frac{1}{p-1}}|\overline{u}|^{q\alpha}\left(\int_0^{kT} f(t)dt\right)^{q}\\ &\quad +2^{\alpha}\|\tilde{u}\|_{\infty}^{\alpha+1}\int_0^{kT} f(t)dt+\|\tilde{u}\|_{\infty}\int_0^{kT} g(t)dt.\end{aligned}$$

Let $\varepsilon=1/(2^{\alpha}2pc_k)$, by (3) we have

$$\begin{aligned}\left|\int_0^{kT}(\nabla H(t,u(t)),\widetilde{u}(t))\,dt\right| &\leqslant \frac{1}{2p}\int_0^{kT}|\dot{u}(t)|^p dt+C_1|\overline{u}|^{q\alpha}+C_2\left(\int_0^{kT}|\dot{u}(t)|^p dt\right)^{\frac{\alpha+1}{p}}\\ &\quad +C_3\left(\int_0^{kT}|\dot{u}(t)|^p dt\right)^{\frac{1}{p}}\end{aligned}$$

for all $u\in W_{kT}^{1,p}$ and some positive constants C_1, C_2 and C_3. From (l_2) and (3) we obtain

$$\begin{aligned}\int_0^{kT}(\nabla G(u(t)),\tilde{u}(t))\,dt &= \int_0^{kT}(\nabla G(u(t))-\nabla G(\overline{u}),\tilde{u}(t))\,dt\\ &\geqslant -r\int_0^{kT}|\tilde{u}(t)|^2 dt\\ &\geqslant -rkT\|\tilde{u}\|_{\infty}^2\\ &\geqslant -rkT\left(c_k\int_0^{kT}|\dot{u}(t)|^p dt\right)^{\frac{2}{p}}\end{aligned}$$

for all $u\in W_{kT}^{1,p}$. Hence one has

$$\begin{aligned}\|\tilde{u}_n\| &\geqslant |\langle\varphi_k'(u_n),\tilde{u}_n\rangle|\\ &= \left|\int_0^{kT}|\dot{u}_n(t)|^p dt+\int_0^{kT}(\nabla F(t,u_n(t)),\tilde{u}_n(t))\,dt\right|\\ &= \left|\int_0^{kT}|\dot{u}_n(t)|^p dt+\int_0^{kT}(\nabla G(u_n(t)),\tilde{u}_n(t))\,dt+\int_0^{kT}(\nabla H(t,u_n(t)),\tilde{u}_n(t))\,dt\right|\end{aligned}$$

$$\geqslant \frac{2p-1}{2p}\int_0^{kT}|\dot{u}_n(t)|^p dt - C_1|\bar{u}_n|^{q\alpha} - C_2\left(\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{\frac{\alpha+1}{p}}$$
$$-C_3\left(\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p}} - rkT\left(c_k\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{\frac{2}{p}}$$

for large n. By (5) and the above inequality we have

$$C|\bar{u}_n|^{\frac{q\alpha}{p}} \geqslant \left(\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p}} - C_4 \tag{6}$$

for some constants $C>0$, $C_4>0$ and all large n, which implies that

$$\|\tilde{u}_n\|_\infty \leqslant C_5(|\bar{u}_n|^{\frac{q\alpha}{p}}+1)$$

for all large n and some positive constant C_5 by (3). Then one has

$$|u_n(t)| \geqslant |\bar{u}_n| - |\tilde{u}_n(t)| \geqslant |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geqslant |\bar{u}_n| - C_5\left(|\bar{u}_n|^{\frac{q\alpha}{p}}+1\right)$$

for all large n and every $t\in[0,kT]$.

If $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar{u}_n| \to \infty \quad \text{as} \quad n\to\infty. \tag{7}$$

Hence

$$|u_n(t)| \geqslant \frac{1}{2}|\bar{u}_n| \tag{8}$$

for all large n and every $t\in[0,kT]$ because of $|u_n(t)| \geqslant |\bar{u}_n| - C_5\left(|\bar{u}_n|^{\frac{q\alpha}{p}}+1\right)$.

Set $\delta = \text{meas}(E/2)$. It follows from (l_7) and Lemma 1 that there exists a subset E_δ of E with meas$(E\backslash E_\delta)<\delta$ such that

$$|x|^{-q\alpha}F(t,x) \to -\infty \quad \text{as} \quad |x|\to\infty$$

uniformly for all $t\in E_\delta$, which implies that

$$\text{meas}(E_\delta) = \text{meas}(E) - \text{meas}(E\backslash E_\delta) > \delta > 0, \tag{9}$$

and for every $\beta>0$, there exists $M\geqslant 1$ such that

$$|x|^{-q\alpha}F(t,x) \leqslant -\beta \tag{10}$$

for all $|x|\geqslant M$ and all $t\in E_\delta$. By (8) and (7), one has

$$|u_n(t)| \geqslant M \tag{11}$$

for large n and every $t \in [0,kT]$. It follows from (l_3), (6), (8) and (9)-(11) that

$$\begin{aligned}\varphi_k(u_n) &\leqslant \left(C|\bar{u}_n|^{\frac{q\alpha}{p}} + C_4\right)^p + \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)dt - \int\limits_{E_\delta} \beta|u_n(t)|^{q\alpha}dt \\ &\leqslant \left(C|\bar{u}_n|^{\frac{q\alpha}{p}} + C_4\right)^p + \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)dt - 2^{-q\alpha}|\bar{u}_n|^{q\alpha}\delta\beta\end{aligned}$$

for all large n. Hence we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-q\alpha}\varphi_k(u_n) \leqslant C^p - 2^{-q\alpha}\delta\beta.$$

By the arbitrariness of $\beta > 0$, one has

$$\limsup_{n\to\infty} |\bar{u}_n|^{-q\alpha}\varphi_k(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi_k(u_n)$. Hence $\{|\bar{u}_n|\}$ is bounded. Furthermore, $\{u_n\}$ is bounded by (5) and (6). Then we can use a same argument as in [15] to show that φ_k satisfies the $(P.S.)$ condition. $\square$

Now we prove our Theorem 3 first.

PROOF OF THEOREM 3. It follows from Lemma 2 that φ_k satisfies the $(P.S.)$ condition. Now we prove that φ_k satisfies the other conditions of the Saddle Point Theorem (see Theorem 4.7 in [5]). Let $\widetilde{W}_{kT}^{1,p}$ be the subspace of $W_{kT}^{1,p}$ given by

$$\widetilde{W}_{kT}^{1,p} = \{u \in W_{kT}^{1,p} \mid \bar{u} = 0\}.$$

Set

$$e_k(t) = k(\cos k^{-1}\omega t)x_0$$

for all $t \in \mathbb{R}$ and some $x_0 \in \mathbb{R}^N$ with $|x_0| = 1$, where $\omega = \frac{2\pi}{T}$. Then we have

$$\dot{e}_k(t) = -\omega(\sin k^{-1}\omega t)x_0$$

for all $t \in \mathbb{R}$. By the Saddle Point Theorem we only need to prove

(i) $\varphi_k(u) \to +\infty$ as $\|u\| \to \infty$ in $\widetilde{W}_{kT}^{1,p}$,

(ii) $\varphi_k(x+e_k) \to -\infty$ as $|x| \to \infty$ in $\mathbb{R}^N$.

It follows from (l_3) and (10) that

$$\begin{aligned}\varphi_k(x+e_k) &= \frac{1}{p}\int_0^{kT} |\dot{e}_k(t)|^p dt + \int_0^{kT} F(t,x+e_k)dt \\ &\leqslant \frac{1}{p}\omega^p kT + \int_0^{kT} F\left(t,x+k(\cos k^{-1}\omega t)x_0\right)dt \\ &\leqslant \frac{1}{p}\omega^p kT + \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)dt - \beta\int\limits_{E_\delta} \left|x+k(\cos k^{-1}\omega t)x_0\right|^{q\alpha} dt\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{p}\omega^p kT + \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)dt - \beta M^{q\alpha}\mathrm{meas}(E_\delta)\\
&\leqslant \frac{1}{p}\omega^p kT + \int\limits_{[0,kT]\setminus E_\delta} \gamma(t)dt - \beta\,\mathrm{meas}(E_\delta)
\end{aligned}
$$

for all $|x| \geqslant M+k$, which implies (ii) by the arbitrariness of β.

It follows from (l_5) and (3) that

$$
\begin{aligned}
\Big|\int_0^{kT}[H(t,u(t))-H(t,0)]\,dt\Big| &= \Big|\int_0^{kT}\int_0^1 \big(\nabla H(t,su(t)),u(t)\big)dsdt\Big|\\
&\leqslant \int_0^{kT}\int_0^1 f(t)|su(t)|^\alpha|u(t)|dsdt\\
&\qquad + \int_0^{kT}\int_0^1 g(t)|u(t)|dsdt\\
&\leqslant \int_0^{kT} f(t)|u(t)|^\alpha|u(t)|dt + \int_0^{kT} g(t)|u(t)|dt\\
&\leqslant \|u\|_\infty^{\alpha+1}\int_0^{kT} f(t)dt + \|u\|_\infty\int_0^{kT} g(t)dt\\
&\leqslant C_6\Big(\int_0^{kT}|\dot u(t)|^p dt\Big)^{\frac{\alpha+1}{p}} + C_7\Big(\int_0^{kT}|\dot u(t)|^p dt\Big)^{\frac{1}{p}}
\end{aligned}
$$

for all $u \in \widetilde{W}_{kT}^{1,p}$ and some positive constants C_6 and C_7. By (l_2) and (3) we have

$$
\begin{aligned}
\int_0^{kT}[G(u(t))-G(0)]\,dt &= \int_0^{kT}\int_0^1 (\nabla G(su(t))-\nabla G(0),u(t))\,dsdt\\
&= \int_0^{kT}\int_0^1 \frac{1}{s}(\nabla G(su(t))-\nabla G(0),su(t))\,dsdt\\
&\geqslant \int_0^{kT}\int_0^1 \frac{1}{s}\left(-rs^2|u(t)|^2\right)dsdt\\
&= -\frac{r}{2}\int_0^{kT}|u(t)|^2dt\\
&\geqslant -\frac{rkT}{2}\left(c_k\int_0^{kT}|\dot u(t)|^p dt\right)^{\frac{2}{p}}
\end{aligned}
$$

for all $u \in \widetilde{W}_{kT}^{1,p}$. Hence one has

$$
\begin{aligned}
\varphi_k(u) - \int_0^{kT} F(t,0)dt &= \frac{1}{p}\int_0^{kT}|\dot u(t)|^p dt + \int_0^{kT}[G(u(t))-G(0)]\,dt\\
&\quad + \int_0^{kT}[H(t,u(t))-H(t,0)]\,dt\\
&\geqslant \frac{1}{p}\int_0^{kT}|\dot u(t)|^p dt - C_6\left(\int_0^{kT}|\dot u(t)|^p dt\right)^{\frac{\alpha+1}{p}}
\end{aligned}
$$

$$-C_7\left(\int_0^{kT}|\dot{u}(t)|^p dt\right)^{\frac{1}{p}}-\frac{rkT}{2}\left(c_k\int_0^{kT}|\dot{u}(t)|^p dt\right)^{\frac{2}{p}}$$

for all $u\in\widetilde{W}_{kT}^{1,p}$, which implies (i) by (5). So there exists a critical point $u_k\in W_{kT}^{1,p}$ for φ_k such that

$$-\infty<\inf_{\widetilde{W}_{kT}^{1,p}}\varphi_k\leqslant\varphi_k(u_k)\leqslant\sup_{\mathbb{R}^N+e_k}\varphi_k.$$

For fixed $x\in\mathbb{R}^N$, set

$$A_k=\{t\in[0,kT]\,\big|\,\big|x+k(\cos k^{-1}\omega t)x_0\big|\leqslant M\}.$$

Then we have

$$\operatorname{meas}(A_k)\leqslant\frac{k\delta}{2}\tag{12}$$

for all large k, where δ is the same as the one in Lemma 2. In fact, if $\operatorname{meas}(A_k)>\frac{k\delta}{2}$, there exists $t_1\in A_k$ such that

$$\frac{1}{8}k\delta\leqslant t_1\leqslant\frac{1}{2}kT-\frac{1}{8}k\delta\tag{13}$$

or

$$\frac{1}{2}kT+\frac{1}{8}k\delta\leqslant t_1\leqslant kT-\frac{1}{8}k\delta.\tag{14}$$

Moreover, there exists $t_2\in A_k$ such that

$$|t_2-t_1|\geqslant\frac{1}{8}k\delta\tag{15}$$

and

$$|t_2-(kT-t_1)|\geqslant\frac{1}{8}k\delta.\tag{16}$$

It follows from (16) that

$$\left|\frac{1}{2}(k^{-1}t_1+k^{-1}t_2)-\frac{1}{2}T\right|\geqslant\frac{1}{16}\delta.\tag{17}$$

By (13) and (14), one has

$$\frac{1}{16}\delta\leqslant\frac{1}{2}(k^{-1}t_1+k^{-1}t_2)\leqslant T-\frac{1}{16}\delta.\tag{18}$$

From (17) and (18) we obtain

$$\left|\sin\left(\frac{1}{2}(k^{-1}t_1+k^{-1}t_2)\omega\right)\right|\geqslant\sin\left(\frac{1}{16}\omega\delta\right).$$

Furthermore, by (15) we have

$$\begin{aligned}&\left|\cos(k^{-1}\omega t_1)-\cos(k^{-1}\omega t_2)\right|\\&\quad=2\left|\sin\left(\frac{1}{2}(k^{-1}t_1+k^{-1}t_2)\omega\right)\right|\left|\sin\left(\frac{1}{2}(k^{-1}t_1-k^{-1}t_2)\omega\right)\right|\\&\quad\geqslant 2\sin^2\left(\frac{1}{16}\omega\delta\right).\end{aligned}$$

But due to $t_1,\ t_2\in A_k$, one has

$$\begin{aligned}&\left|\cos(k^{-1}\omega t_1)-\cos(k^{-1}\omega t_2)\right|\\&\qquad=\frac{1}{k}\left|x+k(\cos k^{-1}\omega t_1)x_0-\left(x+k(\cos k^{-1}\omega t_2)x_0\right)\right|\\&\qquad\leqslant\frac{2M}{k},\end{aligned}$$

which is a contradiction for large k. Hence (12) holds. Let

$$E_k=\bigcup_{j=0}^{k-1}(jT+E_\delta).$$

Then it follows from (12) that

$$\mathrm{meas}(E_k\backslash A_k)\geqslant\frac{1}{2}k\delta$$

for large k. By (10) and (l_3) we have

$$\begin{aligned}k^{-1}\varphi_k(x+e_k)&\leqslant\frac{1}{p}\omega^pT+k^{-1}\int_0^{kT}F\left(t,x+k(\cos k^{-1}\omega t)x_0\right)dt\\&\leqslant\frac{1}{p}\omega^pT+k^{-1}\int\limits_{[0,kT]\backslash(E_k\backslash A_k)}\gamma(t)dt-k^{-1}\beta\,\mathrm{meas}(E_k\backslash A_k)\\&\leqslant\frac{1}{p}\omega^pT+\int_0^T|\gamma(t)|dt-\frac{1}{2}\delta\beta\end{aligned}$$

for every $x\in\mathbb{R}^N$ and all large k. Hence one has

$$\sup_{x\in\mathbb{R}^N}k^{-1}\varphi_k(x+e_k)\leqslant\frac{1}{p}\omega^pT+\int_0^T|\gamma(t)|dt-\frac{1}{2}\delta\beta$$

for all large k, which implies that

$$\limsup_{k\to\infty}\sup_{x\in\mathbb{R}^N}k^{-1}\varphi_k(x+e_k)\leqslant\frac{1}{p}\omega^pT+\int_0^T|\gamma(t)|dt-\frac{1}{2}\delta\beta.$$

By the arbitrariness of β, we obtain

$$\limsup_{k\to\infty}\sup_{x\in\mathbb{R}^N}k^{-1}\varphi_k(x+e_k)=-\infty,$$

which follows that

$$\limsup_{k\to\infty} k^{-1}\varphi_k(u_k) = -\infty. \tag{19}$$

Now we prove that $\|u_k\|_\infty \to \infty$ as $k \to \infty$. If not, going to a subsequence if necessary, we may assume that $\|u_k\|_\infty \leqslant C_8$ for all $k \in N$ and some positive constant C_8. Hence we have

$$\begin{aligned} k^{-1}\varphi_k(u_k) \geqslant k^{-1}\int_0^{kT} F(t,u_k(t))dt &\geqslant -k^{-1}\max_{0\leqslant s\leqslant C_8} a(s)\int_0^{kT} b(t)dt \\ &= -\max_{0\leqslant s\leqslant C_8} a(s)\int_0^T b(t)dt. \end{aligned}$$

It follows that $\liminf_{k\to\infty} k^{-1}\varphi_k(u_k) > -\infty$, which contradicts (19). Therefore we complete our proof. □

Then we prove our Theorems 1 and 2.

PROOF OF THEOREMS 1 AND 2. Theorem 1 follows from Theorem 3 by letting $\alpha = 0$. Theorem 3 implies Theorem 2 because (l_3) follows from (l_6) and (A). In fact, by (l_6) there exists $M > 0$ such that

$$|x|^{-q\alpha}F(t,x) \leqslant 0$$

for all $|x| \geqslant M$ and a.e. $t \in [0,T]$, which implies that $F(t,x) \leqslant 0$ for all $|x| \geqslant M$ and a.e. $t \in [0,T]$. It follows from (A) that

$$F(t,x) \leqslant \max_{0\leqslant s\leqslant M} a(s)b(t)$$

for all $|x| \leqslant M$ and a.e. $t \in [0,T]$. Now (l_3) holds with

$$\gamma(t) = \max_{0\leqslant s\leqslant M} a(s)b(t).$$

Hence Theorem 2 follows from Theorem 3. □

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Kun Liao and Chun-Lei Tang
School of Mathematics and Statistics,
Southwest University,
Chongqing - 400715
People's Republic of China
e-mail: tangcl@swu.edu.cn