

SUBHARMONIC SOLUTIONS FOR NONAUTONOMOUS SUBLINEAR p -HAMILTONIAN SYSTEMS

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Abstract. Some existence theorems are obtained for subharmonic solutions of nonautonomous p -Hamiltonian systems by the minimax methods in critical point theory.

1. Introduction and main results

Consider the p -Hamiltonian systems

$$\frac{d}{dt} (|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t, u(t)), \quad \text{a.e. } t \in \mathbb{R}, \quad (1)$$

where $p > 2$ and $F : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$. In this paper, we always assume:

(A) $F(t, x)$ is T -periodic ($T > 0$) and measurable in t for each $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t) \quad \text{and} \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

When $p = 2$, (1) reduces to the following second-order Hamiltonian systems

$$\ddot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in \mathbb{R}. \quad (2)$$

It has been proved that problem (2) has infinitely subharmonic solutions under suitable conditions (see [2, 3, 8, 11, 12]). In recent years, authors have devoted to the existence of T -periodic solutions for problem (1) (see [1, 6, 9, 13, 14, 15]). But there were a few papers about the infinitely subharmonic solutions for problem (1). Ma and Zhang in [4] studied the existence of subharmonic solutions for problem (1) by using the Generalized Mountain Pass Theorem, the subharmonic solutions for problem (1) under nonsmooth potential are considered in [7] and [13].

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Recently, Ye and Tang in [16] consider the existence of T -periodic solutions for problem (2) under conditions that

$$F(t, x) = G(x) + H(t, x),$$

where ∇H is sublinear, that is, there exist $f, g \in L^1(0, T; \mathbb{R}^+)$ and $\alpha \in [0, 1)$ such that

$$|\nabla H(t, x)| \leq f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and there exists $r < \frac{4\pi^2}{T^2}$ such that

$$(\nabla G(x) - \nabla G(y), x - y) \geq -r|x - y|^2$$

for all $x, y \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Motivated by the results in [16], in the present paper, we obtain some new results of infinitely many subharmonic solutions for p -Hamiltonian systems (1) with $\alpha \in [0, p - 1)$ by using the Saddle Point Theorem.

Our main results are the following theorems.

THEOREM 1. *Suppose that $F(t, x)$ satisfies (A) and the following conditions:*

(l₁) *there exists $g \in L^1(0, T; \mathbb{R}^+)$ such that*

$$|\nabla H(t, x)| \leq g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T],$$

(l₂) *there exists $r \in \mathbb{R}$ such that*

$$(\nabla G(x) - \nabla G(y), x - y) \geq -r|x - y|^2$$

for all $x, y \in \mathbb{R}^N$ and a.e. $t \in [0, T]$,

(l₃) *there exists $\gamma \in L^1(0, T)$ such that*

$$F(t, x) \leq \gamma(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T],$$

(l₄) *there exists a subset E of $[0, T]$ with $\text{meas}(E) > 0$ such that*

$$F(t, x) \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty \quad \text{for a.e. } t \in E.$$

Then problem (1) has a kT -periodic solution $u_k \in W_{kT}^{1,p}$ for every positive integer k such that $\|u_k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$, where $\|u_k\|_\infty = \max_{0 \leq t \leq kT} |u_k(t)|$, and

$$W_{kT}^{1,p} = \{u : [0, kT] \rightarrow \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(0) = u(kT), \\ \text{and } \dot{u} \in L^p(0, kT; \mathbb{R}^N)\}$$

is a Banach space with the norm

$$\|u\| = \left(\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{1}{p}}.$$

REMARK 1. Let $F(t, x) = G(x) + H(t, x)$ with $G(x) = -r \cos x_1$, where x_1 is the first coordinate of x , and

$$H(t, x) = -|\sin \omega t| \ln(1 + |x|^p)$$

for all $x \in \mathbb{R}^N$ and $t \in [0, T]$. Then F satisfies our Theorem 1 but does not satisfy the assumptions in [4] and [13].

THEOREM 2. Suppose that $F(t, x)$ satisfies (A), (l_2) and the following conditions:

(l_5) there exist $f, g \in L^1(0, T; \mathbb{R}^+)$ and $\alpha \in [0, p-1)$ such that

$$|\nabla H(t, x)| \leq f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$,

(l_6) there holds true

$$|x|^{-q\alpha} F(t, x) \rightarrow -\infty \text{ as } |x| \rightarrow \infty \text{ uniformly for a.e. } t \in [0, T],$$

where α is the same as in (l_5) and $q = \frac{p}{p-1}$. Then problem (1) has a kT -periodic solution $u_k \in W_{kT}^{1,p}$ for every positive integer k such that $\|u_k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$.

REMARK 2. Let $F(t, x) = G(x) + H(t, x)$ with $G(x) = -\frac{r}{2}|x_1|^2$, where x_1 is the first coordinate of x , and $H(t, x) = -|x|^{1+\alpha}$, where $0 < \alpha < p-1$. Then F satisfies our Theorem 2 but does not satisfy the assumptions of [4] and [13].

We shall prove a more general result than Theorems 1 and 2.

THEOREM 3. Suppose that $F(t, x)$ satisfies (A), (l_2) , (l_3) , (l_5) and the following condition:

(l_7) there exists a subset E of $[0, T]$ with $\text{meas}(E) > 0$ such that

$$|x|^{-q\alpha} F(t, x) \rightarrow -\infty \text{ as } |x| \rightarrow \infty \text{ for a.e. } t \in E.$$

Then problem (1) has a kT -periodic solution $u_k \in W_{kT}^{1,p}$ for every positive integer k such that $\|u_k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$.

REMARK 3. When $p = 2$, the conditions of Theorem 3 are similar to the ones of Theorem 3 in [16]. But in [16], the authors studied the existence of T -periodic solutions for problem (2) while the infinitely many subharmonic solutions for problem (1) are considered in our paper. Since $\|u_k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$, then $\{\|u_k\|_\infty\}$ has a subsequence with infinitely distinct elements, so the infinitely many solutions for problem (1) are indeed distinct.

2. Proof of theorems

For $u \in W_{kT}^{1,p}$, let

$$\bar{u} = \frac{1}{kT} \int_0^{kT} u(t) dt \quad \text{and} \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then due to Proposition 1.1 in [5], there exists a constant c_k such that

$$\|\tilde{u}\|_\infty^p \leq c_k \int_0^{kT} |\dot{u}(t)|^p dt, \quad (3)$$

and

$$\int_0^{kT} |\tilde{u}(t)|^p dt \leq c_k \int_0^{kT} |\dot{u}(t)|^p dt. \quad (4)$$

It follows from (A) that the functional φ_k on $W_{kT}^{1,p}$ given by

$$\varphi_k(u) = \frac{1}{p} \int_0^{kT} |\dot{u}(t)|^p dt + \int_0^{kT} F(t, u(t)) dt$$

is continuously differentiable on $W_{kT}^{1,p}$ (see Theorem 1.4 in [5]). Moreover, one has

$$\langle \varphi_k'(u), v \rangle = \int_0^{kT} |\dot{u}(t)|^{p-2} (\dot{u}(t), \dot{v}(t)) dt + \int_0^{kT} (\nabla F(t, u(t)), v(t)) dt$$

for all $u, v \in W_{kT}^{1,p}$. It is well known that the kT -periodic solutions of problem (1) correspond to the critical points of the functional φ_k .

For convenience to quote we state an analog of Egorov's theorem (see Lemma 2 in [10]), in which we replace F by $-F$.

LEMMA 1. (see [10]) *Suppose that F satisfies (A) and E is a measurable subset of $[0, T]$. Assume that*

$$F(t, x) \rightarrow -\infty \quad \text{as} \quad |x| \rightarrow \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$ there exists a subset E_δ of E with $\text{meas}(E \setminus E_\delta) < \delta$ such that

$$F(t, x) \rightarrow -\infty \quad \text{as} \quad |x| \rightarrow \infty$$

uniformly for all $t \in E_\delta$.

LEMMA 2. *Assume that $F(t, x)$ satisfies (A), (I_2) , (I_3) , (I_5) and (I_7) . Then φ_k satisfies the (P.S.) condition, that is, $\{u_n\}$ has a convergent subsequence whenever it satisfies $\varphi_k'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varphi_k(u_n)\}$ is bounded.*

Proof. By (4), we have

$$\left(\int_0^{kT} |\dot{u}_n(t)|^p dt \right)^{\frac{1}{p}} \leq \|\tilde{u}_n\| \leq (1 + c_k)^{\frac{1}{p}} \left(\int_0^{kT} |\dot{u}_n(t)|^p dt \right)^{\frac{1}{p}} \quad (5)$$

for all n . For any $\varepsilon > 0$, it follows from (5) and Young's inequality that

$$\begin{aligned} \left| \int_0^{kT} (\nabla H(t, u(t)), \tilde{u}(t)) dt \right| &\leq \int_0^{kT} f(t) |\bar{u} + \tilde{u}(t)|^\alpha |\tilde{u}(t)| dt + \int_0^{kT} g(t) |\tilde{u}(t)| dt \\ &\leq \int_0^{kT} 2^\alpha f(t) (|\bar{u}|^\alpha + |\tilde{u}(t)|^\alpha) |\tilde{u}(t)| dt + \int_0^{kT} g(t) |\tilde{u}(t)| dt \\ &\leq 2^\alpha (|\bar{u}|^\alpha + \|\tilde{u}\|_\infty^\alpha) \|\tilde{u}\|_\infty \int_0^{kT} f(t) dt + \|\tilde{u}\|_\infty \int_0^{kT} g(t) dt \\ &\leq 2^\alpha \varepsilon \|\tilde{u}\|_\infty^p + 2^\alpha \varepsilon^{-\frac{1}{p-1}} |\bar{u}|^{q\alpha} \left(\int_0^{kT} f(t) dt \right)^q \\ &\quad + 2^\alpha \|\tilde{u}\|_\infty^{\alpha+1} \int_0^{kT} f(t) dt + \|\tilde{u}\|_\infty \int_0^{kT} g(t) dt. \end{aligned}$$

Let $\varepsilon = 1/(2^{2\alpha} 2p c_k)$, by (3) we have

$$\begin{aligned} \left| \int_0^{kT} (\nabla H(t, u(t)), \tilde{u}(t)) dt \right| &\leq \frac{1}{2p} \int_0^{kT} |\dot{u}(t)|^p dt + C_1 |\bar{u}|^{q\alpha} + C_2 \left(\int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{\alpha+1}{p}} \\ &\quad + C_3 \left(\int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{1}{p}} \end{aligned}$$

for all $u \in W_{kT}^{1,p}$ and some positive constants C_1 , C_2 and C_3 . From (2) and (3) we obtain

$$\begin{aligned} \int_0^{kT} (\nabla G(u(t)), \tilde{u}(t)) dt &= \int_0^{kT} (\nabla G(u(t)) - \nabla G(\bar{u}), \tilde{u}(t)) dt \\ &\geq -r \int_0^{kT} |\tilde{u}(t)|^2 dt \\ &\geq -rkT \|\tilde{u}\|_\infty^2 \\ &\geq -rkT \left(c_k \int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{2}{p}} \end{aligned}$$

for all $u \in W_{kT}^{1,p}$. Hence one has

$$\begin{aligned} \|\tilde{u}_n\| &\geq |\langle \Phi'_k(u_n), \tilde{u}_n \rangle| \\ &= \left| \int_0^{kT} |\dot{u}_n(t)|^p dt + \int_0^{kT} (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \right| \\ &= \left| \int_0^{kT} |\dot{u}_n(t)|^p dt + \int_0^{kT} (\nabla G(u_n(t)), \tilde{u}_n(t)) dt + \int_0^{kT} (\nabla H(t, u_n(t)), \tilde{u}_n(t)) dt \right| \end{aligned}$$

$$\begin{aligned} &\geq \frac{2p-1}{2p} \int_0^{kT} |\dot{u}_n(t)|^p dt - C_1 |\bar{u}_n|^{q\alpha} - C_2 \left(\int_0^{kT} |\dot{u}_n(t)|^p dt \right)^{\frac{\alpha+1}{p}} \\ &\quad - C_3 \left(\int_0^{kT} |\dot{u}_n(t)|^p dt \right)^{\frac{1}{p}} - rkT \left(c_k \int_0^{kT} |\dot{u}_n(t)|^p dt \right)^{\frac{2}{p}} \end{aligned}$$

for large n . By (5) and the above inequality we have

$$C |\bar{u}_n|^{\frac{q\alpha}{p}} \geq \left(\int_0^{kT} |\dot{u}_n(t)|^p dt \right)^{\frac{1}{p}} - C_4 \quad (6)$$

for some constants $C > 0$, $C_4 > 0$ and all large n , which implies that

$$\|\tilde{u}_n\|_\infty \leq C_5 (|\bar{u}_n|^{\frac{q\alpha}{p}} + 1)$$

for all large n and some positive constant C_5 by (3). Then one has

$$|u_n(t)| \geq |\bar{u}_n| - |\tilde{u}_n(t)| \geq |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geq |\bar{u}_n| - C_5 \left(|\bar{u}_n|^{\frac{q\alpha}{p}} + 1 \right)$$

for all large n and every $t \in [0, kT]$.

If $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar{u}_n| \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (7)$$

Hence

$$|u_n(t)| \geq \frac{1}{2} |\bar{u}_n| \quad (8)$$

for all large n and every $t \in [0, kT]$ because of $|u_n(t)| \geq |\bar{u}_n| - C_5 \left(|\bar{u}_n|^{\frac{q\alpha}{p}} + 1 \right)$.

Set $\delta = \text{meas}(E/2)$. It follows from (l_7) and Lemma 1 that there exists a subset E_δ of E with $\text{meas}(E \setminus E_\delta) < \delta$ such that

$$|x|^{-q\alpha} F(t, x) \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty$$

uniformly for all $t \in E_\delta$, which implies that

$$\text{meas}(E_\delta) = \text{meas}(E) - \text{meas}(E \setminus E_\delta) > \delta > 0, \quad (9)$$

and for every $\beta > 0$, there exists $M \geq 1$ such that

$$|x|^{-q\alpha} F(t, x) \leq -\beta \quad (10)$$

for all $|x| \geq M$ and all $t \in E_\delta$. By (8) and (7), one has

$$|u_n(t)| \geq M \quad (11)$$

for large n and every $t \in [0, kT]$. It follows from (I_3) , (6), (8) and (9)-(11) that

$$\begin{aligned} \varphi_k(u_n) &\leq \left(C|\bar{u}_n|^{\frac{q\alpha}{p}} + C_4 \right)^p + \int_{[0, kT] \setminus E_\delta} \gamma(t) dt - \int_{E_\delta} \beta |u_n(t)|^{q\alpha} dt \\ &\leq \left(C|\bar{u}_n|^{\frac{q\alpha}{p}} + C_4 \right)^p + \int_{[0, kT] \setminus E_\delta} \gamma(t) dt - 2^{-q\alpha} |\bar{u}_n|^{q\alpha} \delta \beta \end{aligned}$$

for all large n . Hence we have

$$\limsup_{n \rightarrow \infty} |\bar{u}_n|^{-q\alpha} \varphi_k(u_n) \leq C^p - 2^{-q\alpha} \delta \beta.$$

By the arbitrariness of $\beta > 0$, one has

$$\limsup_{n \rightarrow \infty} |\bar{u}_n|^{-q\alpha} \varphi_k(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi_k(u_n)$. Hence $\{|\bar{u}_n|\}$ is bounded. Furthermore, $\{u_n\}$ is bounded by (5) and (6). Then we can use a same argument as in [15] to show that φ_k satisfies the (P.S.) condition. \square

Now we prove our Theorem 3 first.

PROOF OF THEOREM 3. It follows from Lemma 2 that φ_k satisfies the (P.S.) condition. Now we prove that φ_k satisfies the other conditions of the Saddle Point Theorem (see Theorem 4.7 in [5]). Let $\tilde{W}_{kT}^{1,p}$ be the subspace of $W_{kT}^{1,p}$ given by

$$\tilde{W}_{kT}^{1,p} = \{u \in W_{kT}^{1,p} \mid \bar{u} = 0\}.$$

Set

$$e_k(t) = k(\cos k^{-1} \omega t) x_0$$

for all $t \in \mathbb{R}$ and some $x_0 \in \mathbb{R}^N$ with $|x_0| = 1$, where $\omega = \frac{2\pi}{T}$. Then we have

$$\dot{e}_k(t) = -\omega(\sin k^{-1} \omega t) x_0$$

for all $t \in \mathbb{R}$. By the Saddle Point Theorem we only need to prove

- (i) $\varphi_k(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$ in $\tilde{W}_{kT}^{1,p}$,
- (ii) $\varphi_k(x + e_k) \rightarrow -\infty$ as $|x| \rightarrow \infty$ in \mathbb{R}^N .

It follows from (I_3) and (10) that

$$\begin{aligned} \varphi_k(x + e_k) &= \frac{1}{p} \int_0^{kT} |\dot{e}_k(t)|^p dt + \int_0^{kT} F(t, x + e_k) dt \\ &\leq \frac{1}{p} \omega^p kT + \int_0^{kT} F(t, x + k(\cos k^{-1} \omega t) x_0) dt \\ &\leq \frac{1}{p} \omega^p kT + \int_{[0, kT] \setminus E_\delta} \gamma(t) dt - \beta \int_{E_\delta} |x + k(\cos k^{-1} \omega t) x_0|^{q\alpha} dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{p} \omega^p kT + \int_{[0, kT] \setminus E_\delta} \gamma(t) dt - \beta M^{q\alpha} \text{meas}(E_\delta) \\
&\leq \frac{1}{p} \omega^p kT + \int_{[0, kT] \setminus E_\delta} \gamma(t) dt - \beta \text{meas}(E_\delta)
\end{aligned}$$

for all $|x| \geq M + k$, which implies (ii) by the arbitrariness of β .

It follows from (l₅) and (3) that

$$\begin{aligned}
\left| \int_0^{kT} [H(t, u(t)) - H(t, 0)] dt \right| &= \left| \int_0^{kT} \int_0^1 (\nabla H(t, su(t)), u(t)) ds dt \right| \\
&\leq \int_0^{kT} \int_0^1 f(t) |su(t)|^\alpha |u(t)| ds dt \\
&\quad + \int_0^{kT} \int_0^1 g(t) |u(t)| ds dt \\
&\leq \int_0^{kT} f(t) |u(t)|^\alpha |u(t)| dt + \int_0^{kT} g(t) |u(t)| dt \\
&\leq \|u\|_\infty^{\alpha+1} \int_0^{kT} f(t) dt + \|u\|_\infty \int_0^{kT} g(t) dt \\
&\leq C_6 \left(\int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{\alpha+1}{p}} + C_7 \left(\int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{1}{p}}
\end{aligned}$$

for all $u \in \widetilde{W}_{kT}^{1,p}$ and some positive constants C_6 and C_7 . By (l₂) and (3) we have

$$\begin{aligned}
\int_0^{kT} [G(u(t)) - G(0)] dt &= \int_0^{kT} \int_0^1 (\nabla G(su(t)) - \nabla G(0), u(t)) ds dt \\
&= \int_0^{kT} \int_0^1 \frac{1}{s} (\nabla G(su(t)) - \nabla G(0), su(t)) ds dt \\
&\geq \int_0^{kT} \int_0^1 \frac{1}{s} (-rs^2 |u(t)|^2) ds dt \\
&= -\frac{r}{2} \int_0^{kT} |u(t)|^2 dt \\
&\geq -\frac{rkT}{2} \left(c_k \int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{2}{p}}
\end{aligned}$$

for all $u \in \widetilde{W}_{kT}^{1,p}$. Hence one has

$$\begin{aligned}
\varphi_k(u) - \int_0^{kT} F(t, 0) dt &= \frac{1}{p} \int_0^{kT} |\dot{u}(t)|^p dt + \int_0^{kT} [G(u(t)) - G(0)] dt \\
&\quad + \int_0^{kT} [H(t, u(t)) - H(t, 0)] dt \\
&\geq \frac{1}{p} \int_0^{kT} |\dot{u}(t)|^p dt - C_6 \left(\int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{\alpha+1}{p}}
\end{aligned}$$

$$-C_7 \left(\int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{1}{p}} - \frac{rkT}{2} \left(c_k \int_0^{kT} |\dot{u}(t)|^p dt \right)^{\frac{2}{p}}$$

for all $u \in \widetilde{W}_{kT}^{1,p}$, which implies (i) by (5). So there exists a critical point $u_k \in W_{kT}^{1,p}$ for φ_k such that

$$-\infty < \inf_{\widetilde{W}_{kT}^{1,p}} \varphi_k \leq \varphi_k(u_k) \leq \sup_{\mathbb{R}^N + e_k} \varphi_k.$$

For fixed $x \in \mathbb{R}^N$, set

$$A_k = \{t \in [0, kT] \mid |x + k(\cos k^{-1} \omega t)x_0| \leq M\}.$$

Then we have

$$\text{meas}(A_k) \leq \frac{k\delta}{2} \quad (12)$$

for all large k , where δ is the same as the one in Lemma 2. In fact, if $\text{meas}(A_k) > \frac{k\delta}{2}$, there exists $t_1 \in A_k$ such that

$$\frac{1}{8}k\delta \leq t_1 \leq \frac{1}{2}kT - \frac{1}{8}k\delta \quad (13)$$

or

$$\frac{1}{2}kT + \frac{1}{8}k\delta \leq t_1 \leq kT - \frac{1}{8}k\delta. \quad (14)$$

Moreover, there exists $t_2 \in A_k$ such that

$$|t_2 - t_1| \geq \frac{1}{8}k\delta \quad (15)$$

and

$$|t_2 - (kT - t_1)| \geq \frac{1}{8}k\delta. \quad (16)$$

It follows from (16) that

$$\left| \frac{1}{2}(k^{-1}t_1 + k^{-1}t_2) - \frac{1}{2}T \right| \geq \frac{1}{16}\delta. \quad (17)$$

By (13) and (14), one has

$$\frac{1}{16}\delta \leq \frac{1}{2}(k^{-1}t_1 + k^{-1}t_2) \leq T - \frac{1}{16}\delta. \quad (18)$$

From (17) and (18) we obtain

$$\left| \sin \left(\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2)\omega \right) \right| \geq \sin \left(\frac{1}{16}\omega\delta \right).$$

Furthermore, by (15) we have

$$\begin{aligned} & |\cos(k^{-1}\omega t_1) - \cos(k^{-1}\omega t_2)| \\ &= 2 \left| \sin\left(\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2)\omega\right) \right| \left| \sin\left(\frac{1}{2}(k^{-1}t_1 - k^{-1}t_2)\omega\right) \right| \\ &\geq 2 \sin^2\left(\frac{1}{16}\omega\delta\right). \end{aligned}$$

But due to $t_1, t_2 \in A_k$, one has

$$\begin{aligned} & |\cos(k^{-1}\omega t_1) - \cos(k^{-1}\omega t_2)| \\ &= \frac{1}{k} \left| x + k(\cos k^{-1}\omega t_1)x_0 - (x + k(\cos k^{-1}\omega t_2)x_0) \right| \\ &\leq \frac{2M}{k}, \end{aligned}$$

which is a contradiction for large k . Hence (12) holds. Let

$$E_k = \bigcup_{j=0}^{k-1} (jT + E_\delta).$$

Then it follows from (12) that

$$\text{meas}(E_k \setminus A_k) \geq \frac{1}{2}k\delta$$

for large k . By (10) and (13) we have

$$\begin{aligned} k^{-1}\varphi_k(x + e_k) &\leq \frac{1}{p}\omega^p T + k^{-1} \int_0^{kT} F(t, x + k(\cos k^{-1}\omega t)x_0) dt \\ &\leq \frac{1}{p}\omega^p T + k^{-1} \int_{[0, kT] \setminus (E_k \setminus A_k)} \gamma(t) dt - k^{-1}\beta \text{meas}(E_k \setminus A_k) \\ &\leq \frac{1}{p}\omega^p T + \int_0^T |\gamma(t)| dt - \frac{1}{2}\delta\beta \end{aligned}$$

for every $x \in \mathbb{R}^N$ and all large k . Hence one has

$$\sup_{x \in \mathbb{R}^N} k^{-1}\varphi_k(x + e_k) \leq \frac{1}{p}\omega^p T + \int_0^T |\gamma(t)| dt - \frac{1}{2}\delta\beta$$

for all large k , which implies that

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^N} k^{-1}\varphi_k(x + e_k) \leq \frac{1}{p}\omega^p T + \int_0^T |\gamma(t)| dt - \frac{1}{2}\delta\beta.$$

By the arbitrariness of β , we obtain

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^N} k^{-1}\varphi_k(x + e_k) = -\infty,$$

which follows that

$$\limsup_{k \rightarrow \infty} k^{-1} \varphi_k(u_k) = -\infty. \quad (19)$$

Now we prove that $\|u_k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$. If not, going to a subsequence if necessary, we may assume that $\|u_k\|_\infty \leq C_8$ for all $k \in N$ and some positive constant C_8 . Hence we have

$$\begin{aligned} k^{-1} \varphi_k(u_k) &\geq k^{-1} \int_0^{kT} F(t, u_k(t)) dt \geq -k^{-1} \max_{0 \leq s \leq C_8} a(s) \int_0^{kT} b(t) dt \\ &= - \max_{0 \leq s \leq C_8} a(s) \int_0^T b(t) dt. \end{aligned}$$

It follows that $\liminf_{k \rightarrow \infty} k^{-1} \varphi_k(u_k) > -\infty$, which contradicts (19). Therefore we complete our proof. \square

Then we prove our Theorems 1 and 2.

PROOF OF THEOREMS 1 AND 2. Theorem 1 follows from Theorem 3 by letting $\alpha = 0$. Theorem 3 implies Theorem 2 because (l_3) follows from (l_6) and (A). In fact, by (l_6) there exists $M > 0$ such that

$$|x|^{-q\alpha} F(t, x) \leq 0$$

for all $|x| \geq M$ and a.e. $t \in [0, T]$, which implies that $F(t, x) \leq 0$ for all $|x| \geq M$ and a.e. $t \in [0, T]$. It follows from (A) that

$$F(t, x) \leq \max_{0 \leq s \leq M} a(s) b(t)$$

for all $|x| \leq M$ and a.e. $t \in [0, T]$. Now (l_3) holds with

$$\gamma(t) = \max_{0 \leq s \leq M} a(s) b(t).$$

Hence Theorem 2 follows from Theorem 3. \square

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