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NONLINEAR DEGENERATE DIFFUSION PROBLEMS WITH A SINGULARITY

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Abstract. We consider a class of degenerate nonlinear diffusion problems with a singularity in a finite value $M > 0$ of the unknown v. For such problems, we introduce a notion of renormalized entropy solution which (under a particular "growth" assumptions on the diffusion term) can reach the value M. We prove the existence of such a solution for the stationary equation with L^1 data.

1. Introduction

We consider a class of diffusion problems, in the stationary case with a singularity with respect to the unknown of the type:

$$P_{b,g}(f,a)\begin{cases} b(v)-\operatorname{div}A(v,\nabla g(v))=f & \text{in } \Omega,\\ v=a & \text{on } \Gamma:=\partial\Omega,\end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with regular boundary if $N>1$, $M>0$, $f\in L^1(\Omega)$, $a:\Gamma\to\mathbb{R}$ is measurable with $g(a)=0$ a.e. on Γ and $b:(-\infty,M]\to\mathbb{R}$ is nondecreasing, continuous such that $b(0)=0$.

The function $g:(-\infty,M)\to\mathbb{R}$ has a flat region $[A_1,A_2]$ with $A_1\leqslant 0\leqslant A_2<M$ on which it keeps a constant value and satisfies:

$$\begin{cases} g \text{ is continuous, nondecreasing, locally Lipschitz on } (-\infty,M),\\ C^1 \text{ and strictly increasing in } [A_2,M) \text{ with } \lim\limits_{r\to M^-} g'(r)=+\infty.\end{cases} \tag{1.1}$$

The vector field $A:(-\infty,M)\times\mathbb{R}^N\to\mathbb{R}^N$ is a Caratheodory function which satisfies the following conditions.

The growth condition: there exists $p>1$ such that

$$|A(r,\xi)-A(r,0)|\leqslant C\Big(\Big|\frac{Mr}{M-r^+}\Big|\Big)|\xi|^{p-1} \quad \text{for all } (r,\xi)\in(-\infty,M)\times\mathbb{R}^N \text{ a.e. on } \Omega, \tag{1.2}$$

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where $C:\mathbb{R}_+\to\mathbb{R}_+$ is increasing with $C(0)=0$;

The weak coerciveness condition:

$$(A(r,\xi)-A(r,0))\cdot\xi\geqslant\lambda(r)|\xi|^p\quad\text{for all } r\in(-\infty,M),\ \xi\in\mathbb{R}^N, \tag{1.3}$$

where $\lambda:(-\infty,M)\to(0,\infty)$ satisfies

$$\lambda_k:=\inf_{\{r\in(-\infty,M];-k\leqslant b(r)\}}\lambda(r)>0\ \text{ for all } k>0;$$

The monotonicity condition:

$$(A(r,\xi)-A(s,\eta))\cdot(\xi-\eta)\geqslant(B(g(r))-B(g(s)))(1+|\xi|^p+|\eta|^p) \tag{1.4}$$

for all $r,s\in\mathbb{R}$, $\xi,\eta\in\mathbb{R}^N$, and for some function $B:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which is locally Lipschitz continuous on $(-\infty,A_1[\cup[A_2,M)$.

Hypothesis (1.4) implies in particular that

$$(A(r,\xi)-A(r,\eta))\cdot(\xi-\eta)\geqslant 0 \text{ for all } r\in(-\infty,M)$$

and assumption (1.1) implies that $a\in[A_1,A_2]$ a.e. on Γ and that $\overline{A}(a)\in L^1(\Gamma)$, where $\overline{A}:\mathbb{R}\times\partial\Omega\to\mathbb{R}$ is defined by

$$\overline{A}(s):=\sup\left\{|A(r,0)\cdot\vec{\eta}(x)|,\ r\in[-s^-,s^+]\right\}.$$

Here, by $\vec{\eta}(x)$, we denote the out unit normal to Ω in $x\in\partial\Omega$.

We will investigate the question of existence in two cases:

(a)

$$\int_{A_2}^{M}(C(\frac{Mr}{M-r^+}))^{\frac{p}{p-1}}dg(r)<\infty. \tag{1.5}$$

(b)

$$\int_{A_2}^{M}(\lambda(r))^{\frac{1}{p-1}}dg(r)=+\infty, \tag{1.6}$$

Assumption (1.5) implies in particular that $\lim_{r\to M-}g(r)<+\infty$ and we denote also by g the continuous extension of g on $(-\infty,M]$. In case **(b)**, we construct by approximation a solution v such that $v<M$ a.e. in Ω but in case **(a)**, for $f\in L^1(\Omega)$, the solution may reach the value M and the behaviour on the subset $\{v=M\}$ has to be specified.

As g is assumed to be constant in $[A_1,A_2]$, the problem is ill-posed even in the variational setting and the weak entropy solution in the sense of [28], [29] is more suitable in order to assure uniqueness results. Furthermore, it is well known (at least as far as a reader which is familiar with hyperbolic problems is concerned) that the condition on the boundary can not be assumed pointwise but has to be understood as an entropy condition on the boundary (see [31], [17], [6] and the bibliography therein).

Let us also emphasize that due the the luck of regularity of the data f and a which are only assumed to be L^1, we can not prove the existence and uniqueness of a weak solution in the usual distributional sense. Our aim, is to establisch these results in the framework of renormalised entropy solutions as defined in [4], [19] and [18].

For simplicity, we focus on the case where

$$\lim_{r\to-\infty}(b+g)(r) = -\infty. \tag{1.7}$$

This forces the renormalized entropy solution to avoid the value $-\infty$ and we do not have to handle its behaviour on this set. The reader interested in the case where $\lim_{r\to-\infty}(b+g)(r) = -\infty$ in referred to [5].

More results on similar problems can be found among other manuscripts in [14], [15] and the bibliography therein. For more results on degenerate diffusion problems, the reader is referred to [17], [32], [33], [2] and [8].

In a forthcoming paper, the second author studies the corresponding evolution problem and proves similar existence and partial uniqueness results.

The outline of the paper is as follows: In the following section, after a short introduction of our notations, we give our concept of renormalized entropy solution with a few comments and we present the main results. Finally, in Section 3, we establish the existence result in the two cases **(a)** and **(b)**.

2. Notations, definitions and main results

We denote by $\mathscr{M}(\overline{\Omega})$ the set of Radon measures on Ω and by $M(\Gamma)$ the set of measurable functions on Γ with values in $\mathbb{R}$. For any measurable function $v:\Omega\to\mathbb{R}$, for any $s\in\overline{\mathbb{R}}$, we denote by $\chi_{\{v<s\}}$ (resp $\chi_{\{v>s\}}$, $\chi_{\{v=s\}}$) the characteristic function of the set $\{x\in\Omega;\, v(x)<s\}$ (resp, $\{x\in\Omega;\, v(x)>s\}$, $\{x\in\Omega;\, v(x)=s\}$). For all $l,\beta>0$ such that $M-\beta>0$, we define $T_{\beta,l}:\mathbb{R}\to\mathbb{R}$ by $T_{\beta,l}(r)=(M-\beta)\wedge\big(-l\vee r\big)$ and for $n\in\mathbb{N}$ large enough, we define

$$h_n : r\mapsto h_n(r) := \begin{cases} 0 & \text{for } r\leqslant -n-1 \text{ or } M-\frac{1}{n}>r,\\ r+n+1 & \text{for } -n-1<r\leqslant -n,\\ 1 & \text{for } -n<r\leqslant M-\frac{2}{n},\\ -n(r+M-\frac{1}{n}) & \text{for } M-\frac{2}{n}<r\leqslant M-\frac{1}{n}.\end{cases}$$

The operators sign^+ and H_0 are defined as follows: $\operatorname{sign}^+(r)=0$ if $r<0$, $=[0,1]$ if $r=0$ and $=1$ if $r>0$ and

$$H_0(s) := \begin{cases} 1 & \text{if } s>0,\\ 0 & \text{if } s\leqslant 0.\end{cases}$$

Moreover, for $r,k\in\mathbb{R}$, we set $r\vee k=\max(r,k)$, $r\wedge k=\min(r,k)$. By $T^1, T^{1,2}$ and T^2, we denote the truncation functions defined successively by

$$T^1(r)=r\wedge A_1,\ \ T^{1,2}(r)=A_1\vee r\wedge A_2 \ \text{ and } \ T^2(r)=r\vee A_2$$

and for $k,l \in \mathbb{R}$, for a.e. $x \in \Gamma$, we define

$$\omega^+(x,k,l) := \max_{k \leqslant r,s \leqslant l \vee k} |(A(r,0) - A(s,0)) \cdot \vec{\eta}(x)|,$$
$$\omega^-(x,k,l) := \max_{l \wedge k \leqslant r,s \leqslant k} |(A(r,0) - A(s,0)) \cdot \vec{\eta}(x)|.$$

3. Existence results in case (a)

Throughout this section, we suppose that conditions (1.2)-(1.5) and (1.7) are satisfied.

DEFINITION 3.1. Let $f \in L^1(\Omega)$ and $a : \Gamma \to \mathbb{R}$ be measurable with $g(a) = 0$ a.e. on Γ. A measurable function $v : \Omega \to (-\infty, M]$ is a renormalized entropy solution of the problem $P_{b,g}(f,a)$ if

$$b(v) \in L^1(\Omega),$$
$$g(v)\chi_{\{-k<v<M-\beta\}} \in W_0^{1,p}(\Omega),\ \ \forall k,\beta > 0 \text{ with } \beta < M, \tag{3.1}$$
$$A(v, Dg(v))\chi_{\{-k<v<M\}} \in (L^{p'}(\Omega))^N,\ \ \forall k > 0, \tag{3.2}$$

and there exist two families $(\mu_\beta)_\beta$ and $(\nu_l)_l$ of bounded measures on $\overline{\Omega}$ such that

$$\mu_\beta \in (W^{-1,p'}(\Omega) + L^1(\Omega) + L^1(\Gamma)) \cap \mathscr{M}(\overline{\Omega}), \tag{3.3}$$
$$\mu_\beta(\{v \leqslant M - \beta\}) = 0,\ \nu_l(\{v \geqslant l\}) = 0,$$
$$\lim_{\beta \to 0} \int_\Omega \xi \, d\mu_\beta(v) = 0 \text{ for all } \xi \in \mathscr{D}(\mathbb{R}^N) \text{ with } \operatorname{supp}(\nabla\xi) \subset \{v < M\},$$
$$\lim_{l \to -\infty} \int_\Omega \xi \, d\nu_l(v) = 0 \text{ for all } \xi \in \mathscr{D}(\mathbb{R}^N)$$

and the following inequalities are satisfied. The first one: for all $\beta, k \in \mathbb{R}$ with $M - \beta > k$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $(g(a \wedge (M-\beta)) - g(k))^+ \xi = 0$ a.e. on Γ,

$$\begin{aligned} &-\int_\Omega b(v \wedge (M-\beta))\chi_{\{v\wedge(M-\beta)>k\}}\xi + \int_\Omega \chi_{\{v\wedge(M-\beta)>k\}} f\xi \\ &\quad - \int_\Omega \chi_{\{v\wedge(M-\beta)>k\}}(A(v\wedge(M-\beta), \nabla g(v \wedge (M-\beta))) - A(k,0)) \cdot \nabla\xi \\ &\qquad \geqslant -\int_\Gamma \omega^+(x,k,a\wedge(M-\beta))\xi + \int_\Omega \xi \, d\mu_\beta(v) \end{aligned} \tag{3.4}$$

and the second one: and for all $l \leqslant k < M \in \mathbb{R}$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $(g(k) - g(a \vee l))^+ \xi = 0$ a.e. on Γ,

$$\begin{aligned} &\int_\Omega b(v \vee l)\chi_{\{k>v\vee l\}}\xi + \int_\Omega \chi_{\{k>v\vee l\}}(A(v\vee l, \nabla g(v \vee l)) - A(k,0)) \cdot \nabla\xi \\ &\qquad - \int_\Omega \chi_{\{k>v\vee l\}} f\xi \geqslant -\int_\Gamma \omega^-(x,k,a\vee l)\xi + \int_\Omega \xi \, d\nu_l(v). \end{aligned} \tag{3.5}$$

REMARK 3.2. We emphasize the following four remarks.

1. In (3.2), we denote by $\chi_{\{<-k<v<M\}}A(v,Dg(v))$ (in (3.2)), the measurable field on Ω satisfying

$$\chi_{\{|v|<l\}}\chi_{\{-k<v<M\}}A(v,Dg(v)) = \nabla g(T_l v) \text{ for all } l>0 \text{ with } l<\min(M,k).$$

It makes sense thanks to condition (3.1) on v.

2. As g is strictly increasing on $[A_2,M)$, it follows from the definition that the function $T_{A_2+L}v - T_{A_2}v \in W_0^{1,p}(\Omega)$ for all $L>0$ with $A_2+L<M$.

3. Suppose that g is strictly increasing and assume (1.5). It follows in particular that $T_{\beta,l}v \in W_0^{1,p}(\Omega)$ for all $\beta, l>0$ with $M-\beta>0$. Taking $\xi = \varphi h(v)$ as a test function in (3.4) with $\varphi \in \mathscr{D}(\Omega)$ and $h \in W^{1,\infty}(\mathbb{R})$ such that supp$(h) \subset]-\infty, M[$, letting $\beta \to 0$, we find

$$\int_\Omega \chi_{\{k<v<M\}}(b(v) + (A(v,\nabla g(v)) - A(k,0))\cdot\nabla(h(v)\varphi) - f\varphi) \leqslant 0. \qquad (3.6)$$

Letting $k \to -\infty$, we find

$$\int_\Omega \chi_{\{v<M\}}(b(v) + (A(v,\nabla g(v)) - A(k,0))\cdot\nabla(h(v)\varphi) - f\varphi) \leqslant 0.$$

Similarly, taking $\xi = h(v)\varphi$ as test function in (3.5), letting $l \to -\infty$, for all $k<M$, we find

$$-\int_\Omega \chi_{\{k>v\}}(b(v)\varphi h(v) + (A(v,\nabla g(v)) - A(k,0))\cdot\nabla(h(v)\varphi) - f\varphi) \leqslant 0. \qquad (3.7)$$

Combining (3.6) and (3.7), we get

$$-\int_\Omega \chi_{\{M>v\}}(b(v)\varphi h(v) + (A(v,\nabla g(v)) - A(M,0))\cdot\nabla(h(v)\varphi) - f\varphi) = 0 \qquad (3.8)$$

for all $\varphi \in \mathscr{D}(\varphi)(\Omega)$ $h \in W^{1,\infty}(\Omega)$ such that supp$(h) \subset (-\infty, M)$.

4. Now, taking $\xi = \varphi(1-h_n(v^+)) \in W^{1,p}(\Omega)$, with $\varphi \in \mathscr{D}^+(\Omega)$ and $\nabla\varphi = 0$ a.e. on $v=M$ as test function in (3.8), we get

$$-\int_\Omega \chi_{\{M\geqslant v\}}(b(v)\varphi h(v) + (A(v,\nabla g(v)) - A(M,0))\cdot\nabla(h(v)\varphi) - f\varphi)$$
$$= -\int_{\{v=M\}} b(M)\varphi h(v) + \int_{\{v=M\}} f\varphi h(v).$$

Letting $n \to +\infty$, we find the energie estimate

$$\int_{\{v=M\}} f\varphi - b(M)\int_{\{v=M\}}\varphi = \lim_{n\to+\infty}\int_{\{M_{\frac{2}{n}}\leqslant v\leqslant M-\frac{1}{n}\}} A(v,\nabla g(v))\cdot\nabla v\varphi.$$

Similarly, taking $\xi := \varphi(1-h_n(-v^-))$ with $\varphi \in \mathscr{D}^+(\Omega)$ as test function in (3.8), letting $n \to +\infty$, we find

$$\lim_{n\to+\infty}\int_{\{-n-1\leqslant v\leqslant -n\}} A(v,\nabla g(v))\cdot\nabla\varphi = 0.$$

Here, we denote $r^+ = r\vee 0$ and $r^- = r\wedge 0$.

THEOREM 3.3. *For all $f \in L^1(\Omega)$ and $a : \Gamma \to \mathbb{R}$ measurable with $g(a) = 0$ a.e. on Γ, there exists $v : \Omega \to \overline{\mathbb{R}}$ such that v is a renormalized entropy solution of $P_{b,g}(f,a)$.*

Proceeding as in [5], we prove the following comparison principle.

THEOREM 3.4. *Let $a_i : \Gamma \to \mathbb{R}$ with $g(a_i) = 0$ a.e. on Γ and $f_i \in L^1(\Omega)$, $i = 1,2$. Let v_1 be a renormalized entropy solution of the problem $P_{b,g}(a_1, f_1)$, v_2 be a renormalized entropy solution of the problem $P_{b,g}(a_2, f_2)$. Then, for all $\beta, l > 0$ with $\beta < M$, there exists $\kappa \in L^\infty(\Omega)$ with $\kappa \in sign^+(-l \vee v_1 \wedge (M-\beta) - (-l \vee v_2 \wedge (M-\beta))$ a.e. in Ω such that, for any $\xi \in \mathscr{D}(\mathbb{R}^N)$,*

$$\begin{aligned}
&\int_\Omega (b(T_{\beta,l}v_1) - b(T_{\beta,l}v_2))^+ \xi \\
&\quad + \int_\Omega (A(T_{\beta,l}v_1, \nabla g(T_{\beta,l}v_1)) - A(T_{\beta,l}v_2, \nabla g(T_{\beta,l}v_2))) \cdot \nabla \xi \\
&\qquad \leqslant \int_\Omega \kappa (f_1 - f_2)\xi - 2\int_\Omega \xi \, d(\mu_\beta(v_1) + \nu_{-l}(v_2)) \\
&\qquad\quad + \int_\Gamma \omega^-(x, T_{\beta,l}a_1, T_{\beta,l}a_2)\xi.
\end{aligned} \tag{3.9}$$

As a consequence, we deduce the following "partial uniqueness" result.

THEOREM 3.5. *Let $f \in L^1(\Omega)$ and $a : \Gamma \to \mathbb{R}$ be piecewise continuous with $g(a) = 0$ a.e. on Γ. Let v_i, $i = 1,2$ be a renormalized entropy solutions of $P_{b,g}(f,a)$. Then, $b(v_1) = b(v_2)$ a.e. in Ω.*

The proofs of Theorem 3.4 and Theorem 3.5 follow the same lines as those in [5] and are omitted here for convenience.

3.1. Proof of the existence result

For $\varepsilon > 0$ with $M - \varepsilon > A_2$, let $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ and $A_\varepsilon : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be defined by:

$$g_\varepsilon(r) = \begin{cases} g(r) & \text{if } r \leqslant M - \varepsilon, \\ g(M-\varepsilon) + g'(M-\varepsilon)(r - M + \varepsilon) & \text{if } r \geqslant M - \varepsilon, \end{cases}$$

and

$$A_\varepsilon(r,\xi) = \begin{cases} A(r,\xi) & \text{if } r \leqslant M - \varepsilon, \\ A(M-\varepsilon, \xi) & \text{if } r \geqslant M - \varepsilon. \end{cases}$$

The proof consists in two steps: in a first step, we consider the problem

$$P_{b_\alpha,\varepsilon}(f,a) \; \begin{cases} b_\alpha(v) - \operatorname{div} A_\varepsilon(v, \nabla g_\varepsilon(v)) = f & \text{in } \Omega, \\ g_\varepsilon(v) = g_\varepsilon(a) & \text{on } \Gamma, \end{cases}$$

with $f \in L^\infty(\Omega)$, $a \in L^\infty(\Gamma)$ $g(a) = 0$ and with $b_\alpha(r) = b(r) + \alpha r$ for $r < M$ and $b_\alpha(r) = b(M) + \alpha r$ for $\alpha \geqslant M$, $\alpha > 0$. The operator $v \to -\operatorname{div} A_\varepsilon(v_\varepsilon, \nabla g_\varepsilon(v))$ being coercive, existence and uniqueness results for this problem are already proved in [5]. The definition of the weak entropy solution in this case is given in Proposition 3.6 below. Going to limit with $\varepsilon \to 0$, we prove the existence of a renormalized solution of the problem

$$P_{b_\alpha,g}(f,a)\begin{cases} b_\alpha(v) - \operatorname{div} A(v,\nabla g(v)) = f & \text{in } \Omega \\ g(v) = g(a) & \text{on } \Gamma, \end{cases}$$

In the second step, proceeding by approximation, we pass to the limit with $\alpha \to 0$ and solve the problem $P_{b,g}(f,a)$ in the L^1-setting.

3.1.1. First step

We start with the following existence result proved in [4].

PROPOSITION 3.6. *Let $f \in L^\infty(\Omega)$ and $a \in L^\infty(\Gamma)$ such that $g(a) = 0$ a.e. on Γ. Then, there exists a unique $v_\varepsilon \in L^\infty(\Omega)$ weak entropy solution of $P_{b_\alpha,g_\varepsilon}(f,a)$ i.e. $g_\varepsilon(v_\varepsilon) \in W_0^{1,p}(\Omega)$ and v_ε satisfies the following entropy inequalities. The first one: for all $k \in \mathbb{R}$, for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ such that $\xi \geqslant 0$ and $(-g(k))^+\xi = 0$ a.e. on Γ,*

$$-\int_\Gamma \omega^+(x,k,a)\xi + \int_\Omega b_\alpha(v_\varepsilon)\chi_{\{v_\varepsilon>k\}}\xi \leqslant \int_\Omega \chi_{\{v_\varepsilon>k\}}(f\xi - (A(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon)) - A(k,0))\cdot\nabla\xi), \tag{3.10}$$

and the second one: for all $k \in \mathbb{R}$, for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ such that $\xi \geqslant 0$ and $(g(k))^+\xi = 0$ a.e. on Γ,

$$-\int_\Gamma \omega^-(x,k,a)\xi - \int_\Omega b_\alpha(v_\varepsilon)\chi_{\{k>v_\varepsilon\}}\xi \leqslant -\int_\Omega \chi_{\{k>v_\varepsilon\}}(f\xi - (A(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon)) - A(k,0))\cdot\nabla\xi). \tag{3.11}$$

The weak entropy solution is in particular a weak solution of the problem $b_\alpha(v) - \operatorname{div} A(v,\nabla g(v)) = f$ i.e.

$$\int_\Omega b_\alpha(v)\xi + \int_\Omega A(v,\nabla g_\varepsilon(v))\cdot\nabla\xi = \int_\Omega f\xi \tag{3.12}$$

holds true for all $\xi \in W_0^{1,p}(\Omega)$.

With a particular choice of test functions in (3.12) and thanks to the strict monotonicity of b_α, one can prove that

$$(v_\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(\Omega), \tag{3.13}$$

and for all $l,\beta > 0$ with $M - \beta > 0$,

$$(|\nabla g_\varepsilon(T_{\beta,l}v_\varepsilon)|)_\varepsilon \text{ is bounded in } L^p(\Omega).$$

Following classical arguments, extracting a subsequence if necessary, we can prove that as $\varepsilon \to 0$,

$$g_\varepsilon(v_\varepsilon) \text{ converges to some measurable function } w \text{ a.e. in } \Omega, \tag{3.14}$$

$$g_\varepsilon(T_{\beta,l}v_\varepsilon) \text{ converges to } T_{\beta,l}w \in W_0^{1,p}(\Omega) \begin{cases} \text{weakly in } W_0^{1,p}(\Omega), \\ \text{strongly in } L^p(\Omega), \end{cases} \tag{3.15}$$

and by the growth condition (1.2) and (3.13),

$$\tilde{A}_\varepsilon(v_\varepsilon, \nabla g_\varepsilon(T_{\beta,l}v_\varepsilon)) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ to some } \chi_{\beta,l} \in L^{p'}(\Omega)^N \tag{3.16}$$

for $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $\tilde{A}_\varepsilon(r,\xi) = A_\varepsilon(r,\xi) - A_\varepsilon(r,0)$.

Now, we need to prove some strong compactness on v_ε in L^1_{Loc}. We propose here to use the L^∞ uniform bound on (v_ε) in order to deduce the weak-$*$ convergence of (v_ε) to a function v. Then, going to the limit in the approximate entropy inequalities, we prove that v is a renormalized entropy process solution of $P_{b_\alpha,g}(a,f)$ (see Definition 3.9 below). Finally using a "strong" principle of uniqueness, we show that v is the entropy solution of $P_{b_\alpha,g}(a,f)$ and that the convergence holds strongly in $L^1(\Omega)$.

DEFINITION 3.7. Let Ω be an open subset of $\mathbb{R}^N$ $(N \geqslant 1)$, (u_n) be a bounded sequence of $L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence (u_n) converges towards u in the "nonlinear weak-$*$ sense" if

$$\int_\Omega S(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega S(u(x,\mu))\psi(x)\,dx\,d\mu \quad \text{as } n \to \infty,$$

for all $\psi \in L^1(\Omega)$, for all $S \in \mathscr{C}(\mathbb{R},\mathbb{R})$.

LEMMA 3.8. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geqslant 1)$ and (u_n) be a bounded sequence of $L^\infty(\Omega)$. Then (u_n) admits a subsequence converging in the nonlinear weak-$*$ sense.*

For the proof of the above lemma, we refer to [26, 20].

According to Lemma 3.8, there exists $v \in L^\infty((0,1) \times \Omega)$ and a subsequence of (v_ε) still denoted by (v_ε) such that

$$(v_\varepsilon) \text{ converges to } v \text{ in the nonlinear weak} - * \text{ sense.} \tag{3.17}$$

We will prove that v is a renormalized entropy process solution of $P_{b_\alpha,g}(a,f)$ in the following sense:

DEFINITION 3.9. A function $u : (0,1) \times \Omega \to (-\infty, M]$ is a renormalized entropy process solution of $P_{b_\alpha,g}(a,f)$ if $u \in L^\infty((0,1) \times \Omega)$ and there exists a measurable function $w : \Omega \to \mathbb{R}$ and a family $(\mu_\beta)_\beta$ of bounded measures on $\overline{\Omega}$ such that

$$w(x) = g(u(\alpha,x)) \text{ a.e. in } \Omega,$$

$$
\begin{gathered}
w \wedge M - \beta \in W_0^{1,p}(\Omega), \quad \text{for all } \beta > 0, \\
A(u, Dw)\chi_{\{-k<u<M\}} \in (L^{p'}(\Omega))^N, \forall k > 0, \\
\mu_\beta \in (W^{-1,p'}(\Omega) + L^1(\Omega) + L^1(\Gamma)) \cap \mathscr{M}(\overline{\Omega}), \\
\mu_\beta(\{v \leqslant M - \beta\}) = 0, \qquad (3.18) \\
\lim_{\beta \to 0} \int_\Omega \xi \, d\mu_\beta(v) = 0 \text{ for all } \xi \in \mathscr{D}(\mathbb{R}^N) \text{ with } \operatorname{supp}(\nabla \xi) \subset \{v < M\}
\end{gathered}
$$

and the following inequalities are satisfied. The first one: for all $\beta, k \in \mathbb{R}$ with $M - \beta > k$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $(g(u \wedge M - \beta) - g(k))^+ \xi = 0$ a.e. on Γ,

$$
\begin{aligned}
&\int_0^1 \int_\Omega b_\alpha(u \wedge (M-\beta))\chi_{\{u \wedge (M-\beta) > k\}} \xi \, d\mu - \int_0^1 \int_\Omega \chi_{\{u \wedge (M-\beta) > k\}} f \xi \\
&\quad + \int_0^1 \int_\Omega (A(u \wedge (M-\beta), \nabla(w \wedge g(M-\beta))) - A(k,0)) \cdot \nabla \xi) \, d\mu \\
&\qquad \leqslant \int_\Gamma \omega^+(x,k,a \wedge (M-\beta))\xi - \int_\Omega \xi \, d\mu_\beta(u)
\end{aligned}
$$

and the second one: for all $l \leqslant k \in \mathbb{R}$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $(g(k) - g(a \vee l))^+ \xi = 0$ a.e. on Γ,

$$
\begin{aligned}
&- \int_0^1 \int_\Omega b_\alpha(u)\chi_{\{k>u\}} \xi \, d\mu - \int_0^1 \int_\Omega \chi_{\{k>u\}} (A(u, \nabla w) - A(k,0)) \cdot \nabla \xi) \, d\mu \\
&\qquad + \int_0^1 \int_\Omega \chi_{\{k>u\}} f \xi \leqslant \int_\Gamma \omega^-(x,k,a)\xi. \qquad (3.19)
\end{aligned}
$$

REMARK 3.10. Inequality (3.19) is well defined. Indeed

$$
\chi_{\{k>u\}} A(u, \nabla w) = \chi_{\{k>u\}} A(u, \nabla(w \wedge g(k))) \in L^{p'}(\Omega)^N.
$$

Taking into account (3.14), (3.15), and as $(g_\varepsilon)_\varepsilon$ converges uniformly on compact subsets to g, it follows that

$$
\begin{gathered}
g_\varepsilon(v_\varepsilon) \text{ converges to } g(v) \text{ a.e. in } \Omega, \qquad (3.20) \\
g_\varepsilon(T_{\beta,l}(v_\varepsilon)))_\varepsilon \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } g(T_{\beta,l}v)
\end{gathered}
$$

and that $g(v)$ is independent of μ. Moreover,

$$
(v_\varepsilon \chi_{\{v \in \mathbb{R} \setminus [A_1, A_2]\}}) \to v \chi_{\{v \in \mathbb{R} \setminus [A_1, A_2]\}} \in L^\infty(\Omega). \qquad (3.21)
$$

Indeed, g^{-1} is locally Lipschitz continuous in the open segment $g(]A_2, M[)$. Moreover, as $g(v_\varepsilon) \to g(v)$ a.e. in Ω, for a.e. $x \in \Omega$, given a fixed $\delta > 0$, there exists $\varepsilon_0 > 0$ small enough such that $|g(v_\varepsilon)(x) - g(v)(x)| \leqslant \delta$ for all $\varepsilon \leqslant \varepsilon_0$. Hence,

$$
|g^{-1}(g(v_\varepsilon(x))) - g^{-1}(g(v(x))| \leqslant C(\delta),
$$

where $C(\delta) > 0$ depends on δ. Thus,

$$\begin{aligned}(|v_\varepsilon - v|\chi_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}})(x) &= (|g^{-1}(g(v_\varepsilon)) - g^{-1}(g(v))|\chi_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}})(x)\\ &\leqslant c(\delta)(|g(v_\varepsilon) - g(v)|\chi_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}})(x)\\ &\leqslant c(\delta)(|g(v_\varepsilon) - g_\varepsilon(v_\varepsilon)|(x)\\ &\qquad + |g_\varepsilon(v_\varepsilon) - g(v)|(x))\chi_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}}(x)\\ &\to 0 \text{ with } \varepsilon \to 0.\end{aligned}$$

Next, we prove that v is less or equal to M: We choose $g_\varepsilon(T_{2M}v_\varepsilon)^+ - g_\varepsilon(T_M v)^+$ as test function in (3.12) to get

$$\begin{aligned}&\int_\Omega A(v_\varepsilon, g_\varepsilon(v_\varepsilon))\cdot\nabla(g_\varepsilon(T_{2M}v_\varepsilon)^+ - g_\varepsilon(T_M v)^+)\\ &\qquad\qquad \leqslant (g_\varepsilon(T_{2M}v_\varepsilon)^+ - g_\varepsilon(T_M v)^+) \parallel f \parallel_{L^1(\Omega)}.\end{aligned}$$

By (1.3) and the divergence Theorem, this implies that

$$\lambda(2M)|\nabla(g(M-\varepsilon) + g'(M-\varepsilon)(v_\varepsilon - M + \varepsilon))|^p \leqslant Mg'(M-\varepsilon) \parallel f \parallel_{L^1(\Omega)}.$$

Then, in view of (3.21) and by Poincaré's inequality together with the fact that $g'(M-\varepsilon) \to +\infty$ as ε tends to 0, we deduce that

$$(T_{2M}v)^+ - (T_{A_2}v)^+ = 0, \text{ a.e. in } \Omega,$$

that is

$$v \leqslant M \text{ a.e. in } \Omega. \tag{3.22}$$

In order to pass to the limit in (3.10) and (3.11), we have to identify $\chi_{\beta,l}$ in (3.16), to define the measure μ_β and to verify the properties (3.3) and (3.18).

We use the argument of Minty Browder in order to prove that

$$\int_\Omega \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(T_{\beta,l}v_\varepsilon))\cdot\nabla\xi \to \int_0^1 \Big(\int_\Omega \tilde{A}(v, \nabla g(T_{\beta,l}v))\cdot\nabla\xi\Big)\,d\mu.$$

Therefore, we need the following convergence result:

$$\lim_{L\to 0}\lim_{\varepsilon\to 0}\int_\Omega \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(T_{\beta,l}v_\varepsilon))\cdot\nabla T_L(g_\varepsilon(T_{\beta,l}v_\varepsilon) - g(T_{\beta,l}v)) = 0. \tag{3.23}$$

In order to prove (3.23), we use the following decomposition of the above integral:

$$\begin{aligned}&\int_\Omega \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(T_{\beta,l}v_\varepsilon))\cdot\nabla T_L(g_\varepsilon(T_{\beta,l}v_\varepsilon) - g(T_{\beta,l}v))\\ &\quad = \int_{\{-l<v_\varepsilon<M-\beta\}} \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))\cdot\nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l}v))\\ &\quad = \int_\Omega \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))\cdot\nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l}v))\end{aligned}$$

$$- \int_{\{-l \geqslant v_\varepsilon\} \cup \{M-\beta \leqslant v_\varepsilon\}} \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v))$$
$$:= \mathscr{T}_1 + \mathscr{T}_2.$$

As v_ε is also a weak solution of $P_{b_\alpha,\varepsilon}(a,f)$,

$$\begin{aligned}
&\lim_{L\to 0}\lim_{\varepsilon\to 0} \mathscr{T}_1 \\
&\quad = \lim_{L\to 0}\lim_{\varepsilon\to 0} \int_\Omega \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)) \\
&\quad = -\lim_{L\to 0}\lim_{\varepsilon\to 0} \Big[\int_\Omega b_\alpha(v_\varepsilon) T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)) - \int_\Omega f T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)) \Big] \\
&\qquad\qquad - \int_\Omega A(v_\varepsilon, 0) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)).
\end{aligned} \tag{3.24}$$

It is clear that the first and second term in the R. H. S of inequality (3.24) converge to 0. In view of (3.22),

$$\begin{aligned}
&\int_\Omega A(v_\varepsilon,0) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)) \\
&\quad = \int_{\{v=M\}} A(v_\varepsilon,0) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(M-\beta)) \\
&\qquad + \int_{\{v\in]A_1,A_2[\}} A(v_\varepsilon,0) \cdot \nabla T_L g_\varepsilon(v_\varepsilon) \\
&\qquad + \int_{\{v\in \mathbb{R}\setminus [A_1,A_2]\}} A(v_\varepsilon,0) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)) \\
&\quad = \int_{\{v=M\}} A(v_\varepsilon,0) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(M-\beta)) \\
&\qquad + \int_{\{v\in]A_1,A_2[, v_\varepsilon \in]A_2,M[\}} A(v_\varepsilon,0) \cdot \nabla T_L g_\varepsilon(v_\varepsilon) \\
&\qquad + \int_{\{v\in]A_1,A_2[, v_\varepsilon \in (-\infty,A_1[\}} A(v_\varepsilon,0) \cdot \nabla T_L g_\varepsilon(v_\varepsilon) \chi_{\{|g_\varepsilon(v_\varepsilon)|\leqslant L\}} \\
&\qquad + \int_{\{\mathbb{R}\setminus [A_1,A_2]\}} A(v_\varepsilon,0) \cdot \nabla T_L(g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)) \\
&\quad = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4.
\end{aligned}$$

Taking into account (3.21) it is not difficult to pass to the limit with $\varepsilon \to 0$ and then with $L \to 0$ in $\mathscr{I}_4$ to find

$$\lim_{L\to 0}\lim_{\varepsilon\to 0} \mathscr{I}_4 = 0.$$

Next, we prove that $(A(v_\varepsilon,0)\chi_{\{v\in]A_1,A_2[, v_\varepsilon \in]A_2,M[\}} \chi_{\{|g_\varepsilon(v_\varepsilon) - g(T_{\beta,l} v)|\leqslant L\}})_\varepsilon$ converges a.e. in Ω and then strongly in $L^{p'}(\Omega)^N$ to $A(A_2,0)\chi_{\{|g(v)-g(T_{\beta,l} v)|\leqslant L\}}$.

Indeed, a simple computation shows that $g_\varepsilon(v_\varepsilon)\chi_{\{v_\varepsilon \in]A_2,M[\}} \in g(]A_2,M[)$ for ε small enough. Moreover, as the restriction of g to $[A_2,M)$ is continuously invertible, taking into account (3.20) and the fact that $g_\varepsilon(r) \to g(r)$ pointwise in $\mathbb{R}$, one has

$$|v_\varepsilon - A_2| = |g^{-1}(g(v_\varepsilon)) - g^{-1}(g(v))|$$

$$\leqslant |g^{-1}(g_\varepsilon(v_\varepsilon)) - g^{-1}(g(v))| \\ + |g^{-1}(g(v_\varepsilon)) - g^{-1}(g_\varepsilon(v_\varepsilon))| \to 0 \text{ a.e. in } \Omega.$$

Here, we denote by g^{-1} the inverse of the restriction of g to $[A_2, M)$. Using similar arguments, we prove that $(A(v_\varepsilon,0)\chi_{\{v\in]A_1,A_2[, v_\varepsilon\in(-\infty,A_1[\}}\chi_{\{|g_\varepsilon(v_\varepsilon)-g(T_{\beta,l}v)|\leqslant L\}})_\varepsilon$ converges strongly in $L^{p'}(\Omega)^N$ to $A(A_1,0)\chi_{\{|g(v)-g(T_{\beta,l}v)|\leqslant L\}}$. This implies that

$$\lim_{L\to 0}\lim_{\varepsilon\to 0}\mathscr{I}_2 = \lim_{L\to 0}\lim_{\varepsilon\to 0}\mathscr{I}_3 = 0.$$

Now, we prove that

$$\lim_{L\to 0}\lim_{\varepsilon\to 0}\mathscr{I}_1 = 0.$$

Indeed,

$$\begin{aligned}\mathscr{I}_1 &= \int_{\{v=M\}} A(v_\varepsilon,0)\cdot\nabla T_L(g_\varepsilon(v_\varepsilon) - g(M-\beta)) \\ &= \int_{\{v=M, v_\varepsilon > M-\varepsilon\}} A(v_\varepsilon,0)\cdot\nabla T_L(g_\varepsilon(v_\varepsilon) - g(M-\beta)) \\ &\quad + \int_{\{v=M, v_\varepsilon\leqslant M-\varepsilon\}} A(v_\varepsilon,0)\cdot\nabla T_L(g(v_\varepsilon) - g(M-\beta)).\end{aligned}$$

As $\lim_{r\to M} g(r) = +\infty$, for $\beta > 0$ fixed, there exists $0 < \alpha < \beta$ such that $g(r) - g(M-\beta) > L$ for all $M-\alpha \leqslant r < M$. This means that the first term in the R.H.S of the above equality converges to 0 with ε. The second term can be estimated as follows:

$$\int_{\{v=M, v_\varepsilon\leqslant M-\varepsilon\}} A(v_\varepsilon,0)\cdot\nabla T_L(g(v_\varepsilon) - g(M-\beta)) \\ \leqslant \int_{\{v=M, v_\varepsilon\leqslant M-\varepsilon\}} A(T_{L+g(M-\beta)}v_\varepsilon,0)\cdot\nabla T_{L+g(M-\beta)}\cdot g(v_\varepsilon)$$

And by (3.21), it is clear that this term converges also to 0 with $\varepsilon\to 0$. Next, we claim that

$$\lim_{L\to 0}\limsup_{\varepsilon\to 0}\mathscr{T}_2 \geqslant 0.$$

Indeed,

$$\begin{aligned}\lim_{L\to 0}\limsup_{\varepsilon\to 0}\mathscr{T}_2 &= \lim_{L\to 0}\limsup_{\varepsilon\to 0}\Big[-\int_{\{g(T_{\beta,l}v)+L>g_\varepsilon(v_\varepsilon)\geqslant g_\varepsilon(M-\beta)\}}\tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla g_\varepsilon(v_\varepsilon) \\ &\qquad -\int_{\{g(T_{\beta,l}v)-L\leqslant g_\varepsilon(v_\varepsilon)<g_\varepsilon(-l)\}}\tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla g_\varepsilon(v_\varepsilon)\Big] \\ &\quad + \lim_{L\to 0}\lim_{\varepsilon\to 0}\int_{\{v_\varepsilon<-l \text{ or } M-\beta<v_\varepsilon \text{ and } |g_\varepsilon(v_\varepsilon)-g(v)|<L\}}\tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla g(T_{\beta,l}v) \\ &= \lim_{L\to 0}\limsup_{\varepsilon\to 0}\Big[-\int_\Omega \tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla(T_{L+g(M-\beta)}g_\varepsilon(v_\varepsilon) - T_{g(M-\beta)}g_\varepsilon(v_\varepsilon))^+\Big]\end{aligned}$$

$$+\lim_{L\to 0}\limsup_{\varepsilon\to 0}\Big[-\int_\Omega \tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla(T_{-g(-l)+L}g_\varepsilon(v_\varepsilon)-T_{-g(-l)}g_\varepsilon(v_\varepsilon))^-\Big]$$
$$+\lim_{L\to 0}\lim_{\varepsilon\to 0}\int_{\{v_\varepsilon<-l \text{ or } M-\beta<v_\varepsilon \text{ and } |g_\varepsilon(v_\varepsilon)-g(v)|<L\}} \tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla g(T_{\beta,l}v)$$
$$:=\mathscr{L}_2^1+\mathscr{L}_2^2+\mathscr{L}_2^3.$$

As v_ε is a weak solution of $P_{b_\alpha,\varepsilon}(a,f)$,

$$\begin{aligned}\mathscr{L}_2^1 &= \lim_{L\to 0}\limsup_{\varepsilon\to 0}\Big[-\int_\Omega b_\alpha(v_\varepsilon)(T_{L+g(M-\beta)}g_\varepsilon(v_\varepsilon)-T_{g(M-\beta)}g_\varepsilon(v_\varepsilon))^+\\ &\qquad +\int_\Omega f(T_{L+g(M-\beta)}g_\varepsilon(v_\varepsilon)-T_{g(M-\beta)}g_\varepsilon(v_\varepsilon))^+\Big]\\ &\qquad -\int_\Omega A(v_\varepsilon,0)\cdot\nabla(T_{L+g(M-\beta)}g_\varepsilon(v_\varepsilon)-T_{g(M-\beta)}g_\varepsilon(v_\varepsilon))^+\\ &= \lim_{L\to 0}\limsup_{\varepsilon\to 0}\Big[-\int_\Omega b_\alpha(v_\varepsilon)(T_{L+g(M-\beta)}g_\varepsilon(v_\varepsilon)-T_{g(M-\beta)}g_\varepsilon(v_\varepsilon))^+\\ &\qquad +\int_\Omega f(T_{L+g(M-\beta)}g_\varepsilon(v_\varepsilon)-T_{g(M-\beta)}g_\varepsilon(v_\varepsilon))^+\Big]\\ &\qquad -\int_\Omega \operatorname{div}\Big(\int_{A_2}^{(T_{L+g(M-\beta)}g_\varepsilon(T^2v_\varepsilon)-T_{g(M-\beta)}g_\varepsilon(T^2v_\varepsilon))^+} A(g_\varepsilon^{-1}(r),0)\,dr\Big)\\ &= 0,\end{aligned}$$

where g_ε^{-1} is the continuous inverse of $(g_\varepsilon)_{/]A_2,+\infty)}$. The term $\mathscr{L}_2^2$ can be estimated in the same way. Now, for $\beta>\varepsilon>0$ and $L>0$ small enough, we denote $\gamma(\beta,l):=g^{-1}(g(M-\beta)+L)$ and $\sigma(\beta,l):=g^{-1}(g(-l)-L)=g_\varepsilon^{-1}(g_\varepsilon(-l)-L)$. Then

$$\begin{aligned}\mathscr{L}_2^3 &= \lim_{L\to 0}\limsup_{\varepsilon\to 0}\int_\Omega(\tilde{A}(v_\varepsilon,\nabla T_{\gamma(\beta,l),\sigma(\beta,l)}g_\varepsilon(v_\varepsilon))-\tilde{A}(v_\varepsilon,\nabla g_\varepsilon(T_{\beta,l}v_\varepsilon)))\cdot\nabla g(T_{\beta,l}v)\\ &= \int_\Omega(-\chi_{\gamma(\beta,l),\sigma(\beta,l)}+\chi_{\beta,l})\cdot\nabla g(T_{\beta,l}v)\\ &= 0.\end{aligned}$$

Indeed, for all $\beta,L>0$,

$$\begin{aligned}\tilde{A}(v_\varepsilon,\nabla T_{\beta,L}g_\varepsilon(v_\varepsilon)) &= \tilde{A}(v_\varepsilon,\nabla T_{\gamma(\beta,l),\sigma(\beta,l)}g_\varepsilon(v_\varepsilon))\chi_{\{g(-l)<g_\varepsilon(v_\varepsilon)|<g(M-\beta)\}}\\ &\qquad +\tilde{A}(v_\varepsilon,0)\chi_{\{g_\varepsilon(v_\varepsilon)\geqslant g(M-\beta) \text{ or } g_\varepsilon(v_\varepsilon)\leqslant g(-l)\}}.\end{aligned}$$

Therefore, going to limit with $\varepsilon\to 0$, by (3.16), (3.17) and (3.20), it follows that

$$\chi_{\beta,l}=\chi_{\gamma(\beta,l),\sigma(\beta,l)}\chi_{\{g(-l)<g(v)<g(M-\beta)\}}+\tilde{A}(v,0)\chi_{\{g(v)\geqslant g(M-\beta) \text{ or } g(v)\leqslant g(-l)\}},$$

a.e. on $\{g(v)\neq g(-l)\}\cap\{g(v)\neq g(M-\beta)\}$. As $\nabla g(v)=0$ on $\{g(v)\neq g(-l)\}\cup\{g(v)\neq g(M-\beta)\}$, this yields

$$\int_\Omega(-\chi_{\gamma(\beta,l),\sigma(\beta,l)}+\chi_{\beta,l})\cdot\nabla g(T_{\beta,l}v)=0.$$

Combining all the estimates, we get (3.23) and by the standard pseudo-monotonicity argument it follows that

$$\int_\Omega \chi_k \cdot \nabla\xi = \int_0^1 \int_\Omega \tilde{A}(v, \nabla T_{\beta,l} g(v)) \cdot \nabla\xi \quad \text{for all } \xi \in \mathscr{D}(\Omega). \tag{3.25}$$

Indeed, for $\xi \in \mathscr{D}(\Omega),\ \xi \geqslant 0,\ \alpha \in \mathbb{R}$,

$$\begin{aligned}
\alpha \int_\Omega \chi_k \nabla\xi \\
&= \lim_{\varepsilon\to 0} \int_\Omega \alpha\tilde{A}(v_\varepsilon, \nabla T_{\beta,l} g_\varepsilon(v_\varepsilon)) \cdot \nabla\xi \\
&\geqslant \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_\Omega \tilde{A}(v_\varepsilon, \nabla T_{\beta,l} g_\varepsilon(v_\varepsilon)) \cdot \nabla(T_L(g_\varepsilon(T_{\beta,l} v_\varepsilon) - T_{\beta,l} g(v)) + \alpha\xi) \\
&= \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{|g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| < L\}} \tilde{A}(v_\varepsilon, \nabla T_{\beta,l} g_\varepsilon(v_\varepsilon)) \\
&\qquad \cdot \nabla(g_\varepsilon(T_{\beta,l} v_\varepsilon) - T_{\beta,l} g(v) + \alpha\xi) \\
&\quad + \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{|g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| \geqslant L\}} \tilde{A}(v_\varepsilon, \nabla T_{\beta,l} g_\varepsilon(v_\varepsilon)) \cdot \nabla(\alpha\xi) \\
&= \mathscr{P}_1 + \mathscr{P}_2.
\end{aligned}$$

By assumption (1.4),

$$\begin{aligned}
\mathscr{P}_1 &\geqslant \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{|g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| < L\}} \tilde{A}(v_\varepsilon, \nabla(g(T_{\beta,l} v) - \alpha\zeta)) \\
&\qquad \cdot \nabla(g_\varepsilon(T_{\beta,l} v_\varepsilon) - T_{\beta,l} g(v) + \alpha\zeta) \\
&= \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{|g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| < L\}} \tilde{A}(v_\varepsilon, \nabla(g(T_{\beta,l} v) - \alpha\zeta)) \cdot \nabla(\alpha\zeta) \\
&\quad + \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{v \in]A_2, M[, |g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| < L\}} \tilde{A}(v_\varepsilon, \nabla(g(T_{\beta,l} v) - \alpha\zeta)) \\
&\qquad \cdot \nabla(g_\varepsilon(T_{\beta,l} v_\varepsilon) - T_{\beta,l} g(v)) \\
&\quad + \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{v \in (-\infty, A_1[, |g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| < L\}} \tilde{A}(v_\varepsilon, \nabla(g(T_{\beta,l} v) - \alpha\zeta)) \cdot \nabla(g_\varepsilon(T_{\beta,l} v_\varepsilon)) \\
&\quad + \lim_{L\to 0} \limsup_{\varepsilon\to 0} \int_{\{v \in]A_1, A_2[, |g_\varepsilon(T_{\beta,l} v_\varepsilon) - g(T_{\beta,l} v)| < L\}} \tilde{A}(v_\varepsilon, \nabla(-\alpha\zeta)) \cdot \nabla(g_\varepsilon(T_{\beta,l} v_\varepsilon)) \\
&= \mathscr{P}_1^1 + \mathscr{P}_1^2 + \mathscr{P}_1^3 + \mathscr{P}_1^4.
\end{aligned}$$

It is not difficult to pass to the limit in the first term in the R.H.S of the above inequality to find

$$\mathscr{P}_1^1 = \int_0^1 (\int_\Omega \tilde{A}(v, \nabla(g(T_{\beta,l} v) - \alpha\zeta)) \cdot \nabla(\alpha\zeta))\, d\mu.$$

The term $\mathscr{P}_1^2$ can be estimated in the same way and it is easy to see that $\mathscr{P}_1^3 = \mathscr{P}_1^4 = 0$. Now, in order to estimate $\mathscr{P}_2$, we use the growth condition (1.2),

$$\mathscr{P}_2 \leqslant \lim_{L\to 0} \limsup_{\varepsilon\to 0} \Big(\int_\Omega |A(v_\varepsilon, \nabla(g(T_{\beta,l}v) - \alpha\zeta)|^{p'} \Big)^{\frac{1}{p'}} \Big(\int_{\{|g_\varepsilon(T_{\beta,l}v_\varepsilon) - g(T_{\beta,l}v)| \geqslant L\}} |\nabla(\alpha\zeta)|^p \Big)^{\frac{1}{p}} = 0.$$

Hence,

$$\alpha \int_\Omega \chi_k \nabla\zeta \geqslant \int_0^1 (\int_\Omega \tilde{A}(v, \nabla(g(T_{\beta,l}v) - \alpha\zeta)) \cdot \nabla(\alpha\zeta))\, d\mu.$$

Dividing by $\alpha > 0$ (resp. $\alpha < 0$), passing to the limit with $\alpha \to 0$, we obtain (3.25).

DEFINITION OF THE MEASURES μ_β. Let us first verify that v_ε satisfies (3.4) and (3.5). For all $k \in I\!R$, for all $\beta > 0$ with $M - \beta \geqslant k$, for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $\xi \geqslant 0$ with $(-g(k))^+\xi = 0$ on Γ,

$$\begin{aligned}
&\int_\Omega \chi_{\{v_\varepsilon \wedge (M-\beta) > k\}} \Big\{ -b_\alpha(v_\varepsilon \wedge (M-\beta))\xi + f\xi \\
&\qquad - (A(v_\varepsilon \wedge (M-\beta), \nabla g_\varepsilon(v_\varepsilon \wedge (M-\beta))) - A(k,0)) \cdot \nabla\xi \Big\} \\
&\qquad\qquad + \int_\Gamma \omega^+(x,k,a \wedge (M-\beta))\xi \\
&= \int_\Omega \chi_{\{v_\varepsilon > k\}} \{ -(b_\alpha(v_\varepsilon) - f)\xi - (A(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - A(k,0)) \cdot \nabla\xi \} \\
&\quad + \int_\Gamma \omega^+(x,k,a)\xi \\
&\quad + \int_\Omega \chi_{\{v_\varepsilon > (M-\beta)\}} \{ (b_\alpha(v_\varepsilon) - b_\alpha(M-\beta))\xi \\
&\qquad\qquad + (A(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - A(M-\beta,0)) \cdot \nabla\xi \} \\
&\quad - \int_\Gamma \omega^+(x,M-\beta,a)\xi \\
&\quad + \int_\Gamma \omega^+(x,k,a \wedge (M-\beta)) - \int_\Gamma \omega^+(x,k,a)\xi + \int_\Gamma \omega^+(x,a,M-\beta)\xi \\
&\geqslant \Big[\int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} \{ (b_\alpha(v_\varepsilon) - b_\alpha(M-\beta))\xi + (A(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - A(M-\beta,0)) \cdot \nabla\xi \} \\
&\quad - \int_\Gamma \omega^+(x,M-\beta,a)\xi \Big] \\
&=: < \mu_\beta(v_\varepsilon), \xi > .
\end{aligned} \tag{3.26}$$

We split the right hand side of inequality (3.26) into

$$\begin{aligned}
&< \mu_\beta(v_\varepsilon), \xi > \\
&\quad = \Big[\int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} \{ (b_\alpha(v_\varepsilon) - f)\xi + (A(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - A(M-\beta,0)) \cdot \nabla\xi \}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_\Gamma \omega^+(x, M-\beta, a)\xi \Big] \\
&\quad + \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} f\xi - \int_\Omega \chi_{\{v_\varepsilon > (M-\beta)\}} b_\alpha(M-\beta)\xi \\
&:= \int_\Omega \xi \, d\tilde{\mu}_\beta + \int_\Omega \chi_{\{v_\varepsilon > (M-\beta)\}} f\xi - \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} b_\alpha(M-\beta)\xi.
\end{aligned}
$$

Thus, $\mu_\beta(v_\varepsilon)$ is the sum of the negative measure $\tilde{\mu}_\varepsilon$ and the operator

$$
\xi \mapsto \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} f\xi - \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} b_\alpha(M-\beta)\xi.
$$

In particular, for any $\xi \in \mathscr{D}(\mathbb{R}^N)$, $0 \leqslant \xi \leqslant 1$,

$$
\int_\Omega \xi d(\mu_\beta(v_\varepsilon))^+ \leqslant \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} f^+\xi + \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} b_\alpha(M-\beta)\xi
$$

and

$$
\begin{aligned}
&\int_\Omega \xi d(\mu_\beta(v_\varepsilon))^- \\
&\qquad \leqslant \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} (|f| + |b_\alpha(v_\varepsilon) - b_\alpha(M-\beta)|)\xi + \int_\Gamma \omega^+(x, M-\beta, a)\xi.
\end{aligned}
$$

It follows that $(\mu_\beta(v_\varepsilon))_\varepsilon$ is uniformly bounded with respect to ε. Therefore, we can extract a subsequence still denoted by $(\mu_\beta(v_\varepsilon))_\beta$ which is convergent with respect to the weak$-*$ topology on $C(\overline{\Omega})$ to some Radon measure $\mu_\beta(v)$. We are going to prove that for $\xi \in \mathscr{D}(\mathbb{R}^N)$ with $\nabla\xi = 0$ on $\{v = M\}$,

$$
\lim_{\beta \to 0} < \mu_\beta(v), \xi >= 0. \tag{3.27}
$$

Indeed, for $M - \beta > A_2$, $\omega^+(x, M-\beta, a) = 0$ a.e. on Γ because $a < A_2$ and as $\lim_{r \to M} b_\alpha(r) = b_\alpha(M) < +\infty$ and $b_\alpha(v) \in L^\infty((0,1) \times \Omega)$,

$$
\begin{aligned}
\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_\Omega (b_\alpha(v_\varepsilon) - b_\alpha(M-\beta))^+\xi &= \lim_{\beta \to 0} \int_0^1 \Big(\int_\Omega (b_\alpha(v) - b_\alpha(M-\beta))^+\xi \Big) d\mu \\
&= \lim_{\beta \to 0} \int_\Omega (b_\alpha(v) - b_\alpha(M-\beta))^+\xi \\
&= \int_\Omega (b_\alpha(M) - b_\alpha(M)) = 0.
\end{aligned}
$$

Moreover, for all $\varepsilon > 0$,

$$
\begin{aligned}
&\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} (A(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - A(M-\beta, 0)) \cdot \nabla\xi \\
&\quad = \lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_\Omega \chi_{\{v_\varepsilon > M-\beta\}} \tilde{A}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \cdot \nabla\xi
\end{aligned}
$$

$$+\lim_{\beta\to 0}\lim_{\varepsilon\to 0}\int_\Omega \chi_{\{v_\varepsilon>M-\beta\}}(A(v_\varepsilon,0)-A(M-\beta,0))\cdot\nabla\xi$$
$$=\mathscr{T}_1+\mathscr{T}_2.$$

In order to estimate $\mathscr{T}_1$, we use hypothesis (1.5). Define

$$w_\varepsilon := \int_{A_2}^{v_\varepsilon\vee A_2}(C(\frac{Mr}{M-r^+}))^{\frac{p}{p-1}}\,dg_\varepsilon(r).$$

Then, by (3.21),

$$w_\varepsilon \to w := \int_{A_2}^{v\vee A_2}(C(\frac{Mr}{M-r^+}))^{\frac{p}{p-1}}\,dg(r) \text{ on } \{v<M\}. \tag{3.28}$$

Using $T_k w_\varepsilon$ as a test function in (3.12), applying the divergence theorem, we get

$$\begin{aligned}&\int_\Omega \tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla T_k(w_\varepsilon)\\ &\qquad=\int_\Omega \chi_{\{v_\varepsilon>A_2\}}A(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla T_k(w_\varepsilon)\\ &\qquad=\int_\Omega(-b_\alpha(v_\varepsilon)+f)T_k w_\varepsilon\\ &\qquad\leqslant k\parallel f\parallel_{L^1(\Omega)}.\end{aligned} \tag{3.29}$$

Hence, by (1.3)

$$\lambda_0\int_\Omega|\nabla g_\varepsilon(v_\varepsilon\vee A_2)|^p\cdot C(\frac{Mv_\varepsilon}{M-v_\varepsilon})^{\frac{p}{p-1}}\leqslant k\parallel f\parallel_{L^1(\Omega)}$$

and by (1.2, it follows

$$\int_{\{|w_\varepsilon|<k\}}|\tilde{A}(v_\varepsilon,\nabla g(v_\varepsilon))|^{p'}\chi_{\{v_\varepsilon>A_2\}}\leqslant k\parallel f\parallel_{L^1(\Omega)} \tag{3.30}$$

with $\frac{1}{p}+\frac{1}{p'}=1$. Therefore,

$$(\tilde{A}(v_\varepsilon,\nabla g(v_\varepsilon))\chi_{\{v_\varepsilon>A_2\}}\chi_{\{|w_\varepsilon|<k\}})_\varepsilon \text{ is bounded in } L^{p'}(\Omega). \tag{3.31}$$

This in turn, implies that

$$h_k(w_\varepsilon)\tilde{A}(v_\varepsilon,\nabla g(v_\varepsilon))\to\psi_k \text{ weakly in } (L^{p'}(\Omega))^N \text{ for all } k>0. \tag{3.32}$$

In order to identify ψ_k on the set $\{A_2<v<M\}$, let h be a smooth function with support in $]A_2+\alpha,M-\eta[$ for some $\alpha,\eta>0$ with $A_2+\alpha<M-\eta$. Then, by (3.25), (3.21) and (3.28),

$$h(v_\varepsilon)h_k(w_\varepsilon)\tilde{A}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\to h(v)h_k(w)\tilde{A}(v,\nabla g(v)) \text{ weakly in } (L^{p'}(\Omega))^N \tag{3.33}$$

as $\varepsilon \to 0$. As α, η are arbitrary, it follows from (3.33) and (3.32) that

$$\psi_k = h_k(w)\tilde{A}(v, \nabla g(v)) \text{ a.e. in } \{A_2 < v < M\}. \tag{3.34}$$

Next, note that for k large enough, $h_k(w) = 1$ on $\{v < M\}$. Indeed, on $\{v < M\}$,

$$\int_{A_2}^{v \vee A_2} C(|r|)^{\frac{p}{p-1}} \, dg(r) \leqslant \int_{A_2}^{M} C(|r|)^{\frac{p}{p-1}} \, dg(r) < +\infty$$

by (1.5). Therefore, for

$$k > \int_{A_2}^{M} C(|r|)^{\frac{p}{p-1}} \, dg(r),$$

$h_k(w) = 1$ and as a consequence of (3.34), one get

$$\tilde{A}(v, \nabla g(v))\chi_{\{A_2 < v < M\}} \in L^{p'}(\Omega)^N. \tag{3.35}$$

This in turn implies that for all $\xi \in \mathscr{D}(\Omega)$ with $\nabla \xi = 0$ on $\{v = M\}$,

$$\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_{\Omega} \chi_{\{v_\varepsilon > M - \beta\}} \tilde{A}(v_\varepsilon, \nabla g(v_\varepsilon)) \cdot \nabla \xi = \lim_{\beta \to 0} \int_{\{M > v > M - \beta\}} \tilde{A}(v, \nabla g(v)) \cdot \nabla \xi = 0.$$

Now, as $A(r, \xi)$ is continuous in r,

$$\begin{aligned}
&\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_{\Omega} \chi_{\{v_\varepsilon > M - \beta\}} (A(v_\varepsilon, 0) - A(M - \beta, 0)) \cdot \nabla \xi = 0 \\
&\quad = \lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_{\Omega} \chi_{\{v < M\}} \chi_{\{v_\varepsilon > M - \beta\}} (A(v_\varepsilon, 0) - A(M - \beta, 0)) \cdot \nabla \xi \\
&\quad = \lim_{\beta \to 0} \int_0^1 (\int_{\Omega} \chi_{\{M > v > M - \beta\}} (A(v, 0) - A(M - \beta, 0)) \cdot \nabla \xi) \, d\mu = 0.
\end{aligned}$$

Thus,

$$\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \int_0^1 \Big(\int_{\Omega} \chi_{\{v > M - \beta\}} (A(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - A(M - \beta, 0)) \cdot \nabla \xi \Big) \, d\mu = 0$$

for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ with $\nabla \xi = 0$ a.e. on $\{v = M\}$. Therefore, combining all the estimates on v_ε, we can pass to the limit with $\varepsilon \to 0$ in inequality (3.26) to obtain (3.4). The second entropy inequality (3.5) can be proved by a classical argument: We choose $\xi = H_\delta(g_\varepsilon(k) - g_\varepsilon(v_\varepsilon))\zeta$ in (3.12) with $\zeta \in \mathscr{D}(\mathbb{R}^N)$ such that $(g(k))^+ \zeta = 0$ a.e. on Γ and we let $\delta, \varepsilon \to 0$ successively.

Hence, v is a weak entropy process solution of $P_{b_\alpha, g}(a, f)$.

Now, in order to prove that v is a week entropy solution of $P_{b_\alpha, g}(a, f)$, we use the following “reinforced ” comparaison result which can be seen as a generalisation of Theorem 3.4 to the entropy process solutions. The reader is referred to [20] in order to verify the technical tools needed when dealing with measure-valued functions.

PROPOSITION 3.11. *Let* $f_i \in L^\infty(\Omega)$, $a_i \in L^\infty(\Gamma)$ *with* $g(a_i) = 0$ *and* $v_i \in L^\infty(\Omega \times (0,1)$ *be a weak entropy process of* $P_{b_\alpha,g}(a_i, f_i)$, $i = 1,2$. *Then there exists* $\kappa \in L^\infty(\Omega \times (0,1))$ *with* $\kappa \in sign^+(v_1 \wedge (M-\beta) - v_2 \wedge (M-\beta))$ *a.e. in* $\Omega \times (0,1)$ *such that for all* $M > \beta > 0$,

$$\int_0^1 \int_\Omega (b_\alpha(v_1(x,\alpha) \wedge (M-\beta)) - b_\alpha(v_2(x,\mu) \wedge (M-\beta)))^+ \xi \, dx d\alpha \, d\mu$$
$$\leqslant \int_0^1 \int_\Omega \kappa(f_1 - f_2)\xi \, dx + \int_0^1 \int_\Gamma \omega^-(x, a_1 \wedge (M-\beta), a_2 \wedge (M-\beta))\xi.$$

In particular, if $f_1 = f_2$ and $a_1 = a_2$, it follows from Proposition 3.11 and the fact that $b_\alpha : \mathbb{R} \to \mathbb{R}$ is strictly increasing that

$$v_1(x,\alpha) \wedge (M-\beta) = v_2(x,\mu) \wedge (M-\beta) \quad \text{for a.e. } (x,\alpha,\mu) \in \Omega \times (0,1) \times (0,1)$$

Defining the function $w(x) = \int_0^1 v_1(x,\alpha)\, d\alpha$, we deduce that $w(x) = v_1(x,\alpha) = v_2(x,\beta)$ for a.e. $(x,\alpha,\beta) \in \Omega \times (0,1) \times (0,1)$.

This ends the first part of the proof. $\diamond$

3.1.2. Second step

The comparison principle is again the main tool in this second step: Let $f \in L^1(\Omega)$ and $a \in M(\Gamma)$ with $g(a) = 0$ a.e. on Γ. For $m,n \in I\!N$, let $f_{m,n} = f \wedge m \vee (-n)$, $a_{m,n} = a \wedge m \vee (-n)$ and define $b_{m,n} : r \mapsto b(r) + \frac{1}{m} r^+ - \frac{1}{n} r^-$. Denote by $v_{m,n}$ the unique weak entropy solution of $P_{b_{m,n},g}(f_{m,n}, a_{m,n})$, which exists by the result of the first step.

Recall that $v_{m,n}$ is a weak solution of $b_{m,n}(v) - \operatorname{div} A(v, \nabla g(v)) = f$ i.e.

$$\int_\Omega b_{m,n}(v_{m,n})\xi + \int_\Omega A(v_{m,n}, \nabla g(v_{m,n})) \cdot \nabla \xi = \int_\Omega f\xi \tag{3.36}$$

for all $\xi \in \mathscr{D}(\Omega)$. By Theorem 3.5, $v_{m_1,n} \leqslant v_{m_2,n}$ for $m_1 \leqslant m_2$ and $v_{m,n_1} \leqslant v_{m,n_2}$ for $n_1 \geqslant n_2$. i.e. $v_{m,n} \downarrow_n v_m$ a.e. on Ω where $v_m : \Omega \to [-\infty, M]$ is measurable. Here, we use the notation $\uparrow_n$ resp. $\downarrow_n$ to denote the convergence of a sequence which is monotone increasing, resp. decreasing in n. Next, we prove that v_m is finite a.e. in Ω: Suppose first that $b(-\infty) := \lim_{r \to -\infty} b(r) > -\infty$. Then, by the range condition (1.7), it follows that $\lim_{r \to -\infty} g(r) = -\infty$.

As $v_{m,n}$ is a week solution of $P_{b_{m,n},g}(a_{m,n}, f_{m,n})$, choosing $g(T_{\beta,l}(-v_{m,n}^-))$ as test function in (3.36), taking into account the growth condition on A, we find

$$\lambda_{b(-\infty)} \int_\Omega |\nabla g(T_{\beta,l}(-v_{m,n}^-))|^p \leqslant |g(-l)| \int_\Omega |f_{m,n}|$$

(see condition (1.2) on A). Hence, by Poincaré's inequality,

$$|\{-v_{m,n}^- \leqslant -l\}| \leqslant \frac{C(1 + |g(-l)|)}{|g(-l)|^p}$$

for some constant C independent of m,n and k. Passing the limit with $n \to \infty$ and then with $k \to -\infty$ in the above inequality, we find that v_m is finite a.e. on Ω. Applying the diagonal principle, we may assume that for some subsequence $(m(n))_n$, we have (with $f_n := f_{m(n),n}$, $a_n = a_{m(n),n}$ and $b_n := b_{m(n),n}$) $f_n \to f$ in $L^1(\Omega)$) and the solution v_n of $P_{b_n,g}(a_n, f_n)$ satisfies: $v_n \to v$ a.e. in Ω, $b(v_n) \to b(v)$ in $L^1(\Omega)$.

The rest of the proof follows the same lines as in the proof of Theorem 1.4 in [6] (see also [5]).

4. Existence results in case (b)

Throughout this section, we suppose that conditions (1.2)-(1.6) and (1.7) are satisfied.

DEFINITION 4.1. Let $f \in L^1(\Omega)$ and $a : \Gamma \to \mathbb{R}$ be measurable with $g(a) = 0$ a.e. on Γ. A measurable function $v : \Omega \to (-\infty, M)$ is said to be a renormalized entropy solution of the problem $P_{b,g}(f,a)$ if

$$b(v) \in L^1(\Omega),$$
$$g(T_{\beta,l}v) \in W_0^{1,p}(\Omega),$$
$$\chi_{\{-l<v<M-\beta\}} \in (L^{p'}(\Omega))^N, \;\; \forall l,\beta > 0,\; 0 < M-\beta, A(v,Dg(v))$$

and there exists two family $(\mu_\beta)_\beta$ and $(\nu_l)_l$ of bounded measures on $\overline{\Omega}$ such that

$$\mu_\beta(\{v \leqslant M-\beta\}) = \nu_l(\{v \geqslant l\}) = \lim_{l\to-\infty} \int_\Omega \xi \, d\nu_l(v) = 0,$$

and the following inequalities are satisfied: for all $\beta, k \in \mathbb{R}$ with $M-\beta > k$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $(g(a \wedge (M-\beta)) - g(k))^+ \xi = 0$ a.e. on Γ,

$$\begin{aligned} -\int_\Omega b(v\wedge(M-\beta))\chi_{\{v\wedge(M-\beta)>k\}}\xi + \int_\Omega \chi_{\{v\wedge(M-\beta)>k\}} f\xi \\ -\int_\Omega \chi_{\{v\wedge(M-\beta)>k\}}(A(v\wedge(M-\beta), \nabla g(v\wedge(M-\beta))) - A(k,0))\cdot\nabla\xi \\ \geqslant -\int_\Gamma \omega^+(x,k,a\wedge(M-\beta))\xi + \int_\Omega \xi \, d\mu_\beta(v) \end{aligned}$$

and for all $l \leqslant k \in \mathbb{R}$, for all $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi \geqslant 0$ and $(g(k) - g(a \vee l))^+ \xi = 0$ a.e. on Γ,

$$\begin{aligned} \int_\Omega b(v\vee l)\chi_{\{k>v\vee l\}}\xi + \int_\Omega \chi_{\{k>v\vee l\}}(A(v\vee l, \nabla g(v\vee l)) - A(k,0))\cdot\nabla\xi \\ -\int_\Omega \chi_{\{k>v\vee l\}} f\xi \geqslant -\int_\Gamma \omega^-(x,k,a\vee l)\xi + \int_\Omega \xi \, d\nu_\beta(v). \end{aligned}$$

THEOREM 4.2. *For all $f \in L^1(\Omega)$ and $a : \Gamma \to \mathbb{R}$ measurable with $g(a) = 0$ a.e. on Γ, there exists $v : \Omega \to \overline{\mathbb{R}}$ such that v is a renormalized entropy solution of $P_{b,g}(f,a)$.*

Moreover, the comparaison result (3.9) and the partial uniqueness result (3.5) hold true.

4.1. Proof of the existence result

We use another method of approximation. We first consider the problem

$$P_{b_\alpha, g\circ(\cdot\wedge(M-\kappa))}(f,a)\ \begin{cases} b_\alpha(v) - \operatorname{div} A(v, \nabla g(v\wedge(M-\kappa))) = f \text{ in } \Omega, \\ v\wedge(M-\kappa) = a \text{ on } \Gamma, \end{cases}$$

with $\kappa > 0$, $f \in L^\infty(\Omega)$, $a \in L^\infty(\Gamma)$ with $g(a) = 0$ and with $b_\alpha(r) = b(r) + \alpha r$, $\alpha > 0$. Existence and uniqueness results for this problem are already proved in [5]. The definition of the weak entropy solution in this case is given in Proposition 4.3 below. Going to limit with $\kappa \to 0$, we prove the existence of a renormalized solution of the problem $P_{b_\alpha,g}(f,a)$ and we continue exactly as in case (a).

PROPOSITION 4.3. *Let $f \in L^\infty(\Omega)$ and $a \in L^\infty(\Gamma)$ such that $g(a) = 0$ a.e on Γ. Then, there exists a unique $v_\kappa \in L^\infty(\Omega)$ weak entropy solution of $P_{b_\alpha, g\circ(.\wedge(M-\kappa))}(f,a)$ i.e. $g(v_\kappa \wedge (M-\kappa)) \in W_0^{1,p}(\Omega)$ and v_κ satisfies the following entropy inequalities. The first one: for all $k \in \mathbb{R}$, for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ such that $\xi \geqslant 0$ and $(-g(k))^+\xi = 0$ a.e. on Γ,*

$$-\int_\Gamma \omega^+(x,k,a)\xi + \int_\Omega b_\alpha(v_\kappa)\chi_{\{v_\kappa > k\}}\xi \leqslant \int_\Omega \chi_{\{v_\kappa>k\}}(f\xi - (A(v_\kappa, \nabla g(v_\kappa\wedge(M-\kappa))) - A(k,0))\cdot\nabla\xi)$$

and the second one: for all $k \in \mathbb{R}$, for all $\xi \in \mathscr{D}(\mathbb{R}^N)$ such that $\xi \geqslant 0$ and $(g(k))^+\xi = 0$ a.e. on Γ,

$$-\int_\Gamma \omega^-(x,k,a)\xi - \int_\Omega b_\alpha(v_\kappa)\chi_{\{k>v_\kappa\}}\xi \leqslant -\int_\Omega \chi_{\{k>v_\kappa\}}(f\xi - (A(v_\kappa, \nabla g(T_{\kappa,l}v_\kappa)) - A(k,0))\cdot\nabla\xi).$$

The weak entropy solution v is in particular a weak solution of the problem $b_\alpha(v) - \operatorname{div} A(v, \nabla g(v\wedge(M-\kappa))) = f$ i.e.

$$\int_\Omega b_\alpha(v)\xi + \int_\Omega A(v, \nabla g(v\wedge(M-\kappa)))\cdot\nabla\xi = \int_\Omega f\xi \tag{4.1}$$

holds true for all $\xi \in W_0^{1,p}(\Omega)$.

With the same arguments as in case **(a)**, we prove that

$$(v_\kappa)_\kappa \text{ is bounded in } L^\infty(\Omega),$$

$$\tilde{A}(v_\kappa, \nabla T_k g(v_\kappa\wedge(M-\kappa))) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ to some } \chi_k \in L^{p'}(\Omega)^N,$$

$$g(v_\kappa) \text{ converges to } g(v) \text{ a.e. in } \Omega,$$
$$(v_\kappa \chi_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}}) \to v\chi_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}}. \text{ a.e. in } \Omega, \tag{4.2}$$

and according to Lemma 3.8, there exists $v \in L^\infty(\Omega\times(0,1)$ and a subsequence of (v_κ) still denoted by (v_κ) such that

$$(v_\kappa) \text{ converges to } v \text{ in the nonlinear weak-}* \text{ sense.} \tag{4.3}$$

We will prove that v is a renormalized entropy process solution of $P_{b_\alpha,g}(a,f)$ in the following sense:

DEFINITION 4.4. A function $u:(0,1)\times\Omega\to(-\infty,M)$ is a renormalized entropy process solution of $P_{b_\alpha,g}(a,f)$ if

$$u \in L^\infty((0,1)\times\Omega)$$

and there exists a measurable function $w:\Omega\to\mathbb{R}$ and a family $(\mu_\kappa)_\kappa$ of bounded measures on $\overline{\Omega}$ such that

$$w(x) = g(u(\alpha,x)) \text{ a.e. in } \Omega,$$
$$w\wedge k \in W_0^{1,p}(\Omega), \quad \text{for all } k>0,$$
$$A(u,Dw)\chi_{\{u<M\}} \in (L^{p'}(\Omega))^N,$$
$$\mu_\kappa \in (W^{-1,p'}(\Omega)+L^1(\Omega)+L^1(\Gamma))\cap\mathscr{M}(\overline{\Omega}), \tag{4.4}$$
$$\mu_\kappa(\{v\leqslant M-\kappa\}) = 0, \tag{4.5}$$
$$\lim_{\kappa\to0}\int_\Omega \xi\, d\mu_\kappa(v) = 0 \text{ for all } \xi\in\mathscr{D}(\mathbb{R}^N), \tag{4.6}$$

and the following inequalities are satisfied: for all $\kappa,k\in\mathbb{R}$ with $M-\kappa>k$, for all $\xi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $\xi\geqslant0$ and $(g(u\wedge(M-\kappa))-g(k))^+\xi=0$ a.e. on Γ,

$$\begin{aligned}\int_0^1\int_\Omega b_\alpha(u\wedge(M-\kappa))\chi_{\{u\wedge(M-\kappa)>k\}}\xi\,d\mu - \int_0^1\int_\Omega \chi_{\{u\wedge(M-\kappa)>k\}}f\xi \\ +\int_0^1\int_\Omega (A(u\wedge(M-\kappa),\nabla(w\wedge g(M-\kappa)))-A(k,0))\cdot\nabla\xi)\,d\mu \\ \leqslant \int_\Gamma \omega^+(x,k,a\wedge(M-\kappa))\xi - \int_\Omega \xi\,d\mu_\kappa(u)\end{aligned} \tag{4.7}$$

and for all $l\leqslant k\in\mathbb{R}$, for all $\xi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $\xi\geqslant0$ and $(g(l))^+\xi=0$ a.e. on Γ,

$$\begin{aligned}-\int_0^1\int_\Omega b_\alpha(u)\chi_{\{l>u\}}\xi\,d\mu - \int_0^1\int_\Omega \chi_{\{l>u\}}(A(u,\nabla w)-A(k,0))\cdot\nabla\xi)\,d\mu \\ +\int_0^1\int_\Omega \chi_{\{l>u\}}f\xi \leqslant \int_\Gamma \omega^-(x,l,a)\xi. \quad \square\end{aligned} \tag{4.8}$$

Let us first prove that $v < M$ a.e. on Ω: Let

$$\omega_\kappa := \int_{A_2}^{A_2 \vee v \wedge (M-\kappa)} \lambda(r)^{\frac{1}{p-1}} g'(r)\, dr.$$

Let us recall that g is increasing in $[A_2, +\infty)$ and therefore $\omega_\kappa \in W^{1,p}(\Omega)$. Taking $T_k \omega_\kappa$ as test function in (4.1), applying the divergence theorem and taking into account assumption (1.3), we get:

$$\int_\Omega |\nabla T_k \omega_\kappa|^p \leqslant k \parallel f \parallel_{L^1(\Omega)} .$$

This implies that $(\nabla T_k \omega_\kappa)_\kappa$ is uniformly bounded in $L^p(\Omega)^N$ and by a classical argument, it follows that $|\{w_\kappa > k\}| \to 0$ with $k \to \infty$. But on the set $\{v = M\}$, $\omega_\kappa \to \infty$ which implies that $v < M$ a.e. in Ω.

Now, we use the Minty Browder argument in order to prove that for $0 < \beta < M$,

$$\int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\beta))) \cdot \nabla \xi \to \int_0^1 \Big(\int_\Omega \tilde{A}(v, \nabla g(v \wedge (M-\beta)) \Big) \cdot \nabla \xi \big) \, d\mu, \quad (4.9)$$

with $\kappa \to 0$. Therefore, we need the following convergence result:

$$\lim_{L\to 0} \lim_{\kappa \to 0} \int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\beta))) \cdot \nabla T_L (g(v_\kappa \wedge (M-\beta)) - g(v \wedge (M-\beta))) = 0. \quad (4.10)$$

In order to prove (4.10), we decompose the above integral as follows for $\kappa < \alpha$:

$$\begin{aligned}
&\int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\beta)) \cdot \nabla T_L (g(v_\kappa \wedge (M-\beta)) - g(v \wedge (M-\beta))) \\
&\quad = \int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\kappa))) \cdot \nabla T_L (g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta))) \\
&\qquad - \int_{\{v_\kappa \geqslant M-\beta\}} \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\kappa))) \cdot \nabla T_L ((g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta))) \\
&\quad := \mathscr{T}_1 + \mathscr{T}_2.
\end{aligned}$$

As v_κ is a weak solution of $P_{b_\alpha, g\circ(\cdot \wedge (M-\kappa))}(a, f)$,

$$\begin{aligned}
\lim_{L\to 0} \lim_{\kappa \to 0} \mathscr{T}_1 &= \lim_{L\to 0} \lim_{\kappa \to 0} \Big[\int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\kappa))) \\
&\qquad\qquad \cdot \nabla T_L \big(g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta)) \big) \Big] \\
&= -\lim_{L\to 0} \lim_{\kappa \to 0} \Big[\int_\Omega b_\alpha(v_\kappa) T_L \big(g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta)) \big) \\
&\qquad - \int_\Omega f T_L (g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta))) \Big] \\
&\qquad - \int_\Omega A(v_\kappa, 0) \cdot \nabla T_L \big(g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta)) \big). \qquad (4.11)
\end{aligned}$$

It is not difficult to pass to the limit in the first and second term in the R.H.S of inequality (4.11). Now, we estimate the last term. In view of (3.22), for $\kappa > 0$ small enough,

$$\begin{aligned}
&\int_\Omega A(v_\kappa,0)\cdot\nabla T_L(g(v_\kappa\wedge(M-\kappa))-g(v\wedge(M-\beta)))\\
&\quad=\int_{\{v\in]A_1,A_2[\}} A(v_\kappa,0)\cdot\nabla T_L g(v_\kappa\wedge(M-\kappa))\\
&\qquad+\int_{\{v\in\mathbb{R}\setminus[A_1,A_2]\}} A(v_\kappa,0)\cdot\nabla T_L(g(v_\kappa\wedge(M-\kappa))-g(v\wedge(M-\beta)))\\
&\quad=\int_{\{v\in]A_1,A_2[,v_\kappa>A_2\}} A(v_\kappa,0)\cdot\nabla T_L g(v_\kappa\wedge(M-\kappa))\\
&\qquad+\int_{\{v\in]A_1,A_2[,v_\kappa<A_1\}} A(v_\kappa,0)\cdot\nabla T_L g(v_\kappa)\\
&\qquad+\int_{\{v\in\mathbb{R}\setminus]A_1,A_2[\}} A(v_\kappa,0)\cdot\nabla T_L(g(v_\kappa\wedge(M-\kappa))-g(v\wedge(M-\beta)))\\
&\quad=\mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3.
\end{aligned}$$

Taking into account (4.2) it is not difficult to pass to the limit with $\kappa\to 0$ and then with $L\to 0$ in $\mathscr{I}_3$ to find

$$\lim_{L\to 0}\lim_{\kappa\to 0}\mathscr{I}_3=0.$$

Moreover,

$$(A(v_\kappa,0)\chi_{\{v\in]A_1,A_2[,v_\kappa>A_2\}}\chi_{\{|g(v_\kappa\wedge(M-\kappa))-g(v\wedge(M-\beta))|\leqslant L\}})_\kappa\to A(A_2,0)$$

strongly in $L^{p'}(\Omega)^N$ and

$$(A(v_\kappa,0)\chi_{\{v\in]A_1,A_2[,v_\kappa\in]-\infty,A_1[\}}\chi_{\{|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|\leqslant L\}})_\kappa\to A(A_1,0)$$

strongly in $L^{p'}(\Omega)^N$. Hence,

$$\lim_{L\to 0}\lim_{\kappa\to 0}\mathscr{I}_1=\lim_{L\to 0}\int_{\{v\in]A_1,A_2[\}} A(A_2,0)\cdot\nabla T_L g(A_2)=0$$

and with the same arguments, we prove that

$$\lim_{L\to 0}\lim_{\kappa\to 0}\mathscr{I}_2=0.$$

Next, we claim that

$$\lim_{L\to 0}\limsup_{\kappa\to 0}\mathscr{T}_2\geqslant 0.$$

Indeed,

$$\begin{aligned}
&\lim_{L\to 0}\limsup_{\kappa\to 0}\mathscr{T}_2\\
&\quad=\lim_{L\to 0}\limsup_{\kappa\to 0}\Big[-\int_{\{v_\kappa\geqslant M-\beta\}}\tilde{A}(v_\kappa,\nabla g(v_\kappa\wedge(M-\beta))
\end{aligned}$$

$$\cdot \nabla T_L(g(v_\kappa \wedge (M-\kappa)) - g(v \wedge (M-\beta))\Big]$$

$$\geqslant \lim_{L\to 0} \limsup_{\kappa \to 0} \Big[- \int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\kappa)) \cdot \nabla T_L(g(v_\kappa \wedge (M-\kappa)) - g(v_\kappa \wedge (M-\beta))\Big]$$
$$+ \lim_{L\to 0} \limsup_{\kappa \to 0} \Big[\int_{\{v_\kappa \geqslant M-\beta\}} \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\kappa)) \cdot \nabla g(v \wedge (M-\beta))\chi_{\{0<g(v_\kappa \wedge (M-\kappa))-g(v\wedge(M-\beta))<L\}}\Big]$$
$$:= \mathscr{L}_2^1 + \mathscr{L}_2^2.$$

As v_κ is a weak solution of $P_{b_\alpha, g\circ(.\wedge(M-\kappa))}(a,f)$,

$$\begin{aligned}
\mathscr{L}_2^1 &= \lim_{L\to 0} \limsup_{\kappa \to 0} \Big[- \int_\Omega b_\alpha(v_\kappa) T_L(g(v_\kappa \wedge (M-\kappa)) - g(v_\kappa \wedge (M-\beta))) \\
&\quad + \int_\Omega f T_L(g(v_\kappa \wedge (M-\kappa)) - g(v_\kappa \wedge (M-\beta))) \\
&\quad - \int_\Omega A(v_\kappa, 0) \cdot \nabla T_L(g(v_\kappa \wedge (M-\kappa)) - g(v_\kappa \wedge (M-\beta)))\Big] \\
&= \lim_{L\to 0} \limsup_{\kappa \to 0} \Big[- \int_\Omega b_\alpha(v_\kappa) T_L(g(v_\kappa \wedge (M-\kappa)) - g(v_\kappa \wedge (M-\beta)) \\
&\quad + \int_\Omega f(T_L(g(v_\kappa \wedge (M-\kappa)) - g(v_\kappa \wedge (M-\beta)) \\
&\quad - \int_\Omega \operatorname{div}\Big(\int_{A_2}^{T_L(g(v_\kappa\wedge(M-\kappa))-g(v_\kappa\wedge(M-\beta))} A(g^{-1}(r),0)\,dr\Big)\Big] = 0,
\end{aligned}$$

where g^{-1} is the continuous inverse of $g_{/]A_2,M[}$. The term $\mathscr{L}_2^2$ can be estimated an in the term $\mathscr{L}_2^3$ in case (**a**).

Combining all the estimates, we get (4.10) and by the standard pseudo monotonicity argument we prove that

$$\int_\Omega \chi_k \cdot \nabla \xi = \int_0^1 \int_\Omega \tilde{A}(v, \nabla T_k g(v \wedge (M-\kappa))) \cdot \nabla \xi \quad \text{for all } \xi \in \mathscr{D}(\Omega). \tag{4.12}$$

Indeed, for $\beta > 0$ with $M-\beta > 0$ and $g(M-\beta) = k$, for $\xi \in \mathscr{D}(\Omega)$, $\xi \geqslant 0$ and $\sigma \in \mathbb{R}$,

$$\begin{aligned}
&\sigma \int_\Omega \chi_{g(M-\beta)} \nabla \xi \\
&\quad = \lim_{\kappa\to 0} \Big[\int_\Omega \sigma \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\beta))) \cdot \nabla \xi \Big] \\
&\quad \geqslant \lim_{L\to 0} \limsup_{\kappa\to 0} \Big[\int_\Omega \tilde{A}(v_\kappa, \nabla g(v_\kappa \wedge (M-\kappa))) \\
&\qquad\qquad \cdot \nabla T_L(g(v_\kappa \wedge (M-\beta)) - g(v \wedge (M-\beta))) + \sigma \xi)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|<L\}}\tilde{A}(v_\kappa,\nabla g(v_\kappa\wedge(M-\beta)))\\
&\qquad\qquad\cdot\nabla(g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))+\alpha\xi)\Big]\\
&\quad+\lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|\geqslant L\}}\tilde{A}(v_\kappa,\nabla g(v_\kappa\wedge(M-\beta)))\cdot\nabla(\sigma\xi)\Big]\\
&= \mathscr{P}_1+\mathscr{P}_2.
\end{aligned}
$$

By assumption (1.4),

$$
\begin{aligned}
\mathscr{P}_1 &\geqslant \lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|<L\}}\tilde{A}(v_\kappa,\nabla(g(v\wedge(M-\beta))-\sigma\zeta))\\
&\qquad\qquad\cdot\nabla\big(g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))+\sigma\zeta\big)\Big]\\
&= \lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|<L\}}\tilde{A}(v_\kappa,\nabla(g(v\wedge(M-\beta))-\sigma\zeta))\cdot\nabla(\alpha\zeta)\Big]\\
&\quad+\lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{v\in]A_2,M[,|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|<L\}}\tilde{A}(v_\kappa,\nabla(g(v\wedge(M-\beta))-\sigma\zeta))\\
&\qquad\qquad\cdot\nabla(g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta)))\Big]\\
&\quad+\lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{v\in]A_1,A_2[,|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|<L\}}\tilde{A}(v_\kappa,\nabla(g(A_1)-\sigma\zeta))\\
&\qquad\qquad\cdot\nabla\big(g(v_\kappa\wedge(M-\beta))-g(A_1)\big)\Big]\\
&\quad+\lim_{L\to 0}\limsup_{\kappa\to 0}\Big[\int_{\{v_\kappa\in]-\infty,A_1[,v\in]A_1,A_2[,|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|<L\}}\tilde{A}(v_\kappa,\nabla g(A_1)-\sigma\zeta))\\
&\qquad\qquad\cdot\nabla(g(v_\kappa\wedge(M-\beta))-g(A_1))\Big]\\
&= \mathscr{P}_1^1+\mathscr{P}_1^2+\mathscr{P}_1^3+\mathscr{P}_1^4.
\end{aligned}
$$

It is not difficult to pass to the limit in the first term in the R.H.S of the above inequality to find

$$
\mathscr{P}_1^1=\int_0^1\Big(\int_\Omega\tilde{A}(v,\nabla(g(v\wedge(M-\beta))-\sigma\zeta))\cdot\nabla(\sigma\zeta)\Big)\,d\mu.
$$

As to $\mathscr{P}_1^2$, this term is equal to 0 by (3.21) and it is easy to see that $\mathscr{P}_1^3=\mathscr{P}_1^4=0$. Now, in order to estimate $\mathscr{P}_2$, we use the growth condition (1.2),

$$
\begin{aligned}
\mathscr{P}_2 &\leqslant \lim_{L\to 0}\limsup_{\kappa\to 0}\Big(\int_\Omega|A(v_\kappa,\nabla(g(v\wedge(M-\beta))-\sigma\zeta)|^{p'}\Big)^{\frac{1}{p'}}\\
&\qquad\qquad\Big(\int_{\{|g(v_\kappa\wedge(M-\beta))-g(v\wedge(M-\beta))|\geqslant L\}}|\nabla(\sigma\zeta)|^p\Big)^{\frac{1}{p}}=0.
\end{aligned}
$$

Hence,

$$\alpha \int_\Omega \chi_k \sigma\zeta \geqslant \int_0^1 (\int_\Omega \tilde{A}(v, \nabla(g(v\wedge(M-\beta)) - \sigma\zeta)) \cdot \nabla(\sigma\zeta))\, d\mu.$$

Dividing by $\sigma > 0$ (resp. $\alpha < 0$), passing to the limit with $\sigma \to 0$, we obtain (3.25).

The rest of the proof follows the same lines as those of in case **(a)**. ◇

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