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OSCILLATION CRITERIA FOR CERTAIN EVEN ORDER DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. By using averaging function and the approach developed by Philos and Kong, Kamenev-type and interval oscillation criteria are established for the even order differential equation with distributed deviating arguments,

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + \int_{\alpha}^{\beta} F[t,\xi,x(g(t,\xi))]d\sigma(\xi) = 0.$$

The obtained results are extensions of existing ones for second order linear differential equations.

1. Introduction

In this paper, we are concerned with the oscillatory properties of the following even order differential equation with distributed deviating arguments

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + \int_{\alpha}^{\beta} F[t,\xi,x(g(t,\xi))]d\sigma(\xi) = 0, \quad n \text{ even}, \tag{1.1}$$

where $t \geqslant t_0 \geqslant 0$ and $p > 0$ is a constant.

Throughout this paper, we assume that the following conditions hold:

(A1) $F \in \mathbf{C}([t_0,\infty)\times[\alpha,\beta]\times\mathbb{R},\mathbb{R})$ and $\operatorname{sign} F(t,\xi,x) = \operatorname{sign} x$ for $t \geqslant t_0$, $\xi \in [\alpha,\beta]$. Moveover, there exist functions $q \in \mathbf{C}([t_0,\infty)\times[\alpha,\beta],\mathbb{R}^+ = (0,\infty))$, and $f \in \mathbf{C}^1(\mathbb{R},\mathbb{R})$ with

$$xf(x) > 0 \quad \text{and} \quad \frac{f'(x)}{|f(x)|^{(p-1)/p}} \geqslant k > 0 \quad \text{for } x \neq 0,$$

such that

$$F(t,\xi,x)x \geqslant q(t,\xi)f(x) \quad \text{for } x > 0,\ t \geqslant t_0,\ \xi \in [\alpha,\beta];$$

(A2) $g \in \mathbf{C}^1([t_0,\infty)\times[\alpha,\beta],\mathbb{R})$, $g(t,\xi) \leqslant t$ for all $t \geqslant t_0$, $\xi \in [\alpha,\beta]$, $g(t,\xi)$ is non-decreasing with respect to t and ξ, respectively, and

$$\lim_{t\to\infty,\ \xi\in[\alpha,\beta]} \inf g(t,\xi) = \infty;$$

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(A3) $r \in \mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$ with $\int^{\infty} r^{-1/p}(s)ds = \infty$, $\liminf_{t\to\infty} r(t) = c > 0$. For any $\varepsilon > 0$, there exists a $t_\varepsilon > t_0$ such that

$$|r'(t)| \leqslant \varepsilon \int_\alpha^\beta q(t,\xi)d\sigma(\xi) \quad \text{for all } t \geqslant t_\varepsilon;$$

(A4) $\sigma \in \mathbf{C}^1([\alpha,\beta],\mathbb{R})$ is nondecreasing, and the integral of Eq.(1.1) is a Riemann-Stieltjes one.

By a solution of Eq.(1.1) we mean a function $x \in \mathbf{C}^{n-1}([T_x,\infty),\mathbb{R})$ for some $T_x \geqslant t_0$ which has the property

$$r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t) \in \mathbf{C}^1([T_x,\infty),\mathbb{R})$$

and satisfies Eq.(1.1) on $[T_x,\infty)$. A nontrivial solution of Eq.(1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1.1) is oscillatory if all of its solutions are oscillatory.

It is clear that Eq.(1.1) includes the following equation

$$(|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + F[t,x(g(t))] = 0. \tag{1.2}$$

The oscillation of Eq.(1.2) was first studied by Agarwal, Grace, O'Regan [1], and also considered by several researchers [17,18]. On the other hand, the recently paper by Wang and Zhang [15] has presented some oscillation criteria for Eq.(1.1). For general interest of the oscillation of high order differential equation, see, for example, [1-5,8,10,12,13,15,17,18,19] and references therein.

As we known, the oscillation results provided in [1,15,17,18] require the integral of the coefficients of Eq.(1.2) on the entire interval $[t_0,\infty)$. But, from the Sturm Separation Theorem, oscillation is only an interval property. More precisely, if the exists a sequence of subset $[a_i,b_i]$ of $[t_0,\infty)$, $b_i < a_{i+1}$, $a_i \to \infty$ as $i \to \infty$, such that for each i there exists a nontrivial solution of equation which has at least two zeros in $[a_i,b_i]$, then every solution of the equation is oscillatory, no matter what the behavior of the coefficients of the equation is on the remaining parts of $[t_0,\infty)$. This idea was used by Kong [9] to establish interval oscillation criteria for second order linear differential equations. Recently, Tang and Yang [13] and Tiryaki, Basci and Gülec [14] have presented several interval oscillation theorems for a class of even order nonlinear damped differential equations and second order functional differential equations, respectively.

In this paper, by using averaging function and the approach developed by Philos [11] and Kong [9], we establish Kamenev-type criteria as well as interval criteria for Eq.(1.1), and extend the results in [9,11] to Eq.(1.1), which improve and complement some existing results in [15]. To show the importance of our main results, two interesting examples are included.

2. Oscillation results

In this section, we shall establish Kamenev-type and interval oscillation criteria for Eq.(1.1). For notational simplicity, we define

$$\varphi(t) = \frac{2^{1-n}}{(n-2)!} r^{-1/p}(t) g'(t,\alpha) g^{n-2}(t,\alpha) \quad \text{and} \quad \mu = \frac{1}{(p+1)^{p+1}} \left(\frac{p}{k}\right)^p.$$

We say that a function $H = H(t,s)$ belongs to a function class $\mathscr{H}$, denoted by $H \in \mathscr{H}$, if $H \in \mathbf{C}(D,[0,\infty))$, where $D = \{(t,s) : t_0 \leqslant s \leqslant t < \infty\}$, which satisfies $H(t,t) = 0$, $H(t,s) > 0$ for $t > s \geqslant t_0$, H has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t}(t,s) = h_1(t,s)H(t,s) \quad \text{and} \quad \frac{\partial H}{\partial s}(t,s) = -h_2(t,s)H(t,s),$$

where $h_1, h_2 \in L_{loc}(D,\mathbb{R})$.

For given functions $h \in \mathbf{C}(D,\mathbb{R})$, $\rho \in \mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$ and $\eta \in \mathbf{C}^1([t_0,\infty),\mathbb{R})$. Let

$$\lambda_1(s,t) = h_1(s,t) + \frac{\rho'(s)}{\rho(s)}, \quad \lambda_2(t,s) = h_2(t,s) - \frac{\rho'(s)}{\rho(s)},$$

and

$$\Theta_1(s,t) = \int_\alpha^\beta q(t,\xi)d\sigma(\xi) - \eta'(s) - \lambda_1(s,t)\eta(s),$$

$$\Theta_2(t,s) = \int_\alpha^\beta q(t,\xi)d\sigma(\xi) - \eta'(s) + \lambda_2(t,s)\eta(s).$$

The following two lemmas will be needed in proving of our main results. The first is the well-known Kiguradze's Lemma [8]. The second can be obtained easily by Kiguradze and Koplatadze's Lemmas, see [10, Chapt. 1].

LEMMA 2.1. ([8]) *Let $u \in \mathbf{C}^n([t_0,\infty),\mathbb{R}_+)$. If $u^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[t^*,\infty)$, then there exist a $t_u \geqslant t_0$ and an integer j, $0 \leqslant j \leqslant n$, with $n+j$ even for $u^{(n)}(t) \geqslant 0$, or $n+j$ odd for $u^{(n)}(t) \leqslant 0$ such that*

$$j > 0 \quad \textit{implies that} \quad u^{(k)}(t) > 0 \quad \textit{for} \quad t \geqslant t_u, \quad k = 0,1,\cdots,j-1,$$

and

$$j \leqslant n-1 \quad \textit{implies that} \quad (-1)^{j+k}u^{(k)}(t) > 0 \quad \textit{for} \quad t \geqslant t_u, \quad k = j, j+1, \cdots, n-1.$$

LEMMA 2.2. ([10]) *If the function $u(t)$ is as in Lemma 2.1 and $u^{(n-1)}(t)u^{(n)}(t) \leqslant 0$ for any $t \geqslant t_u$, then*

$$u(t/2) \geqslant \frac{2^{1-n}}{(n-1)!} t^{n-1} |u^{(n-1)}(t)| \quad \textit{for all large } t.$$

Firstly, we give Kamenev-type criteria for Eq.(1.1).

THEOREM 2.1. *If there exist functions* $\rho \in \mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$, $\eta \in \mathbf{C}^1([t_0,\infty),\mathbb{R})$, *and* $H \in \mathscr{H}$ *such that for any* $T \geqslant t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)\Big[\Theta_2(t,s) - \mu \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}\Big] ds = \infty, \tag{2.1}$$

then Eq. (1.1) *is oscillatory.*

Proof. Suppose to contrary that Eq.(1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t,\xi)) > 0$ for $t \geqslant t_1 \geqslant t_0 \geqslant 0$, $\xi \in [\alpha,\beta]$. Since

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' \leqslant -\int_\alpha^\beta q(t,\xi)f[x(g(t,\xi))]d\sigma(\xi) \leqslant 0,$$

the function $r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. If $x^{(n-1)}(t) < 0$ eventually, then there exists a constant $\delta > 0$ such that

$$-r(t)(-x^{(n-1)}(t))^p \leqslant -\delta^p < 0.$$

Integrating the above inequality from t_1 to t, we get

$$x^{(n-2)}(t) \leqslant x^{(n-2)}(t_1) - \delta \int_{t_1}^t \frac{ds}{r^{1/p}(s)}.$$

By (A3) we find that $x^{(n-2)}(t) < 0$ eventually. But then Lemma 2.1 (note that n is even) implies that $x(t) < 0$ eventually, which is a contradiction. So $x^{(n-1)}(t) > 0$ eventually, then again from Lemma 2.1 we have $x'(t) > 0$ eventually. Thus there exists a $t_2 \geqslant t_1$ such that

$$x'(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geqslant t_2. \tag{2.2}$$

Observing that the function $r(t)(x^{n-1)}(t))^p$ is decreasing for $t \geqslant t_2$, by (A3), there exists a $t_3 \geqslant t_2$ such that

$$(x^{(n-1)}(t))^p \leqslant \frac{r(t_3)}{r(t)}(x^{(n-1)}(t_3))^p \leqslant \frac{r(t_3)}{c}(x^{(n-1)}(t_3))^p \quad \text{for } t \geqslant t_3. \tag{2.3}$$

Note that (A1), (A2) and (2.2), so

$$f(x(g(t,\xi))) \geqslant f(x(g(t,\alpha)) > 0 \quad \text{for } t \geqslant t_3,\ \xi \in [\alpha,\beta]. \tag{2.4}$$

It follows from (1.1) and (2.4) that

$$\big(r(t)(x^{(n-1)}(t))^p\big)' = r'(t)(x^{(n-1)}(t))^p + p\,r(t)(x^{(n-1)}(t))^{p-1}x^{(n)}(t)$$

$$\leqslant -\int_{\alpha}^{\beta} q(t,\xi) f(x(g(t,\xi)))d\sigma(\xi)$$

$$\leqslant -f(x(g(t,\alpha)))\int_{\alpha}^{\beta} q(t,\xi)d\sigma(\xi). \tag{2.5}$$

Now, in view of (A3), let

$$\varepsilon = \frac{c}{2r(t_3)}\frac{f(x(g(t_3,\alpha)))}{(x^{(n-1)}(t_3))^p},$$

there exists a $t_4 \geqslant t_3$ such that, note that (A3), (2.3) and (2.5), for $t \geqslant t_4$,

$$p\,r(t)(x^{(n-1)}(t))^{p-1}x^{(n)}(t) \leqslant |r'(t)|(x^{(n-1)}(t))^p - f(x(g(t,\alpha)))\int_{\alpha}^{\beta} q(t,\xi)d\sigma(\xi)$$

$$\leqslant \Big(\varepsilon\int_{\alpha}^{\beta} q(t,\xi)d\sigma(\xi)\Big)\Big(\frac{r(t_3)}{c}(x^{(n-1)}(t_3))^p\Big) - f(x(g(t,\alpha)))\int_{\alpha}^{\beta} q(t,\xi)d\sigma(\xi)$$

$$= -\frac{1}{2}f(x(g(t,\alpha))\int_{\alpha}^{\beta} q(t,\xi)d\sigma(\xi) \leqslant 0.$$

Thus, we find $x^{(n)}(t) \leqslant 0$ for $t \geqslant t_4$. It is easy to check that we can apply Lemma 2.2 for $x' = u$, and conclude that there exists a $t_5 \geqslant t_4$ such that

$$x'\Big(\frac{1}{2}g(t,\alpha)\Big) \geqslant \frac{2^{2-n}}{(n-2)!}g^{n-2}(t,\alpha)x^{(n-1)}(t) \quad \text{for } t \geqslant t_5, \tag{2.6}$$

since

$$x^{(n-1)}(g(t,s)) \geqslant x^{(n-1)}(t) \text{ for } t \geqslant t_5.$$

Put

$$W(t) = \rho(t)\Big[\frac{r(t)|x^{(n-1)}(t))|^{p-1}x^{(n-1)}(t)}{f(x(g(t,\alpha)/2))} + \eta(t)\Big].$$

Noting that (1.1) and (2.6), we have

$$W'(t) \leqslant -\rho(t)\Big[\int_{\alpha}^{\beta} q(t,\xi)d\sigma(\xi) - \eta'(t)\Big] + \frac{\rho'(t)}{\rho(t)}W(t) - k\rho(t)\varphi(t)\Big|\frac{W(t)}{\rho(t)} - \eta(t)\Big|^{(p+1)/p}. \tag{2.7}$$

Replacing t by s, multiplying by $H(t,s)$, integrating from T to t, we obtain

$$\int_T^t H(t,s)\rho(s)\Theta_2(t,s)ds \leqslant H(t,T)W(T) + \int_T^t H(t,s)\rho(s)|\lambda_2(t,s)|\Big|\frac{W(s)}{\rho(s)} - \eta(s)\Big|ds$$

$$-k\int_T^t H(t,s)\rho(s)\varphi(s)\left|\frac{W(s)}{\rho(s)}-\eta(s)\right|^{(p+1)/p}ds. \tag{2.8}$$

The Young inequality [6, Theorem 61] gives

$$|\lambda_2(t,s)|\left|\frac{W(s)}{\rho(s)}-\eta(s)\right| \leqslant k\varphi(s)\left|\frac{W(s)}{\rho(s)}-\eta(s)\right|^{(p+1)/p}+\mu\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}.$$

Substituting the above inequality into (2.8), we get

$$\int_T^t H(t,s)\rho(s)\Big[\Theta_2(t,s)-\mu\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}\Big]ds \leqslant W(T)H(t,T). \tag{2.9}$$

Dividing by $H(t,T)$, and taking the upper limits as $t\to\infty$. The right hand side is always bounded, which contradicts condition (2.1). This completes the proof. □

As an immediate consequence of Theorem 2.1, we get the following corollary.

COROLLARY 2.1. *Let condition* (2.1) *in Theorem 2.1 be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\Theta_2(t,s)ds=\infty \tag{2.10}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}ds<\infty, \tag{2.11}$$

then conclusion of Theorem 2.1 holds.

COROLLARY 2.2. *Let* ρ *and* η *be as in Theorem 2.1 and* $\lim_{t\to\infty}G(t)=\infty$. *If for some* $\lambda>p$,

$$\limsup_{t\to\infty}\frac{1}{G^\lambda(t)}\int_{t_0}^t [G(t)-G(s)]^\lambda\rho(s)\Theta_2(t,s)ds=\infty, \tag{2.12}$$

and

$$\int^\infty \frac{|\rho'(s)|^{p+1}}{(\rho(s)\varphi(s))^p}ds<\infty, \tag{2.13}$$

where $G(t)=\int_{t_0}^t\varphi(s)\rho^{-1/p}(s)ds$, *then Eq.* (1.1) *is oscillatory.*

Proof. Let $H(t,s) = [G(t) - G(s)]^{\lambda}$, then

$$h_2(t,s) = \frac{\lambda \varphi(s)}{\rho^{1/p}(s)[G(t) - G(s)]}.$$

By the elementary inequality,

$$(X+Y)^{p+1} \leqslant 2^p (X^{p+1} + Y^{p+1}), \ \ X,\, Y \geqslant 0,$$

we obtain

$$\int_T^t H(t,s)\rho(s)\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}ds$$
$$\leqslant 2^p \Big[\int_T^t H(t,s)\rho(s)\frac{|h_2(t,s)|^{p+1}}{\varphi^p(s)}ds + \int_T^t H(t,s)\frac{|\rho'(s)|^{p+1}}{(\rho(s)\varphi(s))^p}ds\Big]. \qquad (2.14)$$

Noting that

$$\int_T^t H(t,s)\rho(s)\frac{|h_2(t,s)|^{p+1}}{\varphi^p(s)}ds = \frac{\lambda^{p+1}}{\lambda - p}\Big[G(t) - G(T)\Big]^{\lambda - p},$$

and by (2.13) and [16, Lemma (14)],

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\frac{|\rho'(s)|^{p+1}}{(\rho(s)\varphi(s))^p}ds = 0.$$

Hence, by (2.14), note that $\lim_{t\to\infty} G(t) = \infty$, we have

$$\lim_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}ds = 0.$$

It following from Corollary 2.1 that Eq.(1.1) is oscillatory. □

REMARK 2.1. Corollary 2.2 improves [15, Theorem 2.2]. □

Next, we give interval oscillation criteria for Eq.(1.1).

THEOREM 2.2. *Let ρ, η and H be as in Theorem 2.1. If for each $T \geqslant t_0$, there exist constants a, b, and c such that $T \leqslant a < c < b$,*

$$\frac{1}{H(b,c)} \int_c^b H(b,s)\rho(s)\Big[\Theta_2(b,s) - \mu \frac{|\lambda_2(b,s)|^{p+1}}{\varphi^p(s)}\Big]ds$$

$$+\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)\Big[\Theta_1(s,a)-\mu\frac{|\lambda_1(s,a)|^{p+1}}{\varphi^p(s)}\Big]ds>0, \qquad (2.15)$$

then Eq. (1.1) *is oscillatory.*

Proof. Proceeding as the proof of Theorem 2.1. For each $T\geqslant t_5$, there exists an interval $[a,b]$ such that (2.7) hold for $t\in[a,b]$. Replacing t to s, multiplying by $H(t,s)$, and integrating from c to t $(t\leqslant b)$, we have

$$\frac{1}{H(t,c)}\int_c^t H(t,s)\rho(s)\Big[\Theta_2(t,s)-\mu\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}\Big]ds\leqslant W(c).$$

Let $t\to b^-$ in above inequality, then

$$\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)\Big[\Theta_2(b,s)-\mu\frac{|\lambda_2(b,s)|^{p+1}}{\varphi^p(s)}\Big]ds\leqslant W(c). \qquad (2.16)$$

On the other hand, replacing t by s in (2.7), multiplying $H(s,t)$ integrating from t $(t\geqslant a)$ to c. Similar to the proof of (2.16), we obtain

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)\Big[\Theta_1(s,a)-\mu\frac{|\lambda_1(s,a)|^{p+1}}{\varphi^p(s)}\Big]ds\leqslant -W(c). \qquad (2.17)$$

Now, (2.16) and (2.17) imply that desired contradiction, which completes the proof. □

COROLLARY 2.3. *Let* ρ,η,H *be as in Theorem 2.1. If for each* $l\geqslant t_0$*,*

$$\limsup_{t\to\infty}\int_l^t H(s,l)\rho(s)\Big[\Theta_1(s,l)-\mu\frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)}\Big]ds>0 \qquad (2.18)$$

and

$$\limsup_{t\to\infty}\int_l^t H(t,s)\rho(s)\Big[\Theta_2(t,s)-\mu\frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}\Big]ds>0, \qquad (2.19)$$

then Eq. (1.1) *is oscillatory.*

Proof. For each $T\geqslant t_0$. Let $l=a=T$ in (2.18). Clearly, we see from (2.18) that there exists a $c>a$ such that

$$\int_a^c H(s,a)\rho(s)\Big[\Theta_1(s,a)-\mu\frac{|\lambda_1(s,a)|^{p+1}}{\varphi^p(s)}\Big]ds>0. \qquad (2.20)$$

Similarly, setting $l=c=T$ in (2.19), it follows that there exists a $b>c$ such that

$$\int_c^b H(b,s)\rho(s)\Big[\Theta_2(b,s)-\mu\frac{|\lambda_2(b,s)|^{p+1}}{\varphi^p(s)}\Big]ds>0. \tag{2.21}$$

So, (2.20) and (2.21) imply that (2.15) in Theorem 2.2 is true, which completes the proof. □

COROLLARY 2.4. *If there exists a function $\rho\in\mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$ such that for each $l\geqslant t_0$, and some $\lambda>p$,*

$$\limsup_{t\to\infty}\frac{1}{G^{\lambda-p}(t)}\int_l^t [G(s)-G(l)]^\lambda\rho(s)\Big(\int_\alpha^\beta q(s,\xi)d\sigma(\xi)\Big)ds>\frac{\mu\lambda^{p+1}}{\lambda-p} \tag{2.22}$$

and

$$\limsup_{t\to\infty}\frac{1}{G^{\lambda-p}(t)}\int_l^t [G(t)-G(s)]^\lambda\rho(s)\Big(\int_\alpha^\beta q(s,\xi)d\sigma(\xi)\Big)ds>\frac{\mu\lambda^{p+1}}{\lambda-p}, \tag{2.23}$$

where $G(t)$ is defined in Corollary 2.2 and $\lim_{t\to\infty}G(t)=\infty$, then Eq. (1.1) is oscillatory.

Proof. Let $H(t,s)=[G(t)-G(s)]^\lambda$ and $\eta(s)=0$, we get

$$h_1(t,s)=\frac{\lambda\varphi(t)}{\rho^{1/p}(t)[G(t)-G(s)]}\quad\text{and}\quad h_2(t,s)=\frac{\lambda\varphi(s)}{\rho^{1/p}(s)[G(t)-G(s)]}.$$

Note that

$$\int_l^t H(s,l)\rho(s)\frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)}ds=\frac{\lambda^{p+1}}{\lambda-p}[G(t)-G(l)]^{\lambda-p}$$

and

$$\int_l^t H(t,s)\rho(s)\frac{|\lambda_2(s,l)|^{p+1}}{\varphi^p(s)}ds=\frac{\lambda^{p+1}}{\lambda-p}[G(t)-G(l)]^{\lambda-p}.$$

In view of $\lim_{t\to\infty}G(t)=\infty$, we have

$$\lim_{t\to\infty}\frac{1}{G^{\lambda-p}(t)}\int_l^t H(s,l)\rho(s)\frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)}ds=\frac{\lambda^{p+1}}{\lambda-p} \tag{2.24}$$

and

$$\lim_{t\to\infty}\frac{1}{G^{\lambda-p}(t)}\int_l^t H(s,l)\rho(s)\frac{|\lambda_2(s,l)|^{p+1}}{\varphi^p(s)}ds=\frac{\lambda^{p+1}}{\lambda-p}. \tag{2.25}$$

From (2.22) and (2.24), we get

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,l)}\int_l^t H(s,l)\rho(s)\Big[\Theta_1(s,l)-\mu\frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)}\Big]ds\\&=\lim_{t\to\infty}\frac{1}{G^{\lambda-p}(t)}\int_l^t[G(s)-G(l)]^\lambda\rho(s)\Big(\int_\alpha^\beta q(s,\xi)d\sigma(\xi)\Big)ds-\frac{\mu\lambda^{p+1}}{\lambda-p}>0,\end{aligned}$$

i.e., (2.18) holds. Similarly, (2.23) implies that (2.22) holds. By Corollary 2.3, Eq.(1.1) is oscillatory. □

REMARK 2.2. From the above oscillation criteria, one can obtain different sufficient conditions for oscillation of Eq.(1.1) by different choices of $H(t,s)$. Following the well known Kamenev-type condition [7], let

$$H(t,s)=(t-s)^\lambda,\quad \lambda>p.$$

As the direct consequences of Theorems 2.1-2.2, we can establish oscillation criteria for Eq.(1.1). Here we omit the details. □

To illustrate the main results obtained in this paper, we consider the following interesting examples.

EXAMPLE 2.1. Consider the delay differential equation

$$(r(t)|x^{(n-1)}(t)|^2x^{(n-1)}(t))'+\int_{1/2}^1 q(t,\xi)x^3(t\xi)d\xi=0,\quad n\text{ even},\tag{2.26}$$

where $t\geqslant 1$, $c>0$,

$$r(t)=t^{-4}+c,\ g(t,\xi)=t\xi,\ f(x)=x^3,\text{ and }q\in C([1,\infty)\times[\tfrac{1}{2},1],\mathbb{R}^+)$$

with

$$\int_{1/2}^1 q(t,\xi)d\xi\geqslant c_1t^{\lambda-1},\ c_1>0,\ \lambda\geqslant 4.$$

For Corollary 2.2, let

$$\rho(t)=t^{-\lambda},\quad \eta(t)=0,\quad H(t,s)=(t-s)^\lambda.$$

Then

$$\lambda_2(t,s)=\frac{\lambda t}{(t-s)s},\quad \varphi(t)=\frac{2^{2(1-n)}}{(n-2)!}(t^{-4}+c)^{-1/3}t^{n-2}\geqslant c_2t^{n-2},$$

where

$$c_2:=\frac{2^{2(1-n)}}{(n-2)!}(\frac{1}{2}+c)^{-1/3}.$$

It follows from [6, Theorem 41] that

$$(t-s)^{\lambda} \geqslant t^{\lambda} - \lambda s t^{\lambda-1} \quad \text{for } t \geqslant s \geqslant 1,$$

we can obtain, for all $T \geqslant 1$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)\Theta_2(t,s)ds \\
&\quad \geqslant \lim_{t\to\infty} \frac{c_1}{(t-T)^{\lambda}} \int_T^t (t-s)^{\lambda} \frac{1}{s} ds \geqslant \lim_{t\to\infty} \frac{c_1}{(t-1)^{\lambda}} \int_T^t \frac{t^{\lambda} - \lambda s t^{\lambda-1}}{s} ds = \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s) \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)} ds \\
&\quad \leqslant \frac{\lambda^4}{c_2^3} \limsup_{t\to\infty} \frac{t^4}{(t-T)^{\lambda}} \int_T^t (t-s)^{\lambda-4} s^{-\lambda-3n+2} ds \\
&\quad \leqslant \frac{\lambda^4}{c_2^3} \limsup_{t\to\infty} \int_T^t s^{-\lambda-3n+2} ds < \infty.
\end{aligned}$$

Thus, all conditions of Corollary 2.2 are satisfied, Eq.(2.26) is oscillatory. □

EXAMPLE 2.2. Consider the even order differential equation

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + \int_{1/2}^{1} q(t,\xi)|x(t\xi)|^{p-1}x(t\xi)d\xi = 0, \quad n \text{ even}, \tag{2.27}$$

where $t \geqslant t_0 > 1$, $r(t)$ satisfies (A3), $g(t,\xi) = t\xi$, $q \in C([t_0,\infty) \times [\frac{1}{2},1], \mathbb{R}^+)$ with

$$\int_{1/2}^{1} q(t,\xi)d\xi \geqslant \frac{\gamma\varphi(t)}{G^{p+1}(t)}, \ \gamma > 0,$$

and $\varphi(t)G^{-(p+1)}(t)$ is decreasing for $t \geqslant t_0$, where $G(t)$ is defined as in Corollary 2.2, and $\lim_{t\to\infty} G(t) = \infty$.

Let $p_0 := \max\{1,p\}$, then we can verify that Eq.(2.27) is oscillatory for $\gamma \geqslant p_0^{p+1}\mu$. Indeed, let $\rho(t) = 1$ and $\eta(t) = 0$. Note that $\lambda > p \geqslant 1$ and from [6, Theorem 41], we have

$$[G(s) - G(l)]^{\lambda} \geqslant G^{\lambda}(s) - \lambda G(l) G^{\lambda-1}(s) \quad \text{for } s \geqslant l \geqslant t_0. \tag{2.28}$$

It follows from $G'(t) = \varphi(t)$ that for each $l \geqslant t_0$,

$$\limsup_{t\to\infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(s)-G(l)]^\lambda \rho(s)\Big(\int_{1/2}^1 q(s,\xi)d\xi\Big)ds$$

$$\geqslant \limsup_{t\to\infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(s)-G(l)]^\lambda \frac{\gamma}{G^{p+1}(s)} dG(s) = \frac{\gamma}{\lambda-p}. \tag{2.29}$$

For any $\gamma > p_0^{p+1}\mu$, there exists $\lambda > p_0$ such that

$$\frac{\gamma}{\lambda-p} > \frac{\mu\lambda^{p+1}}{\lambda-p}.$$

This means (2.22) holds.

On the other hand, note that $\varphi(t)G^{-(p+1)}(t)$ is decreasing, by [9, Lemma 3.1], we have

$$\int_l^t [G(t)-G(s)]^\lambda \frac{\varphi(s)}{G^{p+1}(s)} ds \geqslant \int_l^t [G(s)-G(l)]^\lambda \frac{\varphi(s)}{G^{p+1}(s)} ds. \tag{2.30}$$

By (2.29) and (2.30), condition (2.23) holds for the same λ. Applying Corollary 2.4, we find that Eq.(2.27) is oscillatory if $\gamma > p_0^{p+1}\mu$. □

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