

OSCILLATION CRITERIA FOR CERTAIN EVEN ORDER DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. By using averaging function and the approach developed by Philos and Kong, Kamenev-type and interval oscillation criteria are established for the even order differential equation with distributed deviating arguments,

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + \int_{\alpha}^{\beta} F[t, \xi, x(g(t, \xi))]d\sigma(\xi) = 0.$$

The obtained results are extensions of existing ones for second order linear differential equations.

1. Introduction

In this paper, we are concerned with the oscillatory properties of the following even order differential equation with distributed deviating arguments

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + \int_{\alpha}^{\beta} F[t, \xi, x(g(t, \xi))]d\sigma(\xi) = 0, \quad n \text{ even}, \quad (1.1)$$

where $t \geq t_0 \geq 0$ and $p > 0$ is a constant.

Throughout this paper, we assume that the following conditions hold:

(A1) $F \in C([t_0, \infty) \times [\alpha, \beta] \times \mathbb{R}, \mathbb{R})$ and $\text{sign} F(t, \xi, x) = \text{sign} x$ for $t \geq t_0$, $\xi \in [\alpha, \beta]$. Moreover, there exist functions $q \in C([t_0, \infty) \times [\alpha, \beta], \mathbb{R}^+ = (0, \infty))$, and $f \in C^1(\mathbb{R}, \mathbb{R})$ with

$$xf(x) > 0 \quad \text{and} \quad \frac{f'(x)}{|f(x)|^{(p-1)/p}} \geq k > 0 \quad \text{for } x \neq 0,$$

such that

$$F(t, \xi, x)x \geq q(t, \xi)f(x) \quad \text{for } x > 0, t \geq t_0, \xi \in [\alpha, \beta];$$

(A2) $g \in C^1([t_0, \infty) \times [\alpha, \beta], \mathbb{R})$, $g(t, \xi) \leq t$ for all $t \geq t_0$, $\xi \in [\alpha, \beta]$, $g(t, \xi)$ is non-decreasing with respect to t and ξ , respectively, and

$$\lim_{t \rightarrow \infty} \inf_{\xi \in [\alpha, \beta]} g(t, \xi) = \infty;$$

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(A3) $r \in C^1([t_0, \infty), \mathbb{R}^+)$ with $\int r^{-1/p}(s)ds = \infty$, $\liminf_{t \rightarrow \infty} r(t) = c > 0$. For any $\varepsilon > 0$, there exists a $t_\varepsilon > t_0$ such that

$$|r'(t)| \leq \varepsilon \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) \quad \text{for all } t \geq t_\varepsilon;$$

(A4) $\sigma \in C^1([\alpha, \beta], \mathbb{R})$ is nondecreasing, and the integral of Eq.(1.1) is a Riemann-Stieltjes one.

By a solution of Eq.(1.1) we mean a function $x \in C^{n-1}([T_x, \infty), \mathbb{R})$ for some $T_x \geq t_0$ which has the property

$$r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t) \in C^1([T_x, \infty), \mathbb{R})$$

and satisfies Eq.(1.1) on $[T_x, \infty)$. A nontrivial solution of Eq.(1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1.1) is oscillatory if all of its solutions are oscillatory.

It is clear that Eq.(1.1) includes the following equation

$$(|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + F[t, x(g(t))] = 0. \quad (1.2)$$

The oscillation of Eq.(1.2) was first studied by Agarwal, Grace, O'Regan [1], and also considered by several researchers [17,18]. On the other hand, the recently paper by Wang and Zhang [15] has presented some oscillation criteria for Eq.(1.1). For general interest of the oscillation of high order differential equation, see, for example, [1-5,8,10,12,13,15,17,18,19] and references therein.

As we known, the oscillation results provided in [1,15,17,18] require the integral of the coefficients of Eq.(1.2) on the entire interval $[t_0, \infty)$. But, from the Sturm Separation Theorem, oscillation is only an interval property. More precisely, if there exists a sequence of subset $[a_i, b_i]$ of $[t_0, \infty)$, $b_i < a_{i+1}$, $a_i \rightarrow \infty$ as $i \rightarrow \infty$, such that for each i there exists a nontrivial solution of equation which has at least two zeros in $[a_i, b_i]$, then every solution of the equation is oscillatory, no matter what the behavior of the coefficients of the equation is on the remaining parts of $[t_0, \infty)$. This idea was used by Kong [9] to establish interval oscillation criteria for second order linear differential equations. Recently, Tang and Yang [13] and Tiryaki, Basçi and Güleç [14] have presented several interval oscillation theorems for a class of even order nonlinear damped differential equations and second order functional differential equations, respectively.

In this paper, by using averaging function and the approach developed by Philos [11] and Kong [9], we establish Kamenev-type criteria as well as interval criteria for Eq.(1.1), and extend the results in [9,11] to Eq.(1.1), which improve and complement some existing results in [15]. To show the importance of our main results, two interesting examples are included.

2. Oscillation results

In this section, we shall establish Kamenev-type and interval oscillation criteria for Eq.(1.1). For notational simplicity, we define

$$\varphi(t) = \frac{2^{1-n}}{(n-2)!} r^{-1/p}(t) g'(t, \alpha) g^{n-2}(t, \alpha) \quad \text{and} \quad \mu = \frac{1}{(p+1)^{p+1}} \left(\frac{p}{k}\right)^p.$$

We say that a function $H = H(t, s)$ belongs to a function class \mathcal{H} , denoted by $H \in \mathcal{H}$, if $H \in C(D, [0, \infty))$, where $D = \{(t, s) : t_0 \leq s \leq t < \infty\}$, which satisfies $H(t, t) = 0$, $H(t, s) > 0$ for $t > s \geq t_0$, H has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t}(t, s) = h_1(t, s)H(t, s) \quad \text{and} \quad \frac{\partial H}{\partial s}(t, s) = -h_2(t, s)H(t, s),$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

For given functions $h \in C(D, \mathbb{R})$, $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ and $\eta \in C^1([t_0, \infty), \mathbb{R})$. Let

$$\lambda_1(s, t) = h_1(s, t) + \frac{\rho'(s)}{\rho(s)}, \quad \lambda_2(t, s) = h_2(t, s) - \frac{\rho'(s)}{\rho(s)},$$

and

$$\Theta_1(s, t) = \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) - \eta'(s) - \lambda_1(s, t)\eta(s),$$

$$\Theta_2(t, s) = \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) - \eta'(s) + \lambda_2(t, s)\eta(s).$$

The following two lemmas will be needed in proving of our main results. The first is the well-known Kiguradze’s Lemma [8]. The second can be obtained easily by Kiguradze and Koplatadze’s Lemmas, see [10, Chapt. 1].

LEMMA 2.1. ([8]) *Let $u \in C^n([t_0, \infty), \mathbb{R}_+)$. If $u^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[t^*, \infty)$, then there exist a $t_u \geq t_0$ and an integer j , $0 \leq j \leq n$, with $n + j$ even for $u^{(n)}(t) \geq 0$, or $n + j$ odd for $u^{(n)}(t) \leq 0$ such that*

$$j > 0 \quad \text{implies that} \quad u^{(k)}(t) > 0 \quad \text{for} \quad t \geq t_u, \quad k = 0, 1, \dots, j - 1,$$

and

$$j \leq n - 1 \quad \text{implies that} \quad (-1)^{j+k} u^{(k)}(t) > 0 \quad \text{for} \quad t \geq t_u, \quad k = j, j + 1, \dots, n - 1.$$

LEMMA 2.2. ([10]) *If the function $u(t)$ is as in Lemma 2.1 and $u^{(n-1)}(t)u^{(n)}(t) \leq 0$ for any $t \geq t_u$, then*

$$u(t/2) \geq \frac{2^{1-n}}{(n-1)!} t^{n-1} |u^{(n-1)}(t)| \quad \text{for all large } t.$$

Firstly, we give Kamenev-type criteria for Eq.(1.1).

THEOREM 2.1. *If there exist functions $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$, $\eta \in C^1([t_0, \infty), \mathbb{R})$, and $H \in \mathcal{H}$ such that for any $T \geq t_0$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t H(t, s) \rho(s) \left[\Theta_2(t, s) - \mu \frac{|\lambda_2(t, s)|^{p+1}}{\varphi^p(s)} \right] ds = \infty, \tag{2.1}$$

then Eq. (1.1) is oscillatory.

Proof. Suppose to contrary that Eq.(1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t, \xi)) > 0$ for $t \geq t_1 \geq t_0 \geq 0$, $\xi \in [\alpha, \beta]$. Since

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' \leq - \int_{\alpha}^{\beta} q(t, \xi) f[x(g(t, \xi))] d\sigma(\xi) \leq 0,$$

the function $r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign. If $x^{(n-1)}(t) < 0$ eventually, then there exists a constant $\delta > 0$ such that

$$-r(t)(-x^{(n-1)}(t))^p \leq -\delta^p < 0.$$

Integrating the above inequality from t_1 to t , we get

$$x^{(n-2)}(t) \leq x^{(n-2)}(t_1) - \delta \int_{t_1}^t \frac{ds}{r^{1/p}(s)}.$$

By (A3) we find that $x^{(n-2)}(t) < 0$ eventually. But then Lemma 2.1 (note that n is even) implies that $x(t) < 0$ eventually, which is a contradiction. So $x^{(n-1)}(t) > 0$ eventually, then again from Lemma 2.1 we have $x'(t) > 0$ eventually. Thus there exists a $t_2 \geq t_1$ such that

$$x'(t) > 0 \text{ and } x^{(n-1)}(t) > 0 \text{ for } t \geq t_2. \tag{2.2}$$

Observing that the function $r(t)(x^{(n-1)}(t))^p$ is decreasing for $t \geq t_2$, by (A3), there exists a $t_3 \geq t_2$ such that

$$(x^{(n-1)}(t))^p \leq \frac{r(t_3)}{r(t)} (x^{(n-1)}(t_3))^p \leq \frac{r(t_3)}{c} (x^{(n-1)}(t_3))^p \text{ for } t \geq t_3. \tag{2.3}$$

Note that (A1), (A2) and (2.2), so

$$f(x(g(t, \xi))) \geq f(x(g(t, \alpha))) > 0 \text{ for } t \geq t_3, \xi \in [\alpha, \beta]. \tag{2.4}$$

It follows from (1.1) and (2.4) that

$$(r(t)(x^{(n-1)}(t))^p)' = r'(t)(x^{(n-1)}(t))^p + pr(t)(x^{(n-1)}(t))^{p-1}x^{(n)}(t)$$

$$\begin{aligned} &\leq - \int_{\alpha}^{\beta} q(t, \xi) f(x(g(t, \xi))) d\sigma(\xi) \\ &\leq - f(x(g(t, \alpha))) \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi). \end{aligned} \tag{2.5}$$

Now, in view of (A3), let

$$\varepsilon = \frac{c}{2r(t_3)} \frac{f(x(g(t_3, \alpha)))}{(x^{(n-1)}(t_3))^p},$$

there exists a $t_4 \geq t_3$ such that, note that (A3), (2.3) and (2.5), for $t \geq t_4$,

$$\begin{aligned} p r(t) (x^{(n-1)}(t))^{p-1} x^{(n)}(t) &\leq |r'(t)| (x^{(n-1)}(t))^p - f(x(g(t, \alpha))) \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) \\ &\leq \left(\varepsilon \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) \right) \left(\frac{r(t_3)}{c} (x^{(n-1)}(t_3))^p \right) - f(x(g(t, \alpha))) \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) \\ &= -\frac{1}{2} f(x(g(t, \alpha))) \int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) \leq 0. \end{aligned}$$

Thus, we find $x^{(n)}(t) \leq 0$ for $t \geq t_4$. It is easy to check that we can apply Lemma 2.2 for $x' = u$, and conclude that there exists a $t_5 \geq t_4$ such that

$$x' \left(\frac{1}{2} g(t, \alpha) \right) \geq \frac{2^{2-n}}{(n-2)!} g^{n-2}(t, \alpha) x^{(n-1)}(t) \quad \text{for } t \geq t_5, \tag{2.6}$$

since

$$x^{(n-1)}(g(t, s)) \geq x^{(n-1)}(t) \quad \text{for } t \geq t_5.$$

Put

$$W(t) = \rho(t) \left[\frac{r(t) |x^{(n-1)}(t)|^{p-1} x^{(n-1)}(t)}{f(x(g(t, \alpha)/2))} + \eta(t) \right].$$

Noting that (1.1) and (2.6), we have

$$W'(t) \leq -\rho(t) \left[\int_{\alpha}^{\beta} q(t, \xi) d\sigma(\xi) - \eta'(t) \right] + \frac{\rho'(t)}{\rho(t)} W(t) - k\rho(t)\varphi(t) \left| \frac{W(t)}{\rho(t)} - \eta(t) \right|^{(p+1)/p}. \tag{2.7}$$

Replacing t by s , multiplying by $H(t, s)$, integrating from T to t , we obtain

$$\int_T^t H(t, s) \rho(s) \Theta_2(t, s) ds \leq H(t, T) W(T) + \int_T^t H(t, s) \rho(s) |\lambda_2(t, s)| \left| \frac{W(s)}{\rho(s)} - \eta(s) \right| ds$$

$$-k \int_T^t H(t,s)\rho(s)\varphi(s) \left| \frac{W(s)}{\rho(s)} - \eta(s) \right|^{(p+1)/p} ds. \tag{2.8}$$

The Young inequality [6, Theorem 61] gives

$$|\lambda_2(t,s)| \left| \frac{W(s)}{\rho(s)} - \eta(s) \right| \leq k\varphi(s) \left| \frac{W(s)}{\rho(s)} - \eta(s) \right|^{(p+1)/p} + \mu \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)}.$$

Substituting the above inequality into (2.8), we get

$$\int_T^t H(t,s)\rho(s) \left[\Theta_2(t,s) - \mu \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)} \right] ds \leq W(T)H(t,T). \tag{2.9}$$

Dividing by $H(t,T)$, and taking the upper limits as $t \rightarrow \infty$. The right hand side is always bounded, which contradicts condition (2.1). This completes the proof. \square

As an immediate consequence of Theorem 2.1, we get the following corollary.

COROLLARY 2.1. *Let condition (2.1) in Theorem 2.1 be replaced by*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s)\Theta_2(t,s) ds = \infty \tag{2.10}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\rho(s) \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)} ds < \infty, \tag{2.11}$$

then conclusion of Theorem 2.1 holds.

COROLLARY 2.2. *Let ρ and η be as in Theorem 2.1 and $\lim_{t \rightarrow \infty} G(t) = \infty$. If for some $\lambda > p$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{G^\lambda(t)} \int_{t_0}^t [G(t) - G(s)]^\lambda \rho(s)\Theta_2(t,s) ds = \infty, \tag{2.12}$$

and

$$\int_0^\infty \frac{|\rho'(s)|^{p+1}}{(\rho(s)\varphi(s))^p} ds < \infty, \tag{2.13}$$

where $G(t) = \int_{t_0}^t \varphi(s)\rho^{-1/p}(s)ds$, then Eq. (1.1) is oscillatory.

Proof. Let $H(t, s) = [G(t) - G(s)]^\lambda$, then

$$h_2(t, s) = \frac{\lambda \varphi(s)}{\rho^{1/p}(s)[G(t) - G(s)]}.$$

By the elementary inequality,

$$(X + Y)^{p+1} \leq 2^p(X^{p+1} + Y^{p+1}), \quad X, Y \geq 0,$$

we obtain

$$\begin{aligned} & \int_T^t H(t, s)\rho(s) \frac{|\lambda_2(t, s)|^{p+1}}{\varphi^p(s)} ds \\ & \leq 2^p \left[\int_T^t H(t, s)\rho(s) \frac{|h_2(t, s)|^{p+1}}{\varphi^p(s)} ds + \int_T^t H(t, s) \frac{|\rho'(s)|^{p+1}}{(\rho(s)\varphi(s))^p} ds \right]. \end{aligned} \tag{2.14}$$

Noting that

$$\int_T^t H(t, s)\rho(s) \frac{|h_2(t, s)|^{p+1}}{\varphi^p(s)} ds = \frac{\lambda^{p+1}}{\lambda - p} [G(t) - G(T)]^{\lambda-p},$$

and by (2.13) and [16, Lemma (14)],

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t H(t, s) \frac{|\rho'(s)|^{p+1}}{(\rho(s)\varphi(s))^p} ds = 0.$$

Hence, by (2.14), note that $\lim_{t \rightarrow \infty} G(t) = \infty$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t H(t, s)\rho(s) \frac{|\lambda_2(t, s)|^{p+1}}{\varphi^p(s)} ds = 0.$$

It following from Corollary 2.1 that Eq.(1.1) is oscillatory. \square

REMARK 2.1. Corollary 2.2 improves [15, Theorem 2.2]. \square

Next, we give interval oscillation criteria for Eq.(1.1).

THEOREM 2.2. *Let ρ, η and H be as in Theorem 2.1. If for each $T \geq t_0$, there exist constants a, b , and c such that $T \leq a < c < b$,*

$$\frac{1}{H(b, c)} \int_c^b H(b, s)\rho(s) \left[\Theta_2(b, s) - \mu \frac{|\lambda_2(b, s)|^{p+1}}{\varphi^p(s)} \right] ds$$

$$+ \frac{1}{H(c,a)} \int_a^c H(s,a) \rho(s) \left[\Theta_1(s,a) - \mu \frac{|\lambda_1(s,a)|^{p+1}}{\varphi^p(s)} \right] ds > 0, \quad (2.15)$$

then Eq. (1.1) is oscillatory.

Proof. Proceeding as the proof of Theorem 2.1. For each $T \geq t_5$, there exists an interval $[a, b]$ such that (2.7) hold for $t \in [a, b]$. Replacing t to s , multiplying by $H(t, s)$, and integrating from c to t ($t \leq b$), we have

$$\frac{1}{H(t,c)} \int_c^t H(t,s) \rho(s) \left[\Theta_2(t,s) - \mu \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)} \right] ds \leq W(c).$$

Let $t \rightarrow b^-$ in above inequality, then

$$\frac{1}{H(b,c)} \int_c^b H(b,s) \rho(s) \left[\Theta_2(b,s) - \mu \frac{|\lambda_2(b,s)|^{p+1}}{\varphi^p(s)} \right] ds \leq W(c). \quad (2.16)$$

On the other hand, replacing t by s in (2.7), multiplying $H(s, t)$ integrating from t ($t \geq a$) to c . Similar to the proof of (2.16), we obtain

$$\frac{1}{H(c,a)} \int_a^c H(s,a) \rho(s) \left[\Theta_1(s,a) - \mu \frac{|\lambda_1(s,a)|^{p+1}}{\varphi^p(s)} \right] ds \leq -W(c). \quad (2.17)$$

Now, (2.16) and (2.17) imply that desired contradiction, which completes the proof. \square

COROLLARY 2.3. Let ρ, η, H be as in Theorem 2.1. If for each $l \geq t_0$,

$$\limsup_{t \rightarrow \infty} \int_l^t H(s,l) \rho(s) \left[\Theta_1(s,l) - \mu \frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)} \right] ds > 0 \quad (2.18)$$

and

$$\limsup_{t \rightarrow \infty} \int_l^t H(t,s) \rho(s) \left[\Theta_2(t,s) - \mu \frac{|\lambda_2(t,s)|^{p+1}}{\varphi^p(s)} \right] ds > 0, \quad (2.19)$$

then Eq. (1.1) is oscillatory.

Proof. For each $T \geq t_0$. Let $l = a = T$ in (2.18). Clearly, we see from (2.18) that there exists a $c > a$ such that

$$\int_a^c H(s,a) \rho(s) \left[\Theta_1(s,a) - \mu \frac{|\lambda_1(s,a)|^{p+1}}{\varphi^p(s)} \right] ds > 0. \quad (2.20)$$

Similarly, setting $l = c = T$ in (2.19), it follows that there exists a $b > c$ such that

$$\int_c^b H(b,s)\rho(s) \left[\Theta_2(b,s) - \mu \frac{|\lambda_2(b,s)|^{p+1}}{\varphi^p(s)} \right] ds > 0. \tag{2.21}$$

So, (2.20) and (2.21) imply that (2.15) in Theorem 2.2 is true, which completes the proof. \square

COROLLARY 2.4. *If there exists a function $\rho \in \mathbf{C}^1([t_0, \infty), \mathbb{R}^+)$ such that for each $l \geq t_0$, and some $\lambda > p$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(s) - G(l)]^\lambda \rho(s) \left(\int_\alpha^\beta q(s, \xi) d\sigma(\xi) \right) ds > \frac{\mu \lambda^{p+1}}{\lambda - p} \tag{2.22}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(t) - G(s)]^\lambda \rho(s) \left(\int_\alpha^\beta q(s, \xi) d\sigma(\xi) \right) ds > \frac{\mu \lambda^{p+1}}{\lambda - p}, \tag{2.23}$$

where $G(t)$ is defined in Corollary 2.2 and $\lim_{t \rightarrow \infty} G(t) = \infty$, then Eq. (1.1) is oscillatory.

Proof. Let $H(t,s) = [G(t) - G(s)]^\lambda$ and $\eta(s) = 0$, we get

$$h_1(t,s) = \frac{\lambda \varphi(t)}{\rho^{1/p}(t)[G(t) - G(s)]} \quad \text{and} \quad h_2(t,s) = \frac{\lambda \varphi(s)}{\rho^{1/p}(s)[G(t) - G(s)]}.$$

Note that

$$\int_l^t H(s,l)\rho(s) \frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)} ds = \frac{\lambda^{p+1}}{\lambda - p} [G(t) - G(l)]^{\lambda-p}$$

and

$$\int_l^t H(t,s)\rho(s) \frac{|\lambda_2(s,l)|^{p+1}}{\varphi^p(s)} ds = \frac{\lambda^{p+1}}{\lambda - p} [G(t) - G(l)]^{\lambda-p}.$$

In view of $\lim_{t \rightarrow \infty} G(t) = \infty$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t H(s,l)\rho(s) \frac{|\lambda_1(s,l)|^{p+1}}{\varphi^p(s)} ds = \frac{\lambda^{p+1}}{\lambda - p} \tag{2.24}$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t H(t,s)\rho(s) \frac{|\lambda_2(s,l)|^{p+1}}{\varphi^p(s)} ds = \frac{\lambda^{p+1}}{\lambda - p}. \tag{2.25}$$

From (2.22) and (2.24), we get

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, l)} \int_l^t H(s, l) \rho(s) \left[\Theta_1(s, l) - \mu \frac{|\lambda_1(s, l)|^{p+1}}{\varphi^p(s)} \right] ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(s) - G(l)]^\lambda \rho(s) \left(\int_\alpha^\beta q(s, \xi) d\sigma(\xi) \right) ds - \frac{\mu \lambda^{p+1}}{\lambda - p} > 0, \end{aligned}$$

i.e., (2.18) holds. Similarly, (2.23) implies that (2.22) holds. By Corollary 2.3, Eq.(1.1) is oscillatory. \square

REMARK 2.2. From the above oscillation criteria, one can obtain different sufficient conditions for oscillation of Eq.(1.1) by different choices of $H(t, s)$. Following the well known Kamenev-type condition [7], let

$$H(t, s) = (t - s)^\lambda, \quad \lambda > p.$$

As the direct consequences of Theorems 2.1-2.2, we can establish oscillation criteria for Eq.(1.1). Here we omit the details. \square

To illustrate the main results obtained in this paper, we consider the following interesting examples.

EXAMPLE 2.1. Consider the delay differential equation

$$(r(t)|x^{(n-1)}(t)|^2 x^{(n-1)}(t))' + \int_{1/2}^1 q(t, \xi) x^3(t\xi) d\xi = 0, \quad n \text{ even}, \tag{2.26}$$

where $t \geq 1, c > 0$,

$$r(t) = t^{-4} + c, \quad g(t, \xi) = t\xi, \quad f(x) = x^3, \text{ and } q \in C([1, \infty) \times [\frac{1}{2}, 1], \mathbb{R}^+)$$

with

$$\int_{1/2}^1 q(t, \xi) d\xi \geq c_1 t^{\lambda-1}, \quad c_1 > 0, \quad \lambda \geq 4.$$

For Corollary 2.2, let

$$\rho(t) = t^{-\lambda}, \quad \eta(t) = 0, \quad H(t, s) = (t - s)^\lambda.$$

Then

$$\lambda_2(t, s) = \frac{\lambda t}{(t - s)s}, \quad \varphi(t) = \frac{2^{2(1-n)}}{(n - 2)!} (t^{-4} + c)^{-1/3} t^{n-2} \geq c_2 t^{n-2},$$

where

$$c_2 := \frac{2^{2(1-n)}}{(n - 2)!} \left(\frac{1}{2} + c\right)^{-1/3}.$$

It follows from [6, Theorem 41] that

$$(t - s)^\lambda \geq t^\lambda - \lambda s t^{\lambda-1} \quad \text{for } t \geq s \geq 1,$$

we can obtain, for all $T \geq 1$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t H(t, s) \rho(s) \Theta_2(t, s) ds \\ & \geq \lim_{t \rightarrow \infty} \frac{c_1}{(t - T)^\lambda} \int_T^t (t - s)^\lambda \frac{1}{s} ds \geq \lim_{t \rightarrow \infty} \frac{c_1}{(t - 1)^\lambda} \int_T^t \frac{t^\lambda - \lambda s t^{\lambda-1}}{s} ds = \infty, \end{aligned}$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t H(t, s) \rho(s) \frac{|\lambda_2(t, s)|^{p+1}}{\varphi^p(s)} ds \\ & \leq \frac{\lambda^4}{c_2^3} \limsup_{t \rightarrow \infty} \frac{t^4}{(t - T)^\lambda} \int_T^t (t - s)^{\lambda-4} s^{-\lambda-3n+2} ds \\ & \leq \frac{\lambda^4}{c_2^3} \limsup_{t \rightarrow \infty} \int_T^t s^{-\lambda-3n+2} ds < \infty. \end{aligned}$$

Thus, all conditions of Corollary 2.2 are satisfied, Eq.(2.26) is oscillatory. \square

EXAMPLE 2.2. Consider the even order differential equation

$$(r(t)|x^{(n-1)}(t)|^{p-1}x^{(n-1)}(t))' + \int_{1/2}^1 q(t, \xi)|x(t\xi)|^{p-1}x(t\xi)d\xi = 0, \quad n \text{ even}, \quad (2.27)$$

where $t \geq t_0 > 1$, $r(t)$ satisfies (A3), $g(t, \xi) = t\xi$, $q \in C([t_0, \infty) \times [\frac{1}{2}, 1], \mathbb{R}^+)$ with

$$\int_{1/2}^1 q(t, \xi)d\xi \geq \frac{\gamma \varphi(t)}{G^{p+1}(t)}, \quad \gamma > 0,$$

and $\varphi(t)G^{-(p+1)}(t)$ is decreasing for $t \geq t_0$, where $G(t)$ is defined as in Corollary 2.2, and $\lim_{t \rightarrow \infty} G(t) = \infty$.

Let $p_0 := \max\{1, p\}$, then we can verify that Eq.(2.27) is oscillatory for $\gamma \geq p_0^{p+1} \mu$. Indeed, let $\rho(t) = 1$ and $\eta(t) = 0$. Note that $\lambda > p \geq 1$ and from [6, Theorem 41], we have

$$[G(s) - G(l)]^\lambda \geq G^\lambda(s) - \lambda G(l)G^{\lambda-1}(s) \quad \text{for } s \geq l \geq t_0. \quad (2.28)$$

It follows from $G'(t) = \varphi(t)$ that for each $l \geq t_0$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(s) - G(l)]^\lambda \rho(s) \left(\int_{1/2}^1 q(s, \xi) d\xi \right) ds \\ & \geq \limsup_{t \rightarrow \infty} \frac{1}{G^{\lambda-p}(t)} \int_l^t [G(s) - G(l)]^\lambda \frac{\gamma}{G^{p+1}(s)} dG(s) = \frac{\gamma}{\lambda - p}. \end{aligned} \quad (2.29)$$

For any $\gamma > p_0^{p+1} \mu$, there exists $\lambda > p_0$ such that

$$\frac{\gamma}{\lambda - p} > \frac{\mu \lambda^{p+1}}{\lambda - p}.$$

This means (2.22) holds.

On the other hand, note that $\varphi(t)G^{-(p+1)}(t)$ is decreasing, by [9, Lemma 3.1], we have

$$\int_l^t [G(t) - G(s)]^\lambda \frac{\varphi(s)}{G^{p+1}(s)} ds \geq \int_l^t [G(s) - G(l)]^\lambda \frac{\varphi(s)}{G^{p+1}(s)} ds. \quad (2.30)$$

By (2.29) and (2.30), condition (2.23) holds for the same λ . Applying Corollary 2.4, we find that Eq.(2.27) is oscillatory if $\gamma > p_0^{p+1} \mu$. \square

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