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EXACT NULL CONTROLLABILITY OF ABSTRACT DIFFERENTIAL EQUATIONS BY FINITE–DIMENSIONAL CONTROL AND STRONGLY MINIMAL FAMILIES OF EXPONENTIALS

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Abstract. The exact controllability to the origin for linear evolution control equation is considered. The problem is investigated by its transformation to infinite linear moment problem of generalized exponentials. The existence of solutions of obtained moment problem is investigated for the case when exponentials of a moment problem do not constitute a Riesz basis. The exact controllability of linear control system of neutral type is considered as an example.

1. Introduction

The controllability problem is a classical problem in the control theory. Research in this field has been very intensive, so it is impossible to review all the literature devoted to the controllability. For example one can refer to the book [17] and the paper [15] for an introduction and a survey in the controllability of finite-dimensional systems. One can also see the papers of [6], [22] and the monograph [18] for an introduction to the controllability of PDE's. The different kinds of controllability (both approximate and exact controllability) for infinite dimensional systems with emphasis to retarded systems have been investigated in the monograph [13], and papers [14], [15].

Below we will consider the exact null controllability problem for abstract control differential equations. The large majority of authors investigates abstract control differential equations with bounded input operators. We will consider the situations with unbounded input operator. As a rule unbounded operators appear in boundary control problems.

2. Problem statement

Let X,U be complex Hilbert spaces, where the space U is a finite-dimensional with dimension $r \geqslant 1$, and let A be infinitesimal generator of strongly continuous C_0-semigroups $S(t)$ in X, see [11] and [16]. Consider the abstract control differential equation (see [11], [16]),

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0,\ 0 \leqslant t < +\infty, \tag{2.1}$$

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where $x(t)$, $x_0 \in X, u(t) \in U$, $B: U \to X$ is a linear possibly unbounded operator, $W \subset X \subset V$ are Hilbert spaces with continuous dense injections, where $W = D(A)$ equipped with graphic norm, $V = W^*$, the operator B is a bounded operator from U to V (see more details in [23],[28]).

It is well-known that (see [23], [28], etc.):

• for each $t \geqslant 0$ the operator $S(t)$ has an unique continuous extension $\mathscr{S}(t)$ on the space V and the family of operators $\mathscr{S}(t): V \to V$ is the semigroup in the class C_0 with respect to the norm of V and the corresponding infinitesimal generator $\mathscr{A}$ of the semigroup $\mathscr{S}(t)$ is the closed dense extension of the operator A on the space V with domain $D(\mathscr{A}) = X$;

• the sets of eigenvalues and of generalized eigenvectors of operators $\mathscr{A}, \mathscr{A}^*$ and A, A^* are the same;

• a mild solution $x(t, x_0, u(\cdot))$ of equation (2.1) with initial condition $x(0) = x_0$ is obtained by the following representation formula

$$x(t, x_0, u(\cdot)) = S(t)x_0 + \int_0^t \mathscr{S}(t-\tau)Bu(\tau)d\tau, \tag{2.2}$$

where the integral in (2.2) is understood in the Bochner's sense [11]. To assure

$$x(t, x_0, u(\cdot)) \in X, \ \forall x_0 \in X, u(\cdot) \in L_2^{r\mathrm{loc}}[0, +\infty), t \geqslant 0,$$

we assume that

$$\int_0^t \mathscr{S}(t-\tau)Bu(\tau)d\tau \in X$$

for any $u(\cdot) \in L_2^{\mathrm{loc}}[0, +\infty)$, $t \geqslant 0$, see [23] and [28].

DEFINITION 2.1. Equation (2.1) is said to be exact null-controllable on $[0, t_1]$ by controls vanishing after time moment t_2 if for each $x_0 \in X$ there exists a control $u(\cdot) \in L_2([0, t_2], U), u(t) = 0$ a.e. on $[t_2, +\infty)$ such that

$$x(t_1, x_0, u(\cdot)) = 0. \tag{2.3}$$

The goal of this paper is to establish necessary and sufficient conditions of exact null-controllability for linear evolution control equations with unbounded input operator. The main approach is the transformation of exact null-controllability problem (controllability to the origin) to linear infinite moment problem, which is defined as follows.

Given sequences $\{c_n, \ n \in \mathbb{N}\}$ and $\{x_n \in X, \ n \in \mathbb{N}\}$ find an element $g \in X$ such that

$$c_n = (x_n, g), \ n \in \mathbb{N}. \tag{2.4}$$

The problem formulated above is called the linear moment problem. It has a long history and many applications in geometry, physics, mechanics.

It is well-known, that if the sequence $\{x_n,\ n\in\mathbb{N}\}$ forms a Riesz basis in the closure of its linear span, the linear moment problem (2.4) has a solution if and only if $\sum_{n=1}^{\infty}|c_n|^2<\infty$ and vice-versa [3], [9], [25], [29]. This well-known fact is one of main tools for the controllability analysis of various partial hyperbolic control equations.

However the sequence $\{x_n,\ n\in\mathbb{N}\}$ doesn't need to be a Riesz basis for the solvability of linear moment problem. This case appears under the investigation of the controllability of parabolic control equations or hereditary functional differential control systems. In this paper we consider the null-controllability of control evolution equations for the case when the sequence $\{x_n,\ n\in\mathbb{N}\}$ of the moment problem obtained by the transformation of the source control problem doesn't form a Riesz basis in its closed linear span.

The assumptions on A are listed below.

1. The operators A has purely point spectrum σ with no finite limit points. All the eigenvalues of A have finite multiplicities, bounded from above.

2. There exists $T\geqslant 0$ such that all mild solutions of the equation $\dot{x}(t)=Ax(t)$ are expanded in a series of generalized eigenvectors of the operator A converging uniformly for any $t\in[T_1,T_2], T<T_1<T_2$.

3. Main results

Without lost of generality one can consider $U=\mathbb{R}^r$. In this case the operator $B:U\to X$ is defined by

$$Bu=\sum_{i=1}^{r}b_iu_i,\ \ \forall u=(u_1,...,u_r)\in\mathbb{R}^r,$$

where $b_i\in V,\ i=1,2,...,r$. The operator B is a bounded operator if and only if $b_i\in X,\ i=1,2,...,r$.

Let the eigenvalues $\lambda_j\in\sigma,\ j\in\mathbb{N}$, be enumerated in the order of non-decreasing absolute values, let α_j and q_j be the algebraic and geometric multiplicities[1] of $\lambda_j\in\sigma$ correspondingly, and let

$$\begin{cases}\varphi_{jk}^m,\ j\in\mathbb{N},\ k=1,2,....,\beta_j^m,\ \beta_j^m\leqslant\alpha_j,\\ A\varphi_{j\beta_j^m}^m=\lambda_j\varphi_{j\beta_j^m}^m,\ m=1,2,...,q_j,\end{cases}$$

be the generalized eigenvectors of the operator A, and let

$$\begin{cases}\psi_{jk}^m,\ j\in\mathbb{N},\ k=1,2,....,\beta_j^m,\\ A^*\psi_{j\beta_j^m}^m=\bar{\lambda}_j\psi_{j\beta_j}^m,\end{cases}$$

[1]The geometric multiplicity q_j is the number of Jordan blocks corresponding to $\lambda_j\in\sigma$, and β_j^m, is the dimension of m-th Jordan block, $m=1,2,...,q_j$.

be the generalized eigenvectors of the adjoint operator A^*, chosen such that

$$(\varphi^q_{p\beta_p-l+1,}\psi^m_{jk}) = \delta_{pj}\delta_{lk}\delta_{mq}, \tag{3.1}$$
$$p,j\in\mathbb{N}, q=1,2,...,\gamma_p, l=1,\dots,\beta^q_p,,m=1,2,...,q_j,k=1,\dots,\beta^m_j.$$

We use the following notations:

$$x^m_{jk}(t) = \Big(x\big(t,x_0,u(\cdot)\big),\psi^m_{jk}\Big),\ x_{0jk} = \big(x_0,\psi^m_{jk}\big), \tag{3.2}$$
$$b^m_{jk} = B^*\psi^m_{jk} = \Big(\big(b_1,\psi^m_{jk}\big),...,\big(b_r,\psi^m_{jk}\big)\Big),$$
$$j\in\mathbb{N},\ m=1,2,...,q_j,\ k=1,2,\dots\beta^m_j,$$
$$g^m_{jk}(t) = \exp(\lambda_j t)\sum_{l=0}^{\beta^m_j-k} b^m_{jk+l}\frac{t^l}{l!}, \tag{3.3}$$
$$j\in\mathbb{N},\ k=1,\dots,\beta^m_j,\ m=1,2,...,q_j,\ t\in[0,t_1-T].$$

All scalar products in (3.2) are correctly defined, because $\psi^m_{jk}\in W,\ j\in\mathbb{N},\ m= 1,2,...,q_j,\ k=1,\dots,\beta^m_j,\ b_1,...,b_r\in V=W^*$.

THEOREM 3.1. *For equation (*2.1*) to be exact null-controllable on $[0,t_1],\ t_1>T$, by controls vanishing after time moment t_1-T, it is necessary and sufficient that the following infinite moment problem*

$$x^m_{0jk} = -\int_0^{t_1-T} g^m_{jk}(-t)u(\tau)d\tau,\ j\in\mathbb{N},\ k=1,\dots,\ \beta^m_j,\ m=1,2,...,q_j. \tag{3.4}$$

with respect to $u(\cdot)\in L^r_2[0,t_1-T]$ be solvable for any $x_0\in X$.

Proof. Necessity. If equation (2.1) is exact null-controllable on $[0,t_1],\ t_1>T$, then by Definition 2.1 for each $x_0\in X$ there exists $u(t)\in L^r_2[0,t_2]$ such that

$$S(t_1)x_0 + \int_0^{t_1-T}\mathscr{S}(t_1-\tau)Bu(\tau)d\tau = 0. \tag{3.5}$$

Multiplying both parts of (3.5) to $\psi^m_{jk},\ j\in\mathbb{N},\ k=1,2,....,\beta^m_j,\ m=1,2,...,q_j$, we obtain (3.4).

Sufficiency. Let $x^m_{jk}(t) = \big(x(t),\psi^m_{jk}\big),\ j\in\mathbb{N},\ k=1,2,....,\beta^m_j,\ m=1,2,...,q_j$, where $x(t)$ is a weak solution of equation (2.1). It is easily to show that the sequence

$$\big\{x^m_{jk}(t),\ j\in\mathbb{N},\ m=1,2,...,q_j,\ k=1,2,\dots,\beta^m_j\big\}$$

is the solution of the infinite system

$$\begin{cases}\dot x^m_{jk}(t) = \lambda_j x^m_{jk}(t) + x^m_{jk+1}(t) + b^m_{jk}u(t), & k=1,2,\dots,\beta^m_j-1,\\ \dot x^m_{j\beta_j}(t) = \lambda_j x^m_{j\beta_j}(t) + b^m_{j\beta_j}u(t), & k=\beta^m_j,\end{cases} \tag{3.6}$$

for $j \in \mathbb{N}$, $m = 1,2,...,q_j$, with initial conditions

$$x_{jk}^m(0) = (x_0, \psi_{jk}),\ j \in \mathbb{N},\ k = 1,2,\ldots,\beta_j^m,\ m = 1,2,...,q_j.$$

Let $u^0(t), t \in [0, t_1 - T], u^0(.) \in L_2^r[0, t_1 - T]$ be a solution of moment problem (3.4). If $u(t) = u^0(t)$ a.e. on $[0, t_2]$ then by (3.4) and (3.6) we have

$$x_{jk}^m(t_1 - T) = 0,\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,\ldots,\beta_j^m.$$

Putting $u(t) \equiv 0, t \geqslant t_1 - T$, we obtain by (3.6) that

$$x_{jk}^m(t) \equiv 0,\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,\ldots,\ \beta_j^m,\ \forall t \geqslant t_1,$$

and $x(t)$ is the solution of the equation

$$\dot{x}(t) = Ax(t)$$

for each $t \geqslant t_1$. By Assumption 2 of the operator A (see the top of the 173), we obtain that

$$x(t) \equiv 0, \forall t > t_1.$$

This proves the theorem.

3.1. Solution of moment problem (3.4)

It is well-known, see for instance [9], [25], and [29], that if the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Riesz basis in X, then the linear moment problem

$$c_j = (x_j, g),\ j \in \mathbb{N},$$

has a solution $g \in X$ if and only if

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty.$$

Thus if the sequence of functions

$$\left\{ g_{jk}^m(-t) = \exp(-\lambda_j t) \sum_{l=0}^{\beta_j^m - k} b_{jk+l}^m \frac{(-t)^l}{l!} \right\}, \tag{3.7}$$
$$j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,\ldots,\ \beta_j^m,\ t \in 0, t_1 - T]$$

forms a Riesz basis in $L_2[0, t_1 - T]$, then the moment problem

$$(x_0, \psi_{jk}^m) = -\int_0^{t_1 - T} g_{jk}^m(-t) u(\tau) d\tau,\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,\ldots,\ \beta_j^m \tag{3.8}$$

is solvable for every $x_0 \in X$ if and only if

$$\sum_{j=1}^{\infty} \sum_{m=1}^{\gamma_j} \sum_{k=1}^{\beta_j^m} \left| (x_0, \psi_{jk}^m) \right|^2 < \infty,\ \forall x_0 \in X. \tag{3.9}$$

The above statements are one of main tools for the investigation of the zero controllability of hyperbolic partial control equations.

However moment problem (3.8) may also be solvable when the sequence (3.7) doesn't form a Riesz basis in $L_2[0,t_1-T]$. Below we will try to find more extended controllability conditions which are applicable for the case when the sequence (3.7) doesn't form a Riesz basis in $L_2[0,t_1-T]$.

DEFINITION 3.2. The sequence $\{x_j \in X,\ j \in \mathbb{N},\}$ is said to be minimal, if there no element of the sequence belonging to the closure of the linear span of others. By other words,

$$x_j \notin \overline{\text{span}}\{x_k \in X,\ k=1,2,...,k \neq j\}.$$

Let $\{x_j \in X,\ j \in \mathbb{N},\}$ be a sequence of elements of X, and let

$$G_n = \{(x_i,x_j),\ i,j=1,2,...,n\}$$

be the Gram matrix of n first elements $\{x_1,...,x_n\}$ of above sequence. Denote by $\gamma_n^{\min}$ the minimal eigenvalue of the $n\times n$-matrix G_n. Each minimal sequence $\{x_j \in X,\ j \in \mathbb{N}\}$ is linear independent, hence any first n elements $\{x_1,...,x_n\},\ n \in \mathbb{N}$, of this sequence are linear independent, so $\gamma_n^{\min} > 0,\ \forall n \in \mathbb{N}$. It is easily to show that the sequence $\{\gamma_n^{\min},\ n \in \mathbb{N}\}$ decreases , so there exists $\lim\limits_{n\to\infty} \gamma_n^{\min} \geqslant 0$.

DEFINITION 3.3. The sequence $\{x_j \in X,\ j \in \mathbb{N}\}$ is said to be strongly minimal, if

$$\gamma^{\min} = \lim_{n\to\infty} \gamma_n^{\min} > 0.$$

It is well-known that for Hermitian $n\times n$-matrix

$$G_n = \left\{(x_j,x_k),\ j,k=1,2,...,n\right\}$$

the inequalities

$$\gamma_n^{\min} \sum_{k=1}^{n} |c_k|^2 \leqslant \sum_{j=1}^{n}\sum_{k=1}^{n} c_j(x_j,x_k)\overline{c_k} = \Big\| \sum_{k=1}^{m} c_k x_k \Big\|^2,\ n \in \mathbb{N}, \tag{3.10}$$

hold.

In the sequel the investigation of the controllability problem defined above is based on the following result of Boas [5] (see also [3] and [30]).

THEOREM *Let* $x_j \in X,\ j \in \mathbb{N}$. *The linear moment problem*

$$c_j = (x_j,g),\ j \in \mathbb{N},$$

has a solution $g \in X$ *for each square summable sequence* $\{c_j,\ j \in \mathbb{N}\}$ *if and only if there exists a positive constant* γ *such that all the inequalities*

$$\gamma \sum_{k=1}^{n} |c_k|^2 \leqslant \Big\| \sum_{j=1}^{n} c_j x_j \Big\|^2,\ n \in \mathbb{N}, \tag{3.11}$$

are valid.

From (3.10) and the inequality $\gamma_n^{\min} \geqslant \gamma^{\min} > 0$ it follows that

$$\gamma^{\min} \sum_{k=1}^{n} \left|c_k\right|^2 \leqslant \left\| \sum_{k=1}^{n} c_k x_k \right\|^2. \tag{3.12}$$

Hence the above theorem can be reformulated as follows.

THEOREM 3.4. *The linear moment problem*

$$c_j = (x_j, g), \ j \in \mathbb{N}, \tag{3.13}$$

has a solution $g \in X$ *for any sequence* $\{c_n, \ n \in \mathbb{N}\}$, $\sum_{j=1}^{\infty} |c_k|^2 < \infty$ *if and only if the sequence* $\left\{x_n\right\}_{n \in \mathbb{N}}$ *is strongly minimal.*

4. Solution of the exact null-controllability problem

THEOREM 4.1. *For equation* (2.1) *to be exact null-controllable on* $[0,t_1]$, $t_1 > T$, *by controls vanishing after time moment* $t_1 - T$, *it is necessary, that the sequence* (3.7) *is minimal, and sufficient, that:*

- *the sequence* (3.7) *is strongly minimal,*
- $$\sum_{j=1}^{\infty} \sum_{m=1}^{\gamma_j} \sum_{j=1}^{\beta_j^m} \left| \left(x_0, \psi_{jk}^m\right) \right|^2 < +\infty, \forall x_0 \in X. \tag{4.1}$$

Proof. Necessity. If the problem (3.4) has a solution for any $x_0 \in X$, then it has a solution for any generalized eigenvector φ_{jk}, $j \in \mathbb{N}$, $k = 1,2,...\beta_j^m$, $m = 1,2,...,q_j$, of the operator A, so for each $j \in \mathbb{N}$, $m = 1,2,...,\gamma_j$, $k = 1,2,...\beta_j^m$, there exists a function $u_{jk}(\cdot) \in L_2[0,t_1 - T]$, $j \in \mathbb{N}$, $k = 1,2,...\beta_j$, such that

$$(\varphi_{p\beta_p - l + 1,}^q \psi_{jk}^m) = -\int_{0j}^{t_1 - T} g_{jk}^m(-t) u_{p\beta_p - l + 1}(\tau) d\tau, \tag{4.2}$$
$$p, j \in \mathbb{N}, \ r = 1,2,..,q_p, \ m = 1,2,...,q_j,$$
$$l = 1, \ldots, \beta_p^q, \ k = 1, \ldots, \ \beta_j^m.$$

The sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ of eigenvectors of the operator A is biorthogonal to the sequence $\{\psi_k\}_{k \in \mathbb{N}}$ of eigenvectors of the operator A^*. Hence it follows from (4.2) and (3.1) that

$$\delta_{pj} \delta_{lk} \delta_{qm} = -\int_{0}^{t_1 - T} g_{jk}^m(-t) u_{p\beta_p - l + 1}^q(\tau) d\tau,$$

for $p, j \in \mathbb{N},\ q = 1,2,..,q_p,\ m = 1,2,...,q_j,\ l = 1,\dots,\beta_p^q,\ k = 1,\dots,\beta_j^m$, i.e. the sequence

$$\left\{ -u^q_{p\beta_p - l+1}(\tau),\ t \in [0, t_1 - T],\ p = 1,2,\dots,\ r = 1,2,..,q_p,\ l = 1,\dots,\beta_p^q \right\}$$

is biorthogonal to the sequence

$$\left\{ \int_0^{t_1 - T} g^m_{jk}(-t),\ t \in [0, t_1 - T],\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,\dots,\beta_j^m \right\}.$$

It proves the necessity.

Sufficiency. The sufficiency follows immediately from (4.1) and Theorem 3.4.
It proves the theorem.

4.1. The case when the generalized eigenvectors of the operator A constitute a Riesz basis

One of the important problems of the operator theory is the case when the generalized eigenvectors of the operator A being considered form a Riesz basis in X. The problem of expansion into a Riesz basis of generalized eigenvectors of the operator A is widely investigated in the literature (see, for example, [1], [8], [9], [20] and references therein). Obviously the sequence of these vectors is strongly minimal. In this case one can set $T = 0$, so the Theorems 4.1 can be proven with $T = 0$.

THEOREM 4.2. *Let the sequence of operator A forms a Riesz basis in X. Equation* (2.1) *is exact null-controllable on* $[0, t_1]$, $t_1 > T$, *by controls vanishing after time moment* $t_1 - T$, *if and only if the sequence* (3.7) *is strongly minimal.*

Proof. Let $\left\{ c^m_{jk},\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,...,\beta_j^m \right\}$ be any complex sequence satisfying the condition

$$\sum_{j=1}^{\infty} \sum_{m\gamma=1}^{\gamma_j} \sum_{k=1}^{\beta_j^m} \left| c^m_{jk} \right|^2 < \infty. \tag{4.3}$$

The sequence $\left\{ \varphi^m_{jk},\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,...,\beta_j^m \right\}$ of eigenvectors of the operator A forms a Riesz basis. Therefore, (see [3]), there exists a vector $x_0 \in X$ such that

$$c^m_{jk} = \left(x_0, \psi^m_{jk} \right),\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,...,\beta_j^m, \tag{4.4}$$

where $\left\{ \psi^m_{jk},\ j \in \mathbb{N},\ k = 1,2,...,\beta_j^m \right\}$ are generalized eigenvectors of the operator A^*, so in virtue of Theorem 3.1 the exact null controllability of equation (2.1) is equivalent to the existence of a solution for the linear moment problem

$$c^m_{jk} = \int_0^{t_1 - T} g^m_{jk}(-t) u(\tau) d\tau,\ j \in \mathbb{N}, \tag{4.5}$$

for any complex sequence $\left\{ c^m_{jk},\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,...,\beta_j^m \right\}$ satisfying the condition (4.3).

By above mentioned results of [5] the linear moment problem (4.5) has a solution for any complex sequence $\left\{c_{jk}^{m},\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,...,\beta_j^m\right\}$ satisfying the condition (4.3), if and only if the sequence

$$\begin{gathered}\left\{g_{jk}^{m}(-t),\ t \in [0,t_1 - T]\right\} \\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ k = 1,2,...,\beta_j^m\end{gathered}$$

is strongly minimal. It proves the theorem.

REMARK 4.3. It is clear that if sequence (3.3) is minimal, then it is linear independent. Hence its any subsequence is also linear independent, therefore the linear independence of the sequence

$$\begin{gathered}\left\{f_{j\beta_j^m}^{m}(-t) = b_{j\beta_j^m}^{m} \exp(-\lambda_j t)\right\}, \\ j \in \mathbb{N},\ m = 1,2,...,q_j,\ t \in [0,t_1 - T]\end{gathered} \tag{4.6}$$

is necessary for sequence (3.3) to be linear independent. We have

$$b_{j\beta_j^m}^{m} = \left\{b_{j\beta_j^m 1}^{m}, b_{j\beta_j^m 2}^{m}, ..., b_{j\beta_j^m r}^{m}\right\} \in \mathbb{R}^r,\ j \in \mathbb{N},\ m = 1,...,q_j. \tag{4.7}$$

If there exists a number j such that $r < q_j$, than the number q_j of r-dimensional functions (4.6) will be linear dependent, so moment problem (3.4) will not be solvable for any $x_0 \in X$. Therefore if there exists a moment $t_1 > T$ such that equation (2.1) is exact null-controllable on $[0,t_1]$ by controls vanishing after $t_1 - T$, then $q_j \leqslant r,\ j \in \mathbb{N},$. By other words, equation (2.1) cannot be exact null-controllable on $[0,t_1]$ by controls vanishing after $t_1 - T$, if r (the number of inputs) is less then $\max\limits_{j\in\mathbb{N}} q_j$ (maximal geometrical multiplicity of eigenvalues of the operator A). If $r \geqslant \max\limits_{j\in\mathbb{N}} q_j$, then in accordance with Theorem 4.2 equation (2.1) is exact null-controllable on $[0,t_1]$ by controls vanishing after $t_1 - T$, if and only if the sequence (3.5) is strongly minimal. $\square$

The theory of families of generalized exponents have been developed in [2] (see also [24], [25], [29], and references therein).

In virtue of Theorem 4.2 we have a special interest in the strong minimality property of such families.

4.2. Choice of an operator B for evolution control equations with single control

Let all the geometrical multiplicities $q_j,\ j \in \mathbb{N}$ be equal to 1. In accordance with Remark 4.3 the exact null-controllable equation (2.1) on $[0,t_1]$ by scalar controls ($r = 1$) vanishing after $t_1 - T$, can be considered, and the operator $B : U \to X$ is defined by

$$Bu = bu,\ \forall u \in \mathbb{R},$$

where $b \in V$. The operator b is a bounded operator if and only if $b \in X$. In this case $\beta_j^m = \alpha_j \leqslant \alpha,\ j \in \mathbb{N}$ so the generalized eigenvectors of the operators A and A^* can be written without index m, $b_{jk}^m = b_{jk} = (b, \psi_{jk}),\ k = 1,2,...,\alpha_j,\ j \in \mathbb{N}$, where ψ_{jk}, $j \in \mathbb{N},\ k = 1,2,...,\alpha_j$ are generalized eigenvectors of the adjoint operator A^*.

The problem under consideration is:

for a given operator A, let choose a vector $b \in V$ (i.e. an operator B) such that control evolution equation (2.1) be exact null controllable in accordance with Definition 2.1 (see Page 172).

Denote

$$g = \begin{cases} g_{jk}(-t) = \sum_{l=0}^{\alpha_j - k} b_{jk+l} \exp(-\lambda_j t) \frac{(-t)^l}{l!}, \\ t \in [0, t_1 - T],\ j \in \mathbb{N},\ k = 1,2,...,\alpha_j, \end{cases} \tag{4.8}$$

$$f = \begin{cases} f_{jk}(-t) = \frac{(-t)^{k-1}}{(k-1)!} e^{-\lambda_j t}, \\ t \in [0, t_1 - T],\ j \in \mathbb{N},\ k = 1,2,...,\alpha_j. \end{cases} \tag{4.9}$$

In accordance with Theorem 4.2 equation (2.1) is exact null-controllable on $[0, t_1]$ by scalar controls vanishing after $t_1 - T$,if and only if the sequence (4.8) is strongly minimal.

As it was shown above if sequence (4.8) is strongly minimal, then it is surely linear independent. The necessary and sufficient condition sequence (4.8) to be linear independent is the condition

$$b_{j\alpha_j} \neq 0,\ j \in \mathbb{N}.$$

Hence this condition is necessary for the strong minimality of sequence (4.8).

If $b \in X$ and the vectors $\psi_{jk},\ j \in \mathbb{N},\ k = 1,2,...,\alpha_j$ form a Riesz basis in their closed linear span, then

$$\sum_{j=1}^{\infty} \left| b_{j\alpha_j} \right|^2 \leqslant \sum_{j=1}^{\infty} \sum_{k=1}^{\alpha_j} \left| b_{jk} \right|^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\alpha_j} \left| (b, \psi_{jk}) \right|^2 < +\infty,$$

so $\lim\limits_{j \to \infty} |b_{j\alpha_j}| = \inf\limits_{j \in \mathbb{N}} |b_{j\alpha_j}| = 0$. But if $b \notin X$, than the case $\inf\limits_{j \in \mathbb{N}} |b_{j\alpha_j}| > 0$ can be realized.

Denote

$$B_j = \begin{pmatrix} b_{j1} & b_{j2} & \dots & b_{j\alpha_j} \\ b_{j2} & b_{j3} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{j\alpha_j} & 0 & \dots & 0 \end{pmatrix},\ c_j = \begin{pmatrix} c_{j1} \\ c_{j2} \\ \dots \\ c_{j\alpha_j} \end{pmatrix},\ d_j = B_j c_j,\ j \in \mathbb{N}.$$

From the definition of singular values of matrices [19] it follows that the product of all singular values of the matrix B_j is equal to $\left| b_{j\alpha_j} \right|^{\alpha_j}$, $j \in \mathbb{N}$. Using this property of singular values one can easily prove that if $b_{j\alpha_j} \neq 0, j \in \mathbb{N}$, then for any $j \in \mathbb{N}$ the minimal singular value $\sigma_{\min}(B_j)$ of the symmetric matrix B_j is positive, so

$$\inf_{j \in \mathbb{N}} \left\{ \sigma_{\min}(B_j) \right\} \geqslant 0 \text{ and } \sigma_{\min}\left(B_j\right) \leqslant |b_{j\alpha_j}|^2,\ j \in \mathbb{N}.$$

Hence if $b \in X$, than

$$\inf_{j \in \mathbb{N}} \left\{ \sigma_{\min}(B_j) \right\} \leqslant \inf_{j \in \mathbb{N}} |b_{j\alpha_j}|^2 = 0.$$

However if $b \notin X$, than the case $\inf_{j\in\mathbb{N}} \{\sigma_{\min}(B_j)\} > 0$ can be considered. In this case the strong minimality of sequence (4.8) can be delivered from the strong minimality of sequence (4.9) of generalized exponents.

The next theorem can be useful for the case when the operator A, describing the inner structure of evolution control system is given, but the input device described by the operator B, can be chosen.

THEOREM 4.4. *If*

$$\inf_{j\in\mathbb{N}} \{\sigma_{\min}(B_j)\} = \beta > 0, \tag{4.10}$$

and the sequence (4.9) *of generalized exponents is strongly minimal, then the sequence* (4.8) *is also strongly minimal.*

Proof. From (4.8) it follows that

$$\begin{aligned}\Big\|\sum_{j=1}^{n}\sum_{k=1}^{\alpha_j} c_{jk} g_{jk}(-t)\Big\| &= \Big\|\sum_{j=1}^{n}\sum_{k=1}^{\alpha_j} c_{jk}\sum_{l=0}^{\alpha_j-k} b_{jk+l} f_{jl+1}(-t)\Big\|^2 \\ &= \Big\|\sum_{j=1}^{n}\sum_{l=0}^{\alpha_j-1}\Big(\sum_{k=1}^{\alpha_j-l} c_{jk} b_{jk+l}\Big) f_{jl+1}(-t)\Big\|,\ n\in\mathbb{N}. \end{aligned}\tag{4.11}$$

Since sequence (4.9) is strongly minimal, from (4.11) it follows that there exists a positive number γ such that

$$\gamma\sum_{j=0}^{n}\sum_{l=0}^{\alpha_j-1}\Big|\sum_{k=1}^{\alpha_j-l} c_{jk} b_{jk+l}\Big|^2 \leqslant \Big\|\sum_{j=1}^{n}\sum_{k=1}^{\alpha_j} c_{jk} g_{jk}(-t)\Big\|^2,\ n\in\mathbb{N}, \tag{4.12}$$

where

$$\sum_{l=0}^{\alpha_j-1}\Big|\sum_{k=1}^{\alpha_j-l} c_{jk} b_{jk+l}\Big|^2 = \left\|\begin{pmatrix} b_{j1} & b_{j2} & \dots & b_{j\alpha_j}\\ b_{j2} & b_{j3} & \dots & 0\\ \dots & \dots & \dots & \dots\\ b_{j\alpha_j} & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} c_{j1}\\ c_{j2}\\ \dots\\ c_{j\alpha_j}\end{pmatrix}\right\|^2. \tag{4.13}$$

By above notations of B_j and $c_j, j\in\mathbb{N}$ formula (4.13) can be rewritten as

$$\sum_{l=0}^{\alpha_j-1}\sum_{k=1}^{\alpha_j-l}\left|c_{jk} b_{jk+l}\right|^2 = \left\|B_j c_j\right\|^2,\ j\in\mathbb{N}. \tag{4.14}$$

If we will prove that there exists a positive number $\delta > 0$, such that

$$\delta\sum_{k=1}^{\alpha_j}\left|c_{jk}\right|^2 \leqslant \sum_{l=0}^{\alpha_j-1}\Big|\sum_{k=1}^{\alpha_j-l} c_{jk} b_{jk+l}\Big|^2 \tag{4.15}$$

we obtain by (4.12)-(4.14) that

$$\gamma\delta\sum_{j=0}^{n}\sum_{k=1}^{\alpha_j}|c_{jk}|^2 \leqslant \Big\|\sum_{j=0}^{n}\sum_{k=1}^{\alpha_j} c_{jk} g_{jk}(-t)\Big\|^2, \gamma\delta > 0,\ n\in\mathbb{N}, \tag{4.16}$$

and the proof will be finished. □

It is well-known (see [19]) that

$$\left\|B_jc_j\right\|^2 = \left(c_j^T B_j^T B_j c_j\right) \geqslant \sigma_{\min}^2\left(B_j\right)\|c_j\|^2.$$

Hence it follows from (4.10) that

$$\|B_jc_j\|^2 \geqslant \beta^2\|c_j\|^2,\ j\in\mathbb{N}.$$

Formula (4.14) shows that the last inequality is exactly inequality (4.15) with $\delta = \beta^2$, so (4.16) holds. It proves the theorem. □

Theorem 4.4 establishes the sufficient conditions for exact null controllability provided that the family of generalized exponentials generated by eigenvalues of the operator A is strongly minimal.

The theory of families of generalized exponents have been developed in [2] (see also [24], [25], [29], and references therein). In virtue of Theorem 4.2, we have a special interest in the strong minimality property of such families.

4.2.1. Example of strongly minimal sequence

Using conditions of minimality for families of exponentials and Theorem 1.5 of [7], one can show that the sequence

$$\left\{e^{n^2\pi^2 t},\ n\in\mathbb{N},\ t\in[0,t_1]\right\} \tag{4.17}$$

is strongly minimal for any $t_1>0$. It allows us to establish the exact null controllability for partial heat equations governed by boundary controls.

CLAIM *The sequence* (4.17) *does not satisfy the conditions of* [2] *to constitute a Riesz basis.*

Proof. Let $t_1 = 2t_2$. The series $\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}$ converges and $(n+1)^2-n^2\geqslant 1$, so the sequence

$$\left\{e^{n^2\pi^2 t},\ n\in\mathbb{N},\ t\in[0,t_2]\right\}$$

is minimal (see [7]). In virtue of Theorem 1.5 of [7] for each $\varepsilon>0$, there exists a positive constant K_ε such that the biorthogonal sequence $\left\{w_n(t),\ n\in\mathbb{N},\ t\in[0,t_2]\right\}$ satisfies the condition

$$\left\|w_n(\cdot)\right\| < K_\varepsilon e^{\varepsilon n^2\pi^2},\ n\in\mathbb{N}. \tag{4.18}$$

The positive constant ε can be chosen such that $t_2-\varepsilon>0$.

By the Minkowsky inequality and (4.18) one can show that:

$$\sum_{n=1}^{p}\sum_{m=1}^{p} c_n e^{-n^2\pi^2 t_2}\Big(\int_0^{t_2} w_n(t)w_m(t)dt\Big)e^{-m^2\pi^2 t_2}c_m$$
$$= \int_0^{t_2}\left(\sum_{n=1}^{p} c_n e^{-n^2\pi^2 t_2} w_n(t)dt\right)^2 dt$$

$$\begin{aligned}
&\leqslant \int_0^{t_2} \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} \left| e^{-n^2\pi^2 t_2} w_n(t) \right|^2 dt \\
&= \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} e^{-2n^2\pi^2 t_2} \int_0^{t_2} |w_n(t)|^2 dt \\
&\leqslant \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} e^{-2n^2\pi^2 t_2} \|w_n(\cdot)\|^2 \\
&\leqslant K_\varepsilon^2 \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} e^{-2n^2\pi^2 (t_2-\varepsilon)}.
\end{aligned}$$

The series $\sum_{n=1}^{p} e^{-2n^2\pi^2(t_2-\varepsilon)}$ converges for any t_2, ε, $t_2 > \varepsilon$, so

$$\sum_{n=1}^{p} e^{-2n^2\pi^2(t_2-\varepsilon)} \leqslant M,$$

where M is a positive constant. Hence

$$\sum_{n=1}^{p} \sum_{m=1}^{p} c_n e^{-n^2\pi^2 t_2} \Big(\int_0^{t_2} w_n(t) w_m(t) dt \Big) e^{-m^2\pi^2 t_2} c_m \leqslant K_\varepsilon^2 M \sum_{n=1}^{p} |c_n|^2 \tag{4.19}$$

for every finite sequence $\{c_1, c_2, \ldots, c_p\}$. Obviously the sequence $\{h_n(t)\}_{n\in\mathbb{N}}$, where

$$h_n(t) = \begin{cases} e^{-n^2\pi^2 t_2} w_n(t-t_2), & t \in [t_2, 2t_2], \\ 0, & t \in [0, t_2), \end{cases}$$

is the biorthogonal to the sequence

$$\{e^{n^2\pi^2 t},\ n \in \mathbb{N},\ t \in [0, t_1]\},$$

and

$$\begin{aligned}
\int_0^{t_1} h_n(t) h_m(t) dt &= e^{-n^2\pi^2 t_2} \Big(\int_{t_2}^{2t_2} w_n(t-t_2) w_m(t-t_2) dt \Big) e^{-m^2\pi^2 t_2} \\
&= e^{-n^2\pi^2 t_2} \Big(\int_0^{t_2} w_n(t) w_m(t) dt \Big) e^{-m^2\pi^2 t_2},
\end{aligned}$$

so it follows from (4.19) that

$$\sum_{n=1}^{p} \sum_{m=1}^{p} c_n \Big(\int_0^{t_1} h_n(t) h_m(t) dt \Big) c_m \leqslant K_\varepsilon^2 M \sum_{n=1}^{p} |c_n|^2.$$

Hence (see [12]),

$$\sum_{n=1}^{p} \sum_{m=1}^{p} c_n \Big(\int_0^{2t_1} e^{n^2\pi^2\tau} e^{m^2\pi^2\tau} \Big) c_m d\tau \geqslant \gamma \sum_{n=1}^{p} |c_n|^2,\ p = 1, 2, \ldots, \tag{4.20}$$

for every finite sequence $\{c_1, c_2, \ldots, c_p\}$, where $\gamma = 1/K_\varepsilon^2 M > 0$. It proves that the sequence

$$\{e^{n^2\pi^2 t}, t \in [0, t_1],\ n \in \mathbb{N}\}$$

is strongly minimal for any $t_1 > 0$.

5. Approximation Theorems. Example

As was said at the end of the previous section the condition $\lim\limits_{n\to\infty} \lambda_n^{\min} > 0$ in general can be checked by numerical methods. The problem appears to be rather difficult in general. However there are sequences for which the validity of above inequality can be easily established; for example, every orthonormal sequence is strongly minimal.

Below we will show that if the sequence $\{y_j \in X, j \in \mathbb{N}\}$ can be approximated in the some sense by strongly minimal sequence $\{x_j \in X, j \in \mathbb{N}\}$, then it is also strongly minimal.

THEOREM 5.1. *If the sequence* $\{x_j \in X,\ j \in \mathbb{N}\}$ *is strongly minimal, and the sequence* $\{y_j \in X,\ j \in \mathbb{N}\}$ *is such that*

$$\Big\| \sum_{j=1}^{n} c_j(y_j - x_j) \Big\| \leqslant q \Big\| \sum_{j=1}^{n} c_j x_j \Big\|,\ n \in \mathbb{N}, \tag{5.1}$$

where $\{c_j,\ j \in \mathbb{N}\}$ *is any sequence of complex numbers,* q *is a constant,* $0 < q < 1$, *then the sequence* $\{y_j \in X,\ j \in \mathbb{N}\}$ *also is strongly minimal.*

Proof. Let $\{c_k, k = 1,2,...\}$ be an arbitrary sequence of complex number. Denote:

$$x^0 = \sum_{k=1}^{n} c_k x_k \quad \text{and} \quad x^1 = \sum_{k=1}^{n} c_k (x_k - y_k). \tag{5.2}$$

From (5.2) it follows, that

$$x^0 = x^1 + \sum_{k=1}^{n} c_k y_k,\ n \in \mathbb{N}. \tag{5.3}$$

By (5.1) we obtain

$$\|x^1\| \leqslant q\|x^0\|. \tag{5.4}$$

Hence using (5.4) in (5.3) we obtain

$$\|x^0\| \leqslant \frac{1}{1-q} \Big\| \sum_{k=1}^{n} c_k y_k \Big\|,\ n \in \mathbb{N}. \tag{5.5}$$

Since the sequence $\{x_j \in X,\ j \in \mathbb{N}\}$ is strongly minimal and $x^0 = \sum_{k=1}^{n} c_k x_k$, we have

$$\sum_{k=1}^{n} |c_k|^2 \leqslant \frac{1}{\alpha^2} \|x^0\|^2,\ n \in \mathbb{N}, \tag{5.6}$$

for some $\alpha \in \mathbb{R}, \alpha \neq 0$.

By (5.6) and (5.5) we obtain

$$\alpha^2 \sum_{k=1}^{n} |c_k|^2 \leqslant \frac{1}{1-q} \Big\| \sum_{k=1}^{n} c_k y_k \Big\|,\ \ n \in \mathbb{N},$$

so

$$\alpha^2(1-q)^2\Big(\sum_{k=1}^{n}|c_k|^2\Big) \leqslant \Big\|\sum_{k=1}^{n} c_k y_k\Big\|, \; n\in\mathbb{N}. \tag{5.7}$$

Using in (5.7) the formula

$$\sum_{j=1}^{m}\sum_{k=1}^{m} c_j(y_j,y_k)\overline{c_k} = \Big\|\sum_{j=1}^{m} c_j y_j\Big\|^2,$$

we obtain

$$\gamma\Big(\sum_{k=1}^{n}|c_k|^2\Big) \leqslant \sum_{k=1}^{n}\sum_{l=1}^{n} c_k(y_k,y_l)\overline{c_l}, \tag{5.8}$$

where $\gamma = \alpha^2(1-q)^2 > 0$. Formula (5.8) is equivalent to the strong minimality of the sequence $\{y_k,\, k\in\mathbb{N}\}$. This proves the theorem. $\square$

5.1. Controllability by single control

Let all the geometrical multiplicities $q_j,\; j\in\mathbb{N}$ be equal to 1. In accordance with Remark 4.3 the exact null-controllable equation (2.1) on $[0,t_1]$ by scalar controls ($r = 1$) vanishing after $t_1 - T$, can be considered, and the operator $B : U \to X$ is defined by

$$Bu = bu, \; \forall u \in \mathbb{R},$$

where $b\in V$. The operator b is a bounded operator if and only if $b \in X$. In this case $\beta_j^m = \alpha_j \leqslant \alpha,\; j\in\mathbb{N}$, so the generalized eigenvectors of the operators A and A^* can be written without index m, $b_{jk}^m = b_{jk} = (b,\psi_{jk}),\; k = 1,2,...,\alpha_j,\; j\in\mathbb{N}$, where ψ_{jk}, $j\in\mathbb{N},\; k = 1,2,...,\alpha_j$ are generalized eigenvectors of the adjoint operator A^*.

Denote

$$g = \left\{ \begin{array}{l} g_{jk}(-t) = \sum_{l=0}^{\alpha_j-k} b_{jk+l}\exp(-\lambda_j t)\frac{(-t)^l}{l!}, \\ t\in[0,t_1-T],\; j\in\mathbb{N},\; k=1,2,...,\alpha_j, \end{array} \right\} \tag{5.9}$$

In accordance with Theorem 4.2 equation (2.1) is exact null-controllable on $[0,t_1]$ by scalar controls vanishing after $t_1 - T$, if and only if the sequence (5.9) is strongly minimal.

5.2. Example

Consider the functional differential time-invariant control system of neutral type [4] with scalar control

$$\sum_{j=0}^{m}\big(A_{0j}\dot{x}(t-h_j) + A_{1j}x(t-h_j)\big) = bu(t), \tag{5.10}$$

where $A_{0j},\; A_{1j}, i = 1,...,m$ are $n\times n$-matrices, $\det A_{00} \neq 0,\; b\in\mathbb{R}^n, 0 = h_0 < h_1 < h_m = h$.

Let $X = W_2^1\left([-h,0],\mathbb{C}^n\right)$ denote the Sobolev space of functions $v(\cdot)\in\mathbb{C}^n$ equipped by the norm

$$\|v\|_{w_2} = \left(\int_{-h}^{0}\left(\|\dot v(\theta)\|^2+\|v(\theta)\|^2\right)d\theta\right)^{\frac{1}{2}}. \tag{5.11}$$

The initial conditions for system (5.10) is defined by

$$x(\theta)=\varphi(\theta),\ -h\leqslant\theta\leqslant 0,\ \varphi(\cdot)\in W_2^1\left([-h,0],\mathbb{C}^n\right),\ x(+0)=\varphi(0). \tag{5.12}$$

Let $x_t\in X$ denote the function $x(t+\theta)$, $-h\leqslant\theta\leqslant 0$, $t\geqslant 0$, where $x_0=\varphi(\theta)$, $-h\leqslant\theta\leqslant 0$. Consider the C_0 -semigroup $S(t): X\to X$ of bounded operators defined by (see [10])

$$S(t)\varphi = x_t,\ t\geqslant 0,$$

where $x(t)$ is the strong solution of system

$$\sum_{j=0}^{m}\left(A_{0j}\dot x(t-h_j)+A_{1j}x(t-h_j)\right)=0 \tag{5.13}$$

with initial condition (5.12). The operator B is defined by the pair

$$Bu=(b,0)\,u,\ \forall u\in\mathbb{R},$$

where 0 is the function equals to zero a.e. on $[-h,0]$. The vector $B=(b,0)\ \notin X$, so the operator B is not bounded. One can see that system (5.10) is a partial case of equation (2.1) with single control. The operator A, generating the semigroup $S(t)$, satisfies all the assumptions of page 173; the second assumption is fulfilled with $T=nh$ [4], The eigenvalues of the operator A are exactly zeroes of the quasipolynomial

$$W(\lambda)=\det\sum_{j=0}^{m}\left(A_{0j}\lambda+A_{1j}\right)e^{-\lambda h_j}=0. \tag{5.14}$$

One can prove that exponential solutions corresponding to the eigenvalue λ_j described in [27] (see also [4] and [10]) are exactly generalized eigenvectors of the operator A. Hence from corollary to Theorem 3 of [27] it follows that the generalized eigenvectors of the operator A constitute a Riesz basis of the space $X\ =W_2^1\left([-h,0],\mathbb{C}^n\right)$ provided that in addition to the condition $\det A_{00}\neq 0$ the condition $\det A_{0m}\neq 0$ also holds, and zeroes $\lambda_j, j\in\mathbb{N}$ of quasipolynomial (5.14) constitute a separate set, i.e.

$$\inf_{j,k\in\mathbb{N},\ k\neq j}\left|\lambda_k-\lambda_j\right|>0 \tag{5.15}$$

THEOREM 5.2. *If* $\det A_{00}\neq 0$, $\det A_{0m}\neq 0$ *and the eigenvalues* $\lambda_j, j\in\mathbb{N}$ *of the operator* A *constitute a separate set, then system* (5.10) *is exact null-controllable on* $[0,t_1]$, $t_1>nh$, *by controls vanishing after time moment* t_1-nh, *if and only if the sequence* (5.9) *is strongly minimal*[2].

Condition (5.15) holds for systems (5.10) with commensurable delays ($\lambda_j = jh$ for some $h\in\mathbb{R}$) [4].

[2]Here $b_{jk}=\left(B,\psi_{jk}\right)=b^T c_{jk}$, $j\in\mathbb{N}$, $k=1,\ldots,\beta_j$, where $c_{jk}=\psi_{jk}(0)$ are defined in [10].

6. Comments

There exists the powerful technique of moment problem based on the theory of Riesz basis of exponentials. This theory has a long history [1], [3], [12], [26] and it has been essentially developed in [2], [24], [25], [29], (see also references therein). The approach based on the theory of Riesz basis of exponentials is very useful tool to analyze exact controllability problems for hyperbolic control systems. Unfortunately the corresponding families of exponentials arising in parabolic control systems as well as in functional differential control systems don't constitute a Riesz basis, and the operators generating semigroups of systems under investigations not always has a basis of their generalized eigenvectors. Therefore one needs another approach to investigate exact controllability problem. One of possible approaches has been developed in [21], where the authors have constructed a special Riesz basis based on a Riesz basis of finite-dimensional invariant subspaces allowing to use some special moment problem. The authors of [21] investigate the controllability problem which can be considered as attainability. It differs from the exact null controllability problem considered in the given paper.

Theorem 4.2 allows to investigate exact null-controllability problems in the case when corresponding families of exponential are not Riesz basis in the closure of their linear span. It reduces the exact null-controllability problem to the investigation of the strong minimality property for exponential families. For example, strongly minimal sequence of exponentials (4.17) considered on page 182 is not a basis in the closure of its linear span, but it allows to establish exact null-controllability for classical heat equation (parabolic equation) governed by boundary control (well-known result).

Sometimes the strong minimality property for exponential families can be easily verified and can be applied for exact null-controllability of parabolic control equations (see the example of minimal sequence of exponentials on the Page 182). But as a usual the establishing of the strong minimality of exponentials is not always trivial.

In our private opinion the strongly minimal sequences of exponentials as well as basic families of exponentials play an important role in controllability theory. Conditions for strong minimality of families of exponentials generated by partial differential equations have been investigated in the literature, however the strong minimality of families of exponentials generated by functional-differential control systems are almost not investigated (to the author's knowledge).

Use of asymptotic formulas for zeroes of quasipolynomial (5.14) (see [4]) in Theorem 5.1 may be considered as an approach for establishing the strong minimality of (5.9) generated by linear time-invariant neutral control systems.

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