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ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A p–LAPLACIAN NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH TIME–VARYING OPERATOR

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Abstract. In this paper, a kind of p-Laplacian neutral functional differential equation with time-varying operator as follows

$$\left(\varphi_p\left(u'(t)-\sum_{i=1}^{n}c_i(t)u'(t-r_i)\right)\right)'=f(u'(t))+\beta(t)g(u(t-\gamma(t)))+e(t),$$

is studied and some new results are obtained. It is worth noting that the parameters $c_i(t)(i=1,2,\cdots,n)$ are functions and the coefficient $\beta(t)$ (which is ahead of g) is sign-variable here. It is interesting, but it is so challenging and difficult that few people have discussed it so far.

1. Introduction

In the past few years, the existence of periodic solutions has been studied extensively, see [1, 2, 3, 4, 5, 6, 7, 8] and references therein, especially the existence of periodic solutions to p-Laplacian functional differential equations has received more and more attention, see references [1, 3, 4, 6, 7, 8] for more details. For example, Lu and Gui [6] studied a kind of p-Laplacian Rayleigh differential equation

$$(\varphi_p(y'(t)))'+f(y'(t))+g(y(t-\tau(t)))=e(t). \tag{1.1}$$

Cheung and Ren [1] studied the existence of periodic solutions to a kind of p-Laplacian Rayleigh equation of the form

$$(\varphi_p(x'(t)))'+f(x'(t))+\beta g(x(t-\tau(t)))=e(t), \tag{1.2}$$

where $\beta>0$ is a constant. Lu *et al.* [5] studied a kind of neutral differential equation with deviating arguments as follows

$$(x(t)-cx(t-r)''+f(x'(t))+g(x(t-\tau(t)))=p(t), \tag{1.3}$$

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where $c \neq 1$ is a constant, some results on the difference operator $D: C_T \to C_T, [Dx](t) = x(t) - cx(t-r)$ and the existence of periodic solutions were obtained. Recently, Lu [4] studied a p-Laplacian neutral functional differential equation of the form

$$\begin{aligned}\Big(\varphi_p\big((u(t) - \sum_{j=1}^{n} c_j u(t-r_j))'\big)\Big)'\\ = f(u(t))u'(t) + \alpha(t)g(u(t)) + \sum_{j=1}^{n} \beta_j(t)g(u(t-\gamma_j(t))) + p(t), \quad (1.4)\end{aligned}$$

some results about the difference operator $D: C_T \to C_T, [Dx](t) = x(t) - \sum_{j=1}^{n} c_j x(t-r_j)$ and the existence of periodic solutions were presented.

On the other hand, Liang *et al.* [3] and Wang *et al.* [7] studied the existence of periodic solutions to the following p-Laplacian neutral functional differential equations

$$\Big(\varphi_p\big((x(t) - c(t)x(t-\sigma))^{(n)}\big)\Big)^{(m)} + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t) \qquad (1.5)$$

and

$$\big(\varphi_p(x'(t) - c(t)x'(t-r))\big)' = f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) + e(t), \qquad (1.6)$$

respectively, where $\beta(t)$ is allowed to change sign.

Observing the above equations, it is easy to note that the coefficients which are ahead of function g in Eqs.(1.1-1.4) are constants or the sign-fixed functions. Moreover, the coefficients $c(t)$ of the difference operator in Eq.(1.5) and Eq.(1.6) are functions, but the second term in these equations are $f(x(t))x'(t)$. Therefore, it is easy to obtain the *a priori bounds* of periodic solutions for Eqs(1.1-1.6), which is crucial for the existence of periodic solutions.

To the best of our knowledge, there are few results on the existence of periodic solutions to the following equation of the type

$$\Big(\varphi_p\big(u'(t) - \sum_{i=1}^{n} c_i(t)u'(t-r_i)\big)\Big)' = f(u'(t)) + \beta(t)g(u(t-\gamma(t))) + e(t), \qquad (1.7)$$

where $\varphi_p(u) = |u|^{p-2}u$ for $u \neq 0$ and $\varphi_p(0) = 0, p > 1$; $f, g \in C(\mathbb{R}, \mathbb{R})$; $e(t), \beta(t), \gamma(t)$ are continuous periodic functions defined on $\mathbb{R}$ with period $T > 0$,

$$\int_0^T \beta(t)dt \neq 0, \quad \int_0^T e(t)dt = 0,$$

$c_i(t) \in C^1(\mathbb{R}, \mathbb{R})$ and $c_i(t+T) = c_i(t)$, $(i = 1, 2, \cdots, n)$; $T, r_i (i = 1, 2, \cdots, n)$ are given constants.

It is noted that in Eq.(1.7) the functions $c_i(t), (i = 1, 2, \cdots, n)$ are not constants, the sign of coefficient $\beta(t)$ which is ahead of g can be changed, and the second term is $f(x'(t))$, so it is difficult to estimate *a priori bounds* of periodic solutions. There are two main difficulties. On the one hand, $\int_0^T f(x'(t))dt = 0$ is no longer valid for Eq.(1.7), on the other hand, the difference operator $A : C_T \to C_T, [Ax](t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-r_i)$ has n continuous functions $c_i(t), (i = 1, 2, \cdots, n)$. To obtain *a priori bounds* of periodic solutions and the existence of periodic solutions, one must overcome these difficulties.

Motivated by the above reasons, we will study the properties of time-varying difference operator A and the existence of periodic solutions to Eq.(1.7). Firstly, some new results on the properties of operator A are obtained. Then, based on these properties of operator A and Mawhin continuation theorem, some new results about the existence of periodic solutions are presented. Finally, a numerical example is given to illustrate the availability of the obtained results.

2. Properties of time-varying operator A

In this section, we will investigate some properties of a time-varying operator $A : C_T \to C_T, [Ax](t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-r_i)$ by the knowledge of mathematical analysis. Some new results of the properties of A will be obtained. Let

$$C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t), \text{ for all } t \in \mathbb{R}\},$$

with norm $|\varphi|_0 = \max_{t \in [0,T]} |\varphi(t)|$, for all $\varphi \in C_T$,

$$C_T^1 = \{x : x \in C^1(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t), \text{ for all } t \in \mathbb{R}\},$$

with norm $\|\varphi\| = \max\{|\varphi|_0, |\varphi'|_0\}$, for all $\varphi \in C_T^1$. Therefore, C_T and C_T^1 are both Banach spaces. For $c_i \in C_T^1$, let

$$c_i^0 = \max_{t \in [0,T]} |c_i(t)|, \quad c_i^1 = \max_{t \in [0,T]} |c_i'(t)|, \quad (i = 1, 2, \cdots, n).$$

Define linear operators:

$$A : C_T \to C_T, [Ax](t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-r_i), \text{ for all } t \in [0,T]. \tag{2.1}$$

Throughout this paper, let $\sum_{i=n}^{m} r_i = 0$ if $n > m$.

LEMMA 2.1. *If* $\sum_{i=1}^{n} c_i^0 < 1$, *then* A *has continuous inverse* A^{-1} *on* C_T *with the following properties, where* A *is defined by* (2.1)*:*
(1) *for all* $f \in C_T$,

$$[A^{-1}f](t) = f(t) + \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \prod_{j=1}^{m} c_{i_j}\left(t - \sum_{k=j+1}^{m} r_{i_k}\right) f\left(t - \sum_{s=1}^{m} r_{i_s}\right);$$

(2) *it holds,*

$$\|A^{-1}\| \leqslant \frac{1}{1-\sum\limits_{i=1}^{n} c_i^0};$$

(3) *for all* $f \in C_T$ *and* $p > 1$,

$$\int_0^T \left|[A^{-1}f](t)\right|^p dt \leqslant \frac{1}{\left(1-\sum\limits_{i=1}^{n} c_i^0\right)^p} \int_0^T |f(t)|^p dt;$$

(4) *for all* $f \in C_T^1$,

$$[Af'](t) = [Af]'(t) + \sum_{i=1}^{n} c_i'(t) f(t-r_i);$$

(5) *for all* $f \in C_T^1$,

$$[A^{-1}f]'(t) = [A^{-1}f'](t) + \sum_{m=1}^{\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \left(\prod_{j=1}^{m} c_{i_j}\Big(t - \sum_{k=j+1}^{m} r_{i_k}\Big)\right)' f\Big(t - \sum_{s=1}^{m} r_{i_s}\Big).$$

Proof. Conclusions (1-4) are the direct results of [8]. For conclusion (5), one can obtain it by calculating directly. □

Now suppose that there is an integer $k \in \{1, 2, \cdots, n\}$ such that

$$c_k = \min_{t\in[0,T]} |c_k(t)| > 1 \quad \text{and} \quad \delta := \frac{1}{c_k} + \sum_{i\neq k} \frac{c_i^0}{c_k} < 1.$$

From the definition of A we have

$$[Ax](t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-r_i) = -c_k(t)\left[x(t-r_k) - \frac{x(t)}{c_k(t)} + \sum_{i\neq k} \frac{c_i(t)}{c_k(t)} x(t-r_i)\right].$$

Let $[Ax](t) = f(t)$, $f \in C_T$, that is

$$-c_k(t)\left[x(t-r_k) - \frac{x(t)}{c_k(t)} + \sum_{i\neq k} \frac{c_i(t)}{c_k(t)} x(t-r_i)\right] = f(t),$$

therefore

$$x(t-r_k) - \frac{x(t)}{c_k(t)} + \sum_{i\neq k} \frac{c_i(t)}{c_k(t)} x(t-r_i) = -\frac{f(t)}{c_k(t)}.$$

Doing variable transformation $s = t - r_k$ and replacing s with t finally:

$$x(t) - \frac{1}{c_k(t+r_k)} x(t+r_k) - \sum_{i\neq k} \left(-\frac{c_i(t+r_k)}{c_k(t+r_k)}\right) x(t-r_i+r_k)$$

$$= -\frac{f(t+r_k)}{c_k(t+r_k)} := f_1(t), \tag{2.2}$$

and $f_1(t) \in C_T$ as well. Let $E : C_T \to C_T$ be defined by

$$[Ex](t) = x(t) - \frac{1}{c_k(t+r_k)}x(t+r_k) - \sum_{i\neq k}\left(-\frac{c_i(t+r_k)}{c_k(t+r_k)}\right)x(t-r_i+r_k),$$

and

$$d_k(t) = \frac{1}{c_k(t+r_k)} \quad \text{and} \quad d_i(t) = -\frac{c_i(t+r_k)}{c_k(t+r_k)},\ i = 1,2,\cdots,k-1,k+1,\cdots,n,$$

$$\widetilde{r}_k = -r_k \quad \text{and} \quad \widetilde{r}_i = r_i - r_k,\ i = 1,2,\cdots,k-1,k+1,\cdots,n,$$

$$d_i^0 = \max_{t\in[0,T]} |d_i(t)|,\ i = 1,2,\cdots,n.$$

Then we have

$$[Ex](t) = x(t) - \sum_{i=1}^{n} d_i(t)x(t-\widetilde{r}_i) \quad \text{and} \quad \sum_{i=1}^{n} d_i^0 \leqslant \frac{1}{c_k} + \sum_{i\neq k}\frac{c_i^0}{c_k} = \delta < 1.$$

Therefore, by Lemma 2.1 E has continuous inverse $E^{-1} : C_T \to C_T$ and it follows from the equation (2.2) which is equivalent to $[Ex](t) = f_1(t)$ that $x(t) = [E^{-1}f_1](t)$, and $[E^{-1}f_1](t) \in C_T$ also holds, this implies that A has continuous inverse $A^{-1} : C_T \to C_T$, and

$$[A^{-1}f(t)] = x(t) = [E^{-1}f_1](t).$$

Furthermore, one have following results.

LEMMA 2.2. *If there is an integer $k \in \{1,2,\cdots,n\}$ such that*

$$c_k = \min_{t\in[0,T]} |c_k(t)| > 1 \quad \textit{and} \quad \delta = \frac{1}{c_k} + \sum_{i\neq k}\frac{c_i^0}{c_k} < 1,$$

then A has continuous inverse $A^{-1} : C_T \to C_T$ satisfying:
(1) *for all $f \in C_T$,*

$$[A^{-1}f(t)] = -\frac{f(t+r_k)}{c_k(t+r_k)} + \sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n}\prod_{j=1}^{m} d_{i_j}\Big(t - \sum_{k=j+1}^{m}\widetilde{r}_{i_k}\Big)$$
$$\cdot\left(-\frac{f\big(t-\sum_{s=1}^{m}\widetilde{r}_{i_s}+r_k\big)}{c_k\big(t-\sum_{s=1}^{m}\widetilde{r}_{i_s}+r_k\big)}\right),$$

where $d_i, \widetilde{r}_i (i = 1, 2, \cdots, n)$ *are defined above;*
(2) *it holds,*

$$\|A^{-1}\| \leqslant \frac{1}{c_k - c_k\delta} = \frac{1}{c_k - 1 - \sum\limits_{i \neq k} c_i^0};$$

(3) *for all* $f \in C_T,\ p > 1$,

$$\int_0^T \left|[A^{-1}f](t)\right|^p dt \leqslant \left(\frac{1}{c_k - 1 - \sum\limits_{i \neq k} c_i^0}\right)^p \int_0^T |f(t)|^p dt;$$

(4) *for all* $f \in C_T^1$,

$$[Af'](t) = [Af]'(t) + \sum_{i=1}^n c_i'(t) f(t - r_i);$$

(5) *for all* $f \in C_T^1$,

$$[A^{-1}f]'(t) = [E^{-1}f_1'](t) + \sum_{m=1}^{\infty} \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_m=1}^n \Big(\prod_{j=1}^m d_{i_j}\big(t - \sum_{k=j+1}^m \widetilde{r}_{i_k}\big)\Big)' \cdot f_1\Big(t - \sum_{s=1}^m \widetilde{r}_{i_s}\Big),$$

where $\widetilde{r}_i, d_i\ \ (i = 1, 2, \cdots, n)$ *are defined above and* $f_1(t) = -\frac{f(t+r_k)}{c_k(t+r_k)}$.

Proof. Conclusions (1-3) is in [8] and conclusions (4-5) can be obtained by calculating directly. □

REMARK 2.1. From the lemmas 2.1 and 2.2, if A^{-1} exists, then A^{-1} is linear.

LEMMA 2.3. ([8]) *If* $\sum\limits_{i=1}^n c_i^0 < \frac{1}{2}$ *and* $f(t) \equiv 1$, *then* $[A^{-1}f](t) > 0$, *for all* $t \in \mathbb{R}$.

LEMMA 2.4. ([8]) *If there is an integer* $k \in \{1, 2, \cdots, n\}$ *such that*

$$c_k = \min_{t \in [0,T]} |c_k(t)| > 1 \quad \text{and} \quad \delta c_k^0 < (1 - \delta) c_k,$$

where $\delta = \frac{1}{c_k} + \sum\limits_{i \neq k} \frac{c_i^0}{c_k}$. *Furthermore, if* $f(t) \equiv 1$, *then* $[A^{-1}f](t) \neq 0$, *for all* $t \in \mathbb{R}$.

REMARK 2.2. From the inequality $\delta c_k^0 < (1 - \delta) c_k$ one obtains $\delta < \frac{1}{2}$ easily, so the inverse of A exists.

Moreover, when $f(t) \equiv 1$, for the equation

$$[Ax](t) = x(t) - \sum_{i=1}^n c_i(t) x(t - r_i) \equiv 1, \text{ for all } x(t) \in C_T,\ t \in [0, T], \tag{2.3}$$

we have the following conclusions.

LEMMA 2.5. *If $f(t) \equiv 1$, then the following conclusions hold.*
(1) *If $\sum\limits_{i=1}^{n} c_i(t)$ is not a constant function, for all $c_i(t) \in C_T$ $(i = 1, 2, \cdots, n)$, then $x(t) = [A^{-1}f](t)$ is not a constant function.*
(2) *If $\sum\limits_{i=1}^{n} c_i^0 < \frac{1}{2}$, then $\int_0^T [A^{-1}f](t)dt \neq 0$.*
(3) *If there is an integer $k \in \{1, 2, \cdots, n\}$ such that*

$$c_k = \min_{t\in[0,T]} |c_k(t)| > 1 \quad \text{and} \quad \delta c_k^0 < (1-\delta)c_k,$$

where $\delta = \frac{1}{c_k} + \sum\limits_{i\neq k} \frac{c_i^0}{c_k}$, then $\int_0^T [A^{-1}f](t)dt \neq 0$.

Proof. (1) Assume, by way of contradiction, that the result does not hold, then $x(t) \equiv C$ (C is a constant), $t \in [0, T]$. Therefore

$$C - C\sum_{i=1}^{n} c_i(t) \equiv 1,$$

from (2.3) we know $C \neq 0$, then

$$\sum_{i=1}^{n} c_i(t) \equiv (C-1)/C,$$

which contradicts the assumption of $\sum\limits_{i=1}^{n} c_i(t)$ is not a constant function. This contradiction implies that the conclusion (1) holds.

(2) Similar to the (1), assuming, by way of contradiction, that the result does not hold, then

$$\int_0^T [A^{-1}f](t)dt = \int_0^T x(t)dt = 0,$$

which together with (2.3) yields

$$\int_0^T \left[1 + \sum_{i=1}^{n} c_i(t)x(t-r_i)\right] dt = 0,$$

$$\frac{1}{T}\int_0^T \sum_{i=1}^{n} c_i(t)x(t-r_i)dt = -1,$$

$$1 = \left|\frac{1}{T}\int_0^T \sum_{i=1}^{n} c_i(t)x(t-r_i)dt\right| \leqslant \frac{1}{T}\int_0^T \sum_{i=1}^{n} |c_i(t)||x(t-r_i)|dt \leqslant |x|_0 \sum_{i=1}^{n} c_i^0. \tag{2.4}$$

On the other hand, by the Lemma 2.1 (2) and $f(t) \equiv 1$, we have

$$|x|_0 = \left|[A^{-1}f](t)\right|_0 \leqslant \|A^{-1}\| \cdot |f|_0 = \|A^{-1}\| \leqslant \frac{1}{1 - \sum\limits_{i=1}^{n} c_i^0}. \tag{2.5}$$

In view of $|x|_0 > 0$ and from (2.4) and (2.5), we know $\sum\limits_{i=1}^{n} c_i^0 \geqslant \frac{1}{2}$, which contradicts $\sum\limits_{i=1}^{n} c_i^0 < \frac{1}{2}$. This contradiction implies that the conclusion (2) holds.

(3) By way of contradiction, supposing that the conclusion (3) does not hold, then

$$\int_0^T [A^{-1}f](t)dt = \int_0^T x(t)dt = 0. \tag{2.6}$$

Since $f(t) \equiv 1$, we have

$$\begin{aligned}
[Ax](t) &= x(t) - \sum_{i=1}^{n} c_i(t)x(t-r_i) \\
&= -c_k(t)\left[x(t-r_k) - \frac{x(t)}{c_k(t)} + \sum_{i\neq k} \frac{c_i(t)}{c_k(t)}x(t-r_i)\right] \equiv 1
\end{aligned}$$

that is

$$-c_k(t)\left[x(t-r_k) - \frac{x(t)}{c_k(t)} + \sum_{i\neq k} \frac{c_i(t)}{c_k(t)}x(t-r_i)\right] \equiv 1,$$

so

$$x(t-r_k) - \frac{x(t)}{c_k(t)} + \sum_{i\neq k} \frac{c_i(t)}{c_k(t)}x(t-r_i) \equiv -\frac{1}{c_k(t)}.$$

Doing variable transformation:

$$x(t) \equiv \frac{1}{c_k(t+r_k)}x(t+r_k) - \sum_{i\neq k} \frac{c_i(t+r_k)}{c_k(t+r_k)}x(t-r_i+r_k) - \frac{1}{c_k(t+r_k)},$$

which together with (2.6) we obtain

$$\begin{aligned}
&\frac{1}{T}\int_0^T \frac{x(t+r_k)}{c_k(t+r_k)}dt - \frac{1}{T}\int_0^T \sum_{i\neq k} \frac{c_i(t+r_k)}{c_k(t+r_k)}x(t-r_i+r_k)dt \\
&\qquad\qquad - \frac{1}{T}\int_0^T \frac{1}{c_k(t+r_k)}dt \equiv \frac{1}{T}\int_0^T x(t)dt = 0,
\end{aligned}$$

so

$$\begin{aligned}
\left|\frac{1}{T}\int_0^T \frac{1}{c_k(t+r_k)}dt\right| &\leqslant \frac{1}{T}\int_0^T \left|\frac{x(t+r_k)}{c_k(t+r_k)}\right|dt \\
&\qquad + \frac{1}{T}\int_0^T \sum_{i\neq k}\left|\frac{c_i(t+r_k)}{c_k(t+r_k)}x(t-r_i+r_k)\right|dt \\
&\leqslant |x|_0\left(\frac{1}{c_k} + \sum_{i\neq k}\frac{c_i^0}{c_k}\right) = |x|_0\delta,
\end{aligned}$$

together with $c_k(t)$ does not change sign on $[0,T]$ which from $\min_{t\in[0,T]}|c_k(t)|>1$, we get

$$\frac{1}{c_k^0}\leqslant\left|\frac{1}{T}\int_0^T\frac{1}{c_k(t+r_k)}dt\right|\leqslant|x|_0\delta, \tag{2.7}$$

On the other hand, by the Lemma 2.2(2) and $f(t)\equiv 1$ we obtain

$$|x|_0=|[A^{-1}f](t)|_0\leqslant\|A^{-1}\|\cdot 1\leqslant\frac{1}{c_k(1-\delta)}, \tag{2.8}$$

which together with (2.7), one has

$$\delta c_k^0\geqslant c_k(1-\delta),$$

which contradicts $\delta c_k^0<c_k(1-\delta)$. This contradiction implies that the conclusion (3) holds. □

REMARK 2.3. The second conclusion can be obtained from Lemma 2.3. In fact, from Lemma 2.3 we know, if $\sum\limits_{i=1}^{n}c_i^0<\frac{1}{2}$ and $f(t)\equiv 1$, then $[A^{-1}f](t)>0$, for all $t\in\mathbb{R}$, so $\int_0^T[A^{-1}f](t)dt>0$. In addition, the third conclusion can not be obtained from Lemma 2.4 directly.

In view of Remark 2.1, we can conclude the following results from Lemma 2.5.

LEMMA 2.6. *If $f(t)\equiv C\neq 0$, for the equation*

$$[Ax](t)=x(t)-\sum_{i=1}^{n}c_i(t)x(t-r_i)\equiv C,$$

where $x(t)\in C_T$, then the following conclusions hold.

(1) *If $\sum\limits_{i=1}^{n}c_i^0<\frac{1}{2}$, then $\int_0^T[A^{-1}C](t)dt\neq 0$.*

(2) *If there is an integer $k\in\{1,2,\cdots,n\}$ such that*

$$c_k=\min_{t\in[0,T]}|c_k(t)|>1 \ \textit{and}\ \ \delta c_k^0<(1-\delta)c_k,$$

where $\delta=\frac{1}{c_k}+\sum\limits_{i\neq k}\frac{c_i^0}{c_k}$, then $\int_0^T[A^{-1}C](t)dt\neq 0$.

REMARK 2.4. These results on the properties of time-varying operator A improve the results in previous literatures [3, 4, 5, 7, 8].

3. The existence of periodic solutions

In this section, based on the above properties of operator A in the section 2 and Mawhin's continuation theorem [2], we investigate the existence of periodic solutions to Eq.(1.1).

Firstly, we introduce Mawhin's continuation theorem.

LEMMA 3.1. (Gaines and Mawhin [2]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N(x,\lambda) : \overline{\Omega} \times [0,1] \to Y$ is L-compact on $\overline{\Omega}$. If:*
(1) $Lx \neq \lambda N(x,\lambda),\ for\ all\ x \in \partial\Omega \cap D(L), \lambda \in (0,1)$,
(2) $QN(x,0) \neq 0,\ for\ all\ x \in \partial\Omega \cap kerL$,
(3) $\deg(JQN(\cdot,0),\Omega \cap kerL, 0) \neq 0$ *, where $J : ImQ \to kerL$ is an isomorphism, then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.*

In order to apply Lemma 3.1 to study the existence of $T-$periodic solutions to Eq.(1.1), supposing that $\sum_{i=1}^{n} c_i^0 < 1$, so the problem of existence of $T-$periodic solutions to Eq.(1.1) can be converted to the corresponding problem of the following system:

$$\begin{cases} x_1'(t) = \left[A^{-1}\left(\varphi_q(x_2(\cdot))\right)\right](t), \\ x_2'(t) = f\left(\left[A^{-1}\left(\varphi_q(x_2(\cdot))\right)\right](t)\right) + \beta(t)g(x_1(t-\gamma(t)) + e(t), \end{cases} \tag{3.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. In fact, if $x(\cdot) = (x_1(\cdot), x_2(\cdot))^\top$ is a $T-$periodic solution to Eqs.(3.1), then $x_1(t)$ must be a $T-$periodic solution to Eq.(1.1). Thus, in order to prove that Eq.(1.1) has a $T-$periodic solution, it suffices to show that Eqs.(3.1) has a $T-$periodic solution.

Now, let

$$X = Y = \left\{x = (x_1(\cdot), x_2(\cdot))^\top \in C(R,R^2) : x_1 \in C_T, x_2 \in C_T\right\},$$

$\|\psi\| = \max\{|\psi_1|_0, |\psi_2|_0\}, \forall \psi \in X$ or Y. Then, X and Y are both Banach spaces. Defining a linear operator

$$L : D(L) \cap X \to Y, \quad Lx = x' = (x_1', x_2')^T, \tag{3.2}$$

where $D(L) = \left\{x = (x_1(\cdot), x_2(\cdot))^\top \in C^1(R,R^2) : x_1 \in C_T^1, x_2 \in C_T^1\right\}$ and a nonlinear operator $N : X \to Y$ by setting

$$[Nx](t) = \begin{pmatrix} \left[A^{-1}\left(\varphi_q(x_2(\cdot))\right)\right](t) \\ f\left(\left[A^{-1}\left(\varphi_q(x_2(\cdot))\right)\right](t)\right) + \beta(t)g(x_1(t-\gamma(t)) + e(t) \end{pmatrix} \tag{3.3}$$

and another nonlinear operator $N(\cdot,\lambda) : X \times [0,1] \to Y$ by setting

$$[N(x,\lambda)](t) = \begin{pmatrix} \left[A^{-1}\left(\varphi_q(x_2(\cdot))\right)\right](t) \\ f\left(\lambda\left[A^{-1}\left(\varphi_q(x_2(\cdot))\right)\right](t)\right) + \beta(t)g(x_1(t-\gamma(t)) + e(t) \end{pmatrix}. \tag{3.4}$$

It is easy to see that $N(\cdot,1)=N$ and Eqs.(3.1) can be converted to the abstract equation $Lx=Nx$. Moreover, it follows from the definition of L that $\ker L=R^2$ and $\operatorname{Im}L=\left\{y\in Y:\int_0^T y(s)ds=0\right\}$. So $\operatorname{Im}L$ is closed in Y and $\dim\ker L=\operatorname{codim}\operatorname{Im}L=2$, then the operator L is a Fredholm operator with index zero. Meanwhile, let projectors $P:X\to\ker L$ and $Q:Y\to\operatorname{Im}Q$ be defined by

$$Px=\frac{1}{T}\int_0^T x(t)dt,\qquad Qy=\frac{1}{T}\int_0^T y(t)dt. \tag{3.5}$$

Therefore $\ker L=\operatorname{Im}P$, $\ker Q=\operatorname{Im}L$. Let operator K_P denote the inverse of $L|_{D(L)\cap\ker P}$, then

$$[K_p y](t)=\int_0^T G(t,s)y(s)ds, \tag{3.6}$$

where

$$G(t,s)=\begin{cases}\frac{s}{T}, & 0\leqslant s\leqslant t\leqslant T,\\ \frac{s-T}{T}, & 0\leqslant t\leqslant s\leqslant T.\end{cases} \tag{3.7}$$

From (3.3), (3.4) and (3.5) we know N is L-compact on $\overline{\Omega}$ and $N(\cdot,\lambda)$ is L-compact on $\overline{\Omega}\times[0,1]$, where Ω is an open bounded subset of X.

For the sake of convenience, we list the following conditions.

$[H_1]$ There are constants $r>0$ and $m>0$ such that $|f(x)|\leqslant r|x|^m$, for all $x\in\mathbb{R}$.

$[H_2]$ There are constants $\ell_1>0$ and $\ell_2>0$ such that

$$\ell_1|x|^m\leqslant|g(x)|\leqslant\ell_2|x|^m \text{ and } xg(x)>0, \text{ for all } |x|>0.$$

$[H_3]$ Let $K_1=\sum_{i=1}^n c_i^0\left(1+\sum_{i=1}^n c_i^0\right)^{\frac{p}{q}}$ and

$$K_2=\frac{rDT^{\frac{(p-m)(m+1)}{mp}}}{1-E}+\frac{rT^{2-\frac{m+1}{p}}}{1-E}+|\beta|_0\ell_2T2^m\left[\left(\frac{DT^{\frac{p-m}{mp}}}{1-E}\right)^{m+1}+\left(\frac{T^{\frac{p-1}{p}}}{1-E}\right)^{m+1}\right],$$

where

$$D=\begin{cases}\left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}2^{\frac{1-m}{m}}, & 0<m\leqslant 1,\\ \left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}, & m>1.\end{cases}\quad E=\begin{cases}\left[\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}2^{\frac{1-m}{m}}, & 0<m\leqslant 1,\\ \left[\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}, & m>1.\end{cases}$$

and $\beta^+(t)=\max\{\beta(t),0\}$, $\beta^-(t)=\max\{-\beta(t),0\}$. Supposing the following inequalities hold

$$E<1 \text{ and } \begin{cases}\frac{K_1}{\left(1-\sum_{i=1}^n c_i^0\right)^p}<1, \text{ if } m<p-1,\\ \frac{K_1+K_2}{\left(1-\sum_{i=1}^n c_i^0\right)^p}<1, \text{ if } m=p-1.\end{cases}$$

THEOREM 3.1. *Under the assumptions* $[H_1]$-$[H_3]$ *and* $\sum_{i=1}^n c_i^0<\frac{1}{2}$, *Eq.* (1.7) *has at least one T-periodic solution.*

Proof. Let $\Omega_1=\{x: Lx=\lambda N(x,\lambda),\lambda\in(0,1)\}$. If $x(\cdot)=(u(\cdot),v(\cdot))^\top\in\Omega_1$, then one can obtain from (3.2) and (3.4) that

$$\begin{cases} u'(t)=\lambda\left[A^{-1}(\varphi_q(v(\cdot)))\right](t), \\ v'(t)=\lambda f\left(\lambda\left[A^{-1}(\varphi_q(v(\cdot)))\right](t)\right)+\lambda\beta(t)g(u(t-\gamma(t)))+\lambda e(t). \end{cases} \tag{3.8}$$

From the first equation of (3.8), we have $v(t)=\varphi_p\left(\frac{1}{\lambda}[Au'](t)\right)$, which together with the second equation of (3.8) yields

$$\left[\varphi_p\left([Au'](t)\right)\right]'=\lambda^p f\left(u'(t)\right)+\lambda^p\beta(t)g(u(t-\gamma(t)))+\lambda^p e(t). \tag{3.9}$$

Integrating two sides of Eq.(3.9) on the interval $[0,T]$,

$$\int_0^T f\left(u'(t)\right)dt+\int_0^T\beta(t)g(u(t-\gamma(t)))dt=0. \tag{3.10}$$

Let $\beta^+(t)=\max\{\beta(t),0\},\beta^-(t)=\max\{-\beta(t),0\}$, so $\beta(t)=\beta^+(t)-\beta^-(t)$, furthermore,

$$\int_0^T\beta^+(t)dt\geqslant 0,\int_0^T\beta^-(t)dt\geqslant 0 \text{ and } \int_0^T\beta(t)dt=\int_0^T\beta^+(t)dt-\int_0^T\beta^-(t)dt.$$

Since $\int_0^T\beta(t)dt\neq 0$, $\int_0^T\beta^+(t)dt$ and $\int_0^T\beta^-(t)dt$ need not be all vanish, without loss of generality, supposing $\int_0^T\beta^-(t)dt>0$ (The case of $\int_0^T\beta^+(t)dt>0$ is shown as Remark 3.2). Note that (3.10) implies

$$\int_0^T\beta^-(t)g(u(t-\gamma(t)))dt=\int_0^T f\left(u'(t)\right)dt+\int_0^T\beta^+(t)g(u(t-\gamma(t)))dt.$$

Applying the integrating mean theorem, one can see that there exists a constant $\xi\in[0,T]$ such that

$$g(u(\xi-\gamma(\xi)))\int_0^T\beta^-(t)dt=\int_0^T f\left(u'(t)\right)dt+\int_0^T\beta^+(t)g(u(t-\gamma(t)))dt. \tag{3.11}$$

Now we claim that there exist constants D and E such that

$$|u(\xi-\gamma(\xi))|\leqslant D\left(\int_0^T|u'(t)|^m dt\right)^{\frac{1}{m}}+E|u|_0, \tag{3.12}$$

where

$$D=\begin{cases}\left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}2^{\frac{1-m}{m}}, & 0<m\leqslant 1,\\ \left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}, & m>1.\end{cases}\quad E=\begin{cases}\left[\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}2^{\frac{1-m}{m}}, & 0<m\leqslant 1,\\ \left[\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}, & m>1.\end{cases}$$

Case 1: If $u(\xi-\gamma(\xi))=0$, then (3.12) holds.

Case 2: If $|u(\xi-\gamma(\xi))|>0$, let $I_1=\{t\in[0,T]:u(t-\gamma(t))=0\}, I_2=\{t\in[0,T]: |u(t-\gamma(t))|>0\}$. By (3.11) and assumption $[H_1]$-$[H_2]$ one has $g(0)=0$ and

$$\begin{aligned}
&\ell_1|u(\xi-\gamma(\xi))|^m\int_0^T\beta^-(t)dt\\
&\quad\leqslant |g(u(\xi-\gamma(\xi))|\int_0^T\beta^-(t)dt\\
&\quad\leqslant \int_0^T|f\left(u'(t)\right)|dt+\left(\int_{I_1}+\int_{I_2}\right)\beta^+(t)|g(u(t-\gamma(t)))|dt\\
&\quad\leqslant r\int_0^T|u'(t)|^m dt+\ell_2|u|_0^m\int_0^T\beta^+(t)dt.
\end{aligned}$$

If $0<m\leqslant 1$, then by Jensen inequality, one has

$$\begin{aligned}
|u(\xi-\gamma(\xi))| &\leqslant \left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\int_0^T|u'(t)|^m dt+\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}|u|_0^m\right]^{\frac{1}{m}}\\
&= 2^{\frac{1}{m}}\left[\frac{1}{2}\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\int_0^T|u'(t)|^m dt+\frac{1}{2}\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}|u|_0^m\right]^{\frac{1}{m}}\\
&\leqslant 2^{\frac{1}{m}}\left[\frac{1}{2}\left(\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\int_0^T|u'(t)|^m dt\right)^{\frac{1}{m}}+\frac{1}{2}\left(\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}|u|_0^m\right)^{\frac{1}{m}}\right]\\
&= \left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}2^{\frac{1-m}{m}}\left(\int_0^T|u'(t)|^m dt\right)^{\frac{1}{m}}\\
&\qquad\qquad+\left[\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}2^{\frac{1-m}{m}}|u|_0.
\end{aligned}$$

If $m>1$, then by $(a+b)^{\frac{1}{m}}\leqslant a^{\frac{1}{m}}+b^{\frac{1}{m}}, a,b\in[0,+\infty), m>1$, one has

$$|u(\xi-\gamma(\xi))|\leqslant\left[\frac{r}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}\left(\int_0^T|u'(t)|^m dt\right)^{\frac{1}{m}}+\left[\frac{\ell_2\int_0^T\beta^+(t)dt}{\ell_1\int_0^T\beta^-(t)dt}\right]^{\frac{1}{m}}|u|_0.$$

It is easy to see that (3.12) holds.

Let $\xi-\gamma(\xi)=kT+\overline{\xi}$, where k is an integer and $\overline{\xi}\in[0,T]$, note that (3.12) implies

$$|u(\overline{\xi})|=|u(\xi-\gamma(\xi))|\leqslant D\left(\int_0^T|u'(t)|^m dt\right)^{\frac{1}{m}}+E|u|_0.$$

Therefore,

$$\begin{aligned}
|u(t)| &= \left|u(\overline{\xi})+\int_{\overline{\xi}}^t u'(s)ds\right|\leqslant|u(\overline{\xi})|+\left|\int_{\overline{\xi}}^t|u'(s)|ds\right|\\
&\leqslant D\left(\int_0^T|u'(t)|^m dt\right)^{\frac{1}{m}}+E|u|_0+\int_0^T|u'(t)|dt,\quad\text{for all } t\in[0,T],
\end{aligned}$$

that is

$$|u|_0 \leqslant D\left(\int_0^T |u'(t)|^m dt\right)^{\frac{1}{m}} + E|u|_0 + \int_0^T |u'(t)|dt,$$

then by $E < 1$, one obtains

$$|u|_0 \leqslant \frac{D}{1-E}\left(\int_0^T |u'(t)|^m dt\right)^{\frac{1}{m}} + \frac{1}{1-E}\int_0^T |u'(t)|dt. \tag{3.13}$$

Since

$$\begin{aligned}
\int_0^T [\varphi_p([Au'](t))]'u(t)dt &= u(t)\varphi_p([Au'](t))|_0^T - \int_0^T \varphi_p([Au'](t))u'(t)dt \\
&= -\int_0^T \varphi_p([Au'](t))\Big[[Au'](t) + \sum_{i=1}^n c_i(t)u'(t-r_i)\Big]dt \\
&= -\int_0^T \big|[Au'](t)\big|^p dt - \int_0^T \varphi_p([Au'](t))\sum_{i=1}^n c_i(t)u'(t-r_i)dt,
\end{aligned}$$

which together with Eq.(3.9) yields

$$\begin{aligned}
&\int_0^T \big|[Au'](t)\big|^p dt \\
&\quad= -\int_0^T [\varphi_p([Au'](t))]'u(t)dt - \int_0^T \varphi_p([Au'](t))\sum_{i=1}^n c_i(t)u'(t-r_i)dt \\
&\quad= -\int_0^T \varphi_p([Au'](t))\sum_{i=1}^n c_i(t)u'(t-r_i)dt \\
&\qquad\qquad - \lambda^p\int_0^T [f(u'(t)) + \beta(t)g(u(t-\gamma(t))) + e(t)]u(t)dt \\
&\quad\leqslant \sum_{i=1}^n c_i^0\left(\int_0^T |\varphi_p([Au'](t))|^q dt\right)^{\frac{1}{q}}\left(\int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}} \\
&\qquad\qquad + \left[r\int_0^T |u'(t)|^m dt + |\beta|_0\ell_2 T|u|_0^m + |e|_0 T\right]|u|_0 \\
&\quad\leqslant \sum_{i=1}^n c_i^0\left(\int_0^T \left|u'(t) - \sum_{i=1}^n c_i(t)u'(t-r_i)\right|^p dt\right)^{\frac{1}{q}}\left(\int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}} \\
&\qquad\qquad + \left(r\int_0^T |u'(t)|^m dt + |e|_0 T\right)\left[\frac{D}{1-E}\left(\int_0^T |u'(t)|^m dt\right)^{\frac{1}{m}}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left. + \frac{1}{1-E}\int_0^T |u'(t)|dt\right] \\
&\qquad\qquad + |\beta|_0\ell_2 T\left[\frac{D}{1-E}\left(\int_0^T |u'(t)|^m dt\right)^{\frac{1}{m}} + \frac{1}{1-E}\int_0^T |u'(t)|dt\right]^{m+1}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n} c_i^0 \left[\left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=1}^{n} c_i^0 \left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}} \right]^{\frac{p}{q}} \left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}} \\
&\qquad + \left[\frac{rD}{1-E} T^{\frac{(p-m)(m+1)}{mp}} + \frac{r}{1-E} T^{2-\frac{m+1}{p}} \right] \left(\int_0^T |u'(t)|^p dt \right)^{\frac{m+1}{p}} \\
&\qquad + |\beta|_0 \ell_2 T 2^m \left[\Big(\frac{D}{1-E} \Big)^{m+1} T^{\frac{(p-m)(m+1)}{mp}} \right. \\
&\qquad\qquad \left. + \Big(\frac{1}{1-E} \Big)^{m+1} T^{\frac{(p-1)(m+1)}{p}} \right] \left(\int_0^T |u'(t)|^p dt \right)^{\frac{m+1}{p}} \\
&\qquad + \frac{|e|_0 TD}{1-E} \left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}} T^{\frac{p-m}{mp}} + \frac{|e|_0 T}{1-E} \left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}} T^{1-\frac{1}{p}} \\
&\leqslant K_1 \int_0^T |u'(t)|^p dt + K_2 \left(\int_0^T |u'(t)|^p dt \right)^{\frac{m+1}{p}} + K_3 \left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}},
\end{aligned}
$$

where

$$
\frac{1}{p} + \frac{1}{q} = 1, \;\; K_1 = \sum_{i=1}^{n} c_i^0 \Big(1 + \sum_{i=1}^{n} c_i^0 \Big)^{\frac{p}{q}}, \; K_3 = \frac{|e|_0 D T^{1+\frac{p-m}{mp}}}{1-E} + \frac{|e|_0 T^{2-\frac{1}{p}}}{1-E}
$$

and

$$
K_2 = \frac{rDT^{\frac{(p-m)(m+1)}{mp}}}{1-E} + \frac{rT^{2-\frac{m+1}{p}}}{1-E} + |\beta|_0 \ell_2 T 2^m \left[\Big(\frac{DT^{\frac{p-m}{mp}}}{1-E} \Big)^{m+1} + \Big(\frac{T^{\frac{p-1}{p}}}{1-E} \Big)^{m+1} \right].
$$

Then by Lemma 2.1(3) one gets

$$
\begin{aligned}
\int_0^T |u'(t)|^p dt &= \int_0^T |[A^{-1}Au'](t)|^p dt \\
&\leqslant \frac{1}{\big(1 - \sum_{i=1}^{n} c_i^0\big)^p} \int_0^T |[Au'](t)|^p dt \\
&\leqslant \frac{1}{\big(1 - \sum_{i=1}^{n} c_i^0\big)^p} \left[K_1 \int_0^T |u'(t)|^p dt \right. \\
&\qquad \left. + K_2 \Big(\int_0^T |u'(t)|^p dt \Big)^{\frac{m+1}{p}} + K_3 \Big(\int_0^T |u'(t)|^p dt \Big)^{\frac{1}{p}} \right].
\end{aligned}
$$

Therefore, it follows from $m \leqslant p-1$ and $[H_3]$ that $\int_0^T |u'(t)|^p dt$ is bounded, which means there exists a positive constant M_0 (independent of λ) such that

$$
\int_0^T |u'(t)|^p dt \leqslant M_0. \tag{3.14}
$$

It follows from (3.13) and (3.14) that

$$\begin{aligned}|u|_0 &\leqslant \frac{D}{1-E}\left(\int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}} T^{\frac{p-m}{pm}} + \frac{1}{1-E}\left(\int_0^T |u'(t)|^p dt\right)^{\frac{1}{p}} T^{\frac{p-1}{p}}\\ &\leqslant \frac{D}{1-E} M_0^{\frac{1}{p}} T^{\frac{p-m}{pm}} + \frac{1}{1-E} M_0^{\frac{1}{p}} T^{\frac{p-1}{p}} := M_1.\end{aligned}$$

From the first equation of (3.8) again that

$$\int_0^T \left[A^{-1}\left(\varphi_q(v(\cdot))\right)\right](t)dt = 0,$$

this implies that there exists a constant $\eta \in [0,T]$ such that $A^{-1}(\varphi_q(v(\eta))) = 0,$ which together with the first part of Lemma 2.1,

$$\varphi_q(v(\eta)) + \sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n}\prod_{j=1}^{m} c_{i_j}\left(\eta - \sum_{k=j+1}^{m} r_{i_k}\right)\varphi_q\left(v\left(\eta - \sum_{s=1}^{m} r_{i_s}\right)\right) = 0,$$

therefore

$$|v(\eta)|^{q-1} = |\varphi_q(v(\eta))| \leqslant \sum_{m=1}^{\infty}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n}\prod_{j=1}^{m} c_{i_j}^0 |v|_0^{q-1} \leqslant \frac{\sum\limits_{i=1}^{n} c_i^0}{1-\sum\limits_{i=1}^{n} c_i^0}|v|_0^{q-1},$$

that is

$$|v(\eta)| \leqslant \left[\frac{\sum\limits_{i=1}^{n} c_i^0}{1-\sum\limits_{i=1}^{n} c_i^0}\right]^{\frac{1}{q-1}} |v|_0. \tag{3.15}$$

It follows from the second equation of Eq.(3.8) that

$$\begin{aligned}\int_0^T |v'(t)|dt &\leqslant \int_0^T \left|f\left(u'(t)\right)\right|dt + \int_0^T |\beta(t)g(u(t-\gamma(t)))|dt + \int_0^T |e(t)|dt\\ &\leqslant r\left(\int_0^T |u'(t)|^p dt\right)^{\frac{m}{p}} T^{1-\frac{m}{p}} + |\beta|_0 g_M T + |e|_0 T\\ &\leqslant r M_0^{\frac{m}{p}} T^{1-\frac{m}{p}} + |\beta|_0 g_M T + |e|_0 T := \widetilde{M_0},\end{aligned}$$

where $g_M = \max\limits_{|u|\leqslant M_1} |g(u)|$, together with (3.15) one obtains

$$|v(t)| \leqslant |v(\eta)| + \int_0^T |v'(t)|dt \leqslant \left[\frac{\sum\limits_{i=1}^{n} c_i^0}{1-\sum\limits_{i=1}^{n} c_i^0}\right]^{\frac{1}{q-1}} |v|_0 + \widetilde{M_0}.$$

It follows from $\sum\limits_{i=1}^{n} c_i^0 < \frac{1}{2}$ that $\left[\frac{\sum_{i=1}^{n} c_i^0}{1-\sum_{i=1}^{n} c_i^0}\right]^{\frac{1}{q-1}} < 1$, therefore there exists a positive constant M_2 such that

$$|v|_0 \leqslant M_2.$$

Let $\Omega_2 = \{x : x \in \ker L, QN(x,0) = 0\}$. If $x(\cdot) = (x_1(\cdot), x_2(\cdot))^\top \in \Omega_2$, then x is a constant vector, and

$$QN(x,0) = \begin{pmatrix} \frac{1}{T}\int_0^T \left[A^{-1}\left(\varphi_q(x_2)\right)\right](t)dt \\ \frac{1}{T}\int_0^T [f(0)+\beta(t)g(x_1)+e(t)]dt \end{pmatrix} = 0. \tag{3.16}$$

From assumption $[H_1]$ we know $f(0) = 0$, which together with $\int_0^T e(t)dt = 0$ we have

$$\begin{cases} \frac{1}{T}\int_0^T \left[A^{-1}\left(\varphi_q(x_2)\right)\right](t)dt = 0, \\ \frac{g(x_1)}{T}\int_0^T \beta(t)dt = 0. \end{cases} \tag{3.17}$$

In view of $\int_0^T \beta(t)dt \neq 0$ and the second formula of (3.17), we know $g(x_1) = 0$, which together with $[H_2]$ we obtain $x_1 = 0$. Moreover, it follows from the first formula of (3.17) that $\frac{1}{T}\int_0^T A^{-1}\big(\varphi_q(x_2)\big)dt = 0$, by Lemma 2.6 we obtain $\varphi_q(x_2) = 0$, that is $x_2 = 0$. Therefore $\Omega_2 \subset \Omega_1$.

Now, if we set $\Omega = \{x : x = (u,v)^\top \in X, |u|_0 < M_1 + 1, |v|_0 < M_2 + 1\}$, then $\Omega_2 \subset \Omega_1 \subset \Omega$. So conditions (1) and (2) of Lemma 3.1 are satisfied. Now we prove that condition (3) of Lemma 3.1 is satisfied.

For all $x \in \overline{\Omega} \cap \ker L, \lambda \in [0,1]$, defining

$$H(x,\lambda) = \begin{cases} \lambda JQN(x,0) + (1-\lambda)x, & \text{if } \int_0^T \beta(t)dt > 0, \\ \lambda JQN(x,0) - (1-\lambda)x, & \text{if } \int_0^T \beta(t)dt < 0, \end{cases}$$

where $J : \operatorname{Im} Q \to \ker L$ is a homeomorphism with $J(x_1,x_2) = (x_2,x_1)$. Hence

$$H(x,\lambda) \neq 0, \quad \text{for all } (x,\lambda) \in \partial\Omega \cap \ker L \times [0,1].$$

By the degree theory, one has

$$\begin{aligned} \deg\{JQN(\cdot,0), \Omega \cap \ker L, 0\} &= \deg\{H(\cdot,1), \Omega \cap \ker L, 0\} \\ &= \deg\{H(\cdot,0), \Omega \cap \ker L, 0\} \\ &= \deg\{\pm I, \Omega \cap \ker L, 0\} \neq 0. \end{aligned}$$

Applying Lemma 3.1, proof is completed. □

REMARK 3.1. Noting the definitions of L and $N(\cdot,\cdot)$, one can obtain $\ker L = R^2$ and $g(x_1)$ is a constant, when $x = (x_1,x_2)^\top \in \ker L$. Therefore, it is easy to obtain (3.17) from (3.16), and this also implies that it is easy to obtain the solutions of $QN(x,0)=0$. If one defines $L=(Ax_1',x_2')^\top$, then the priori estimates of $QN(x,0)=0$ will be very difficult, furthermore, it may be difficult to obtain the existence of periodic solutions to Eq.(1.7).

REMARK 3.2. If $\int_0^T \beta^+(t)dt>0$, then it follows from (3.10) that

$$\int_0^T \beta^+(t)g(u(t-\gamma(t)))dt=\int_0^T \beta^-(t)g(u(t-\gamma(t)))dt-\int_0^T f\left(u'(t)\right)dt.$$

By the integrating mean theorem, we also know that there exists a constant $\vartheta\in[0,T]$ such that

$$g(u(\vartheta-\gamma(\vartheta)))\int_0^T \beta^+(t)dt=\int_0^T \beta^-(t)g(u(t-\gamma(t)))dt-\int_0^T f\left(u'(t)\right)dt.$$

The remainder is similarly to Theorem 3.1. Therefore Theorem 3.1 only needs one of $\int_0^T \beta^+(t)dt$ and $\int_0^T \beta^-(t)dt$ is positive.

4. Numerical simulation

EXAMPLE 4.1. Considering the following equation:

$$\left[\varphi_5\left(u'(t)-\left(\frac{1}{12}\cos t\right)u'(t-5)-\left(\frac{1}{12}\sin t\right)u'(t-3)\right)\right]' \\ =5(u'(t))^3+3\left(\sin t-\frac{\sqrt{3}}{2}\right)(u(t-\cos t))^3+\cos t. \quad (4.1)$$

Corresponding to Eq.(1.7), we have $p=5$, $T=2\pi$, $f(x)=5x^3$, $g(x)=3x^3$, $r=5$, $\ell_1=2$, $\ell_2=4$, $m=3$, $c_1(t)=\frac{1}{12}\cos t$, $c_2(t)=\frac{1}{12}\sin t$, $\beta(t)=\sin t-\frac{\sqrt{3}}{2}$, $\gamma(t)=\cos t$, $e(t)=\cos t$. Moreover, $\sum_{i=1}^{2} c_i^0=\frac{1}{6}<\frac{1}{2}$ and it is easy to verify that $[H_1]$ to $[H_3]$ all hold. Thus by applying Theorem 3.1, we know that Eq.(4.1) has at least one $2\pi-$periodic solution.

REMARK 4.1. It is easy to see that the conclusion of Example 4.1 can not be obtained by references [1, 2, 3, 4, 5, 6, 7, 8], and what is more important is that the properties of time-varying operator A improve the results in previous literatures [3, 4, 5, 7, 8].

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