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POSITIVE WEAK RADIAL SOLUTIONS OF NONLINEAR SYSTEMS WITH p–LAPLACIAN

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Abstract. We study the existence and localization results for radial solutions of systems with p-Laplacian. The compression and expansion conditions that are used are related to the first eigenvalue of the p-Laplacian.

1. Introduction

Many authors studied different problems concerning the p-Laplacian equations. For example, in [1] it is studied the existence and multiplicity of positive solutions for the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = Kg(u)+\lambda h(u)+f \ \text{ in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \\ u>0 \quad \text{in } \Omega, \end{cases}$$

where $\lambda > 0, 1 < p < \infty$, assuming that $g : (0,\infty) \to (0,\infty)$ is a continuous nonincreasing function (that may be singular at the origin), $h : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function, K and f are nonnegative functions defined on Ω which satisfy that K is non identically zero, $K \in L^\infty(\Omega)$ and $f \in C(\overline{\Omega})$. The solution is understood in a weak sense, and for the proof of the main results some comparison results for p-Laplacian are used. In [11] it is treated the non-existence of positive radially symmetric solutions for the following p-Laplacian boundary value problem:

$$\begin{cases} \left(|u'|^{p-1} u'\right)' + f\left(t,u(t)\right) = 0, \ a<t<b, \ p>1, \\ u(a)-B_0(u'(a)) = 0, \\ u(b)+B_1(u'(b)) = 0, \end{cases}$$

where $f \in C\left([a,b]\times[0,\infty);(0,\infty)\right)$, $B_0(v)$ and $B_1(v)$ are both increasing, continuous, odd functions defined on $\mathbb{R}$ and there exists a $\theta > 0$ such that

$$0 \leqslant B_i(v) \leqslant \theta v \ \text{ for all } v \geqslant 0 \ \text{ and for } i=0 \text{ or } i=1.$$

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In this paper we are concerned with a boundary value problem for a system of equations with p-Laplacian ($p \geqslant 2$):

$$\begin{cases} -\operatorname{div}(|\nabla u|_s^{p-2}\nabla u) = f(u) \ \text{ for } |x| < T, \\ u > 0 \quad \text{for } 0 < |x| < T, \\ u = 0 \quad \text{for } x = 0, \\ \nabla u = 0 \ \text{ for } |x| = T. \end{cases} \tag{1.1}$$

Here $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$, $u = \big(u_1(x), u_2(x), ..., u_n(x)\big)$ and let

$$|x|_e = \sqrt{\sum_{j=1}^{N} x_j^2} = r$$

be the euclidian norm. Also $\nabla u = \big(\nabla u_1, \nabla u_2, ..., \nabla u_n\big)$ is the gradient of u in the following sense:

$$\nabla u_i = \Big(\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, ..., \frac{\partial u_i}{\partial x_N}\Big) \ \text{ and } \ |\nabla u\,|_s = \sum_{i=1}^{n} |\nabla u_i|_e,$$

and

$$\operatorname{div}(|\nabla u|_s^{p-2}\nabla u) \\ = \big(\operatorname{div}(|\nabla u\,|_s^{p-2}\,\nabla u_1), \operatorname{div}(|\nabla u|_s^{p-2}\nabla u_2), ..., \operatorname{div}(|\nabla u|_s^{p-2}\nabla u_n)\big).$$

Searching a solution $u(x) = v(|x|_e)$, a radial solution of (1.1) can be considered as a solution of the problem (as we will see from the following arguments):

$$\begin{cases} \Big[\big(\sum\limits_{i=1}^{n} |v_i'(r)\,|\,\big)^{p-2} v'(r)\Big]' + \frac{N-1}{r}\Big[\big(\sum\limits_{i=1}^{n} |v_i'(r)|\big)^{p-2} v'(r)\Big] = -f(v), \ 0 < r < T, \\ v'(T) = v(0) = 0, \\ v > 0 \ \text{ on } (0,T). \end{cases} \tag{1.2}$$

Let $g := |\nabla u|_s^{p-2}\nabla u$. We have:

$$\frac{\partial u_i(x)}{\partial x_j} = \frac{\partial v_i(|x\,|_e)}{\partial x_j} = \frac{\partial v_i(|x|_e)}{\partial\,|\,x|_e}\frac{\partial |x|_e}{\partial x_j} = v_i'(r)\frac{x_j}{|x|_e} = \frac{v_i'(r)}{r}x_j.$$

This implies that

$$|\nabla u_i|_e = \sqrt{\sum_{j=1}^{N}(\frac{\partial u_i}{\partial x_j})^2} = \sqrt{\Big[\frac{v_i'(r)}{r}\Big]^2\sum_{j=1}^{N} x_j^2} = \sqrt{\Big[\frac{v_i'(r)}{r}\Big]^2 r^2} = |v_i'(r)|,$$

so we have

$$|\nabla u|_s = \sum_{i=1}^{n} |v_i'(r)|.$$

Then $g = (\sum_{i=1}^{n} |v_i'(r)|)^{p-2}\big(\nabla u_1, \nabla u_2, ..., \nabla u_n\big)$. Denote

$$\begin{aligned} g_i(x) &:= (\sum_{i=1}^{n} |v_i'(r)\ |)^{p-2} \nabla u_i \\ \operatorname{div} g &= \big(\operatorname{div} g_1, \operatorname{div} g_2, ..., \operatorname{div} g_n\big) \\ h_j(x) &:= (\sum_{i=1}^{n} |v_i'(r)|)^{p-2} \frac{\partial u_i}{\partial x_j}. \end{aligned}$$

We have that

$$\operatorname{div} g_i(x) = \sum_{j=1}^{N} \frac{\partial h_j}{\partial x_j},$$

and

$$\frac{\partial h_j}{\partial x_j} = (p-2)\Big(\sum_{i=1}^{n} |v_i'(r)|\Big)^{p-3} \frac{\partial \big(\sum_{i=1}^{n} |v_i'(r)|\big)}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \Big(\sum_{i=1}^{n} |\ v_i'(r)|\Big)^{p-2} \frac{\partial^2 u_i}{\partial x_j^2}.$$

Also, we have

$$\frac{\partial |v_i'(r)|}{\partial x_j} = \frac{\partial |v_i'(r)|}{\partial r} \frac{\partial r}{\partial x_j} = \frac{\partial |v_i'(r)|}{\partial r} \frac{x_j}{r},$$

and since

$$\frac{\partial u_i(x)}{\partial x_j} = v_i'(r) \frac{x_j}{r},$$

we have

$$\begin{aligned} \frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial v_i'(r)}{\partial x_j} \frac{x_j}{r} + v_i'(r) \frac{\partial}{\partial x_j}\left(\frac{x_j}{r}\right) &= \frac{\partial v_i'(r)}{\partial r} \frac{\partial r}{\partial x_j} \frac{x_j}{r} + v_i'(r) \frac{r - \frac{x_j^2}{r}}{r^2} \\ &= v_i''(r) \frac{x_j^2}{r^2} + v_i'(r) \frac{r^2 - x_j^2}{r^3}. \end{aligned}$$

These imply that

$$\frac{\partial h_j}{\partial x_j} = (p-2)\Big(\sum_{i=1}^{n} |v_i'(r)|\Big)^{p-3} \frac{\partial (\sum_{i=1}^{n} |v_i'(r)|)}{\partial r} v_i'(r) \frac{x_j^2}{r^2} + \Big(\sum_{i=1}^{n} |v_i'(r)|\Big)^{p-2} \left[v_i''(r) \frac{x_j^2}{r^2} + v_i'(r) \frac{r^2 - x_j^2}{r^3}\right]$$

and

$$\sum_{j=1}^{N}\frac{\partial h_j}{\partial x_j}=(p-2)\Big(\sum_{i=1}^{n}|v_i'(r)|\Big)^{p-3}\frac{\partial(\sum_{i=1}^{n}|v_i'(r)|)}{\partial r}v_i'(r)\frac{\sum_{j=1}^{N}x_j^2}{r^2}+$$
$$+\Big(\sum_{i=1}^{n}|v_i'(r)|\Big)^{p-2}\Big[v_i''(r)\frac{\sum_{j=1}^{N}x_j^2}{r^2}+v_i'(r)\frac{Nr^2-\sum_{j=1}^{N}x_j^2}{r^3}\Big].$$

Hence we have

$$\sum_{j=1}^{N}\frac{\partial h_j}{\partial x_j}=(p-2)\Big(\sum_{i=1}^{n}|v_i'(r)|\Big)^{p-3}\frac{\partial(\sum_{i=1}^{n}|v_i'(r)|)}{\partial r}v_i'(r)$$
$$+\Big(\sum_{i=1}^{n}|v_i'(r)|\Big)^{p-2}\Big[v_i''(r)+\frac{N-1}{r}v_i'(r)\Big],$$

$$\sum_{j=1}^{N}\frac{\partial h_j}{\partial x_j}=(|\nabla u|_s^{p-2})'v_i'(r)+|\nabla u|_s^{p-2}v_i''(r)+\frac{N-1}{r}|\nabla u|_s^{p-2}v_i'(r).$$

and

$$\operatorname{div}g_i=\sum_{j=1}^{N}\frac{\partial h_j}{\partial x_j}=\big[|\nabla u|_s^{p-2}v_i'(r)\big]'+\frac{N-1}{r}\big[|\nabla u|_s^{p-2}v_i'(r)\big].$$

So (1.1) reduces to (1.2). We will put the problem (1.2) in an equivalent form, as a problem of fixed point.

We make the substitution $w_i(r)=|\nabla u|_s^{p-2}v_i'(r)$, $w=\big(w_1,w_2,...,w_n\big)$. Let $SF:=u_F$ be the unique solution of the problem:

$$\begin{cases}-(|\nabla u|_s^{p-2}v'(r))'=F(r) \text{ for a.e. } r\in[0,T],\\ v(0)=v'(T)=0,\end{cases}\tag{1.3}$$

and

$$S:L^1([0,T];R^n)\to C^1([0,T];R^n),\ \ (SF)(r)=\int_0^r\phi\Big(\int_s^T F(\tau)d\tau\Big)ds$$

with

$$\phi(x)=\begin{cases}|x|^{-\frac{p-2}{p-1}}x & \text{for } x\neq 0,\\ 0 & \text{for } x=0.\end{cases}$$

Since

$$-\big(|\nabla u|_s^{p-2}v'(r)\big)'=-w'(r),$$

we find that $u_{-w'}(r)=(S(-w'))(r)$. Then (1.1) becomes:

$$w_i'(r)+\frac{N-1}{r}w_i(r)=q_i(r):=-f_i(v(r)) \text{ for all } i\in\{1,...,n\}.\tag{1.4}$$

We attach the homogenous equation

$$w_i'(r) + \frac{N-1}{r} w_i(r) = 0$$

whose solution is $r^{1-N}C$.

We search a particular solution of (1.4), in the form $(w_i)_p(r) = C(r)r^{1-N}$. Then

$$(w_i)_p'(r) = C'(r)r^{1-N} - (N-1)C(r)r^{-N} \quad \text{and} \quad C'(r) = r^{N-1}q_i(r),$$

so we find

$$C(r) = \int_{r_0}^{r} t^{N-1} q_i(t)dt \quad \text{and} \quad (w_i)_p(r) = r^{1-N} \int_{r_0}^{r} t^{N-1} q_i(t)dt,$$

and

$$(w_i)(r) = r^{1-N}\left(C + \int_{r_0}^{r} t^{N-1} q_i(t)dt\right)$$

is the general solution of equation (1.4). We have

$$w_i(T) = (\sum_{i=1}^{n} |v_i'(T)|)^{p-2} v'(T) = 0$$

since $v'(T) = 0$. This implies

$$C = -\int_{r_0}^{T} t^{N-1} q_i(t)dt$$

and

$$\begin{aligned}
(w_i)(r) &= -r^{1-N} \int_{r}^{T} t^{N-1} q_i(t)dt = r^{1-N} \int_{T}^{r} t^{N-1} q_i(t)dt, \\
(w_i)'(r) &= r^{1-N} r^{N-1} q_i(r) + (1-N)r^{-N} \int_{T}^{r} t^{N-1} q_i(t)dt \\
&= -f_i(v(r)) - (N-1)r^{-N} \int_{r}^{T} t^{N-1} f_i(v(t))dt
\end{aligned}$$

and

$$(-w_i)'(r) = f_i(v(r)) + (N-1)r^{-N} \int_{r}^{T} t^{N-1} f_i(v(t))dt > 0. \tag{1.5}$$

Define now the operator

$$(Hu)(r) := f(v(r)) + (N-1)r^{-N} \int_{r}^{T} t^{N-1} f(v(t))dt > 0.$$

The problem (1.2) becomes the fixed point problem

$$u = (SH)(u)$$

Let us now present some properties of the operator S.

LEMMA 1.1. (O'Regan and Precup [7])

1^0 *If* $F \in L^1([0,T];\mathbb{R}^n_+)$ *and* $u = SF$, $u = (u_1, u_2, ..., u_n)$, *then for every* j, *the function* u_j *is nonnegative and nondecreasing, and the function* $\mid u(t)|_s = \sum_{j=1}^n u_j(t)$ *is nonnegative, nondecreasing and concave.*

2^0 $S(\lambda F) = \lambda^{\frac{1}{p-1}} SF$ *for all* $F \in L^1([0,T];R^n)$ *and* $\lambda > 0$.

3^0 *If* $0 \leqslant F_1 \leqslant F_2$, *then* $|(SF_1)(t)|_s \leqslant |(SF_2)(t)|_s$ *for all* $t \in [0,T]$.

Now we present the fixed point theorem that we use for our main results.

THEOREM 1.1. (Krasnoselskii [4]) *Let* $(X, |.|)$ *be a normed linear space,* $C \subset X$ *a cone,* $\leqslant$ *the partial order relation induced by* C, $0 < r < R$ *and* $C_{r,R} = \{x \in C : r \leqslant |x| \leqslant R\}$. *Assume that* $N : C_{r,R} \to C$ *is a compact map and one of the following conditions is satisfied:*

(a) $x \nleq Nx$ *for* $|x| = r$ *and* $Nx \nleq x$ *for* $|x| = R$;

(b) $x \nleq Nx$ *for* $|x| = R$ *and* $Nx \nleq x$ *for* $|x| = r$.

Then N *has a fixed point* x *with* $r < |x| < R$.

2. Positive solutions of problem (1.1)

Let us denote by

$$|u|_\infty = \max_{|y|_e \in [0,T]} |v(|y|_e)|_s = \max_{t \in [0,T]} \sum_{j=1}^n |v_j(t)|_e .$$

We search the fixed point of SH in a cone C of X, namely

$$C := \left\{u \in C\left([0,T];\mathbb{R}^n_+\right) : u(0) = 0, |u(t)|_s \geqslant \frac{t}{T}|u|_\infty \text{ for all } t \in [0,T]\right\}.$$

From Lemma 1.1, we have for each $F \in L^1([0,T];\mathbb{R}^n_+)$,
$(SF)(0) = 0, |SF|_\infty = (SF)(T)$ and $|SF|_s$ is a concave function. Consequently

$$|\,(SF)\left(\frac{t}{T}T\right)|_s = |\,(SF)(t)|_s \geqslant \frac{t}{T}|SF|_\infty \text{ for all } t \in [0,T]. \tag{2.1}$$

Now, if $u \geqslant 0$, then by our form of H, we have $Hu \geqslant 0$, and so, by (2.1), $SHu \in C$. Therefore

$$SH(C) \subset C.$$

The Ascoli-Arzela theorem ensures that S is completely continuous from $L^1([0,T];R^n)$ to $C([0,T];R^n)$. Then, by our form of H, the operator SH is completely continuous from $C([0,T];R^n)$ to itself.

THEOREM 2.1. *Let $f \in C(\mathbb{R}^n_+;\mathbb{R}^n_+)$. Assume there are numbers $\alpha, \beta > 0$, $\alpha \neq \beta$ and functions $\varphi, \psi \in L^1([0,T];R_+)$ such that*

$$\frac{S_1}{\alpha^{p-1}} < \left[\int_0^T \left(\int_s^T \tau^{-N}\varphi(\tau)d\tau\right)^{\frac{1}{p-1}} ds\right]^{1-p}, \tag{2.2}$$

and

$$\frac{I_1}{\beta^{p-1}} > \left[\int_0^T \left(\int_s^T \psi(\tau)d\tau\right)^{\frac{1}{p-1}} ds\right]^{1-p}, \tag{2.3}$$

where

$$S_1 := \sup_{x\in\mathbb{R}^n_+,\ |x|\leqslant\alpha,\ r\in(0,T)} \frac{|f(x(r)) + (N-1)r^{-N}\int_r^T t^{N-1}f(x(t))dt|}{r^{-N}\varphi(r)},$$

$$I_1 := \inf_{x\in\mathbb{R}^n_+,\ |x|\in[\frac{r}{T}\beta,\beta],\ r\in(0,T)} \frac{|f(x(r)) + (N-1)r^{-N}\int_r^T t^{N-1}f(x(t))dt|}{\psi(r)}.$$

Here the norm $|.|$ is $|.|_s$.

Then (1.1) has at least one solution $u \in C$ with $|u|_s$ increasing, concave and

$$\min\{\alpha,\beta\} < |u|_\infty < \max\{\alpha,\beta\}.$$

REMARK 2.1. Notice the sup and inf in the above expresions are assumed to be essential, i.e., with respect to all $t \in (0,T)$ except a set of measure zero. We also make the convention that $\frac{f(x(r))}{\varphi(r)} = \infty$ if $\varphi(r) = 0$, and the same for $\frac{f(x(r))}{\psi(r)}$. Then S_1 excludes that φ be zero on a set of positive measure, while if $\psi(r) = 0$ for $t \in [a_0,b_0] \subset [0,T]$, then inf in I_1 will be taken over $t \in (0,T) \setminus [a_0,b_0]$.

Proof. From the assumptions on f we have $H : C([0,T];\mathbb{R}^n_+) \to L^1([0,T];\mathbb{R}^n_+)$ is well defined, continuous and bounded. We shall apply Krasnoselskii's fixed point theorem, Theorem 1.1. Let $u \in C$ with $|u|_\infty = \alpha$. We claim that $u \nleq (SH)(u)$. To prove this, assume the contrary, i.e. $u \leqslant (SH)(u)$. Then we deduce

$$\begin{aligned}
\alpha = |u|_\infty &\leqslant |(SH)(u)|_\infty \\
&= \int_0^T \left(\int_s^T \left|f\big(u(\tau)\big) + (N-1)\tau^{-N}\int_\tau^T t^{N-1}f\big(u(t)\big)dt\right|d\tau\right)^{\frac{1}{p-1}} ds \\
&\leqslant \left(\sup_{x\in\mathbb{R}^n_+,\ |x|\leqslant\alpha,\ r\in(0,T)} \frac{|f(x(r)) + (N-1)r^{-N}\int_r^T t^{N-1}f(x(t))dt|}{r^{-N}\varphi(r)}\right)^{\frac{1}{p-1}} \\
&\qquad\cdot \int_0^T \left(\int_s^T \tau^{-N}\varphi(\tau)d\tau\right)^{\frac{1}{p-1}} ds < \alpha,
\end{aligned}$$

which is a contradicition because of (2.2). So we have that $u \nleq (SH)(u)$.

Next we show that $u \ngeqslant SHu$ for every $u \in C$ with $|u|_\infty = \beta$. Assume the contrary, i.e. $u \geqslant SHu$. Then we obtain

$$\begin{aligned}
\beta = |u|_\infty \geqslant |SHu|_\infty = |\big(SHu\big)(T)|_s \\
&= \int_0^T \Big(\int_s^T |f\big(u(\tau)\big) + (N-1)\tau^{-N}\int_\tau^T t^{N-1} f\big(u(t)\big)dt|d\tau\Big)^{\frac{1}{p-1}} ds \\
&\geqslant \Bigg(\inf_{x\in\mathbb{R}^n_+,\ |x|\in\left[\frac{r}{T}\beta,\ \beta\right],\ r\in(0,T)} \frac{|f(x(r)) + (N-1)r^{-N}\int_r^T t^{N-1} f(x(t))dt|}{\psi(r)}\Bigg)^{\frac{1}{p-1}} \\
&\qquad \cdot \int_0^T \Big(\int_s^T \psi(\tau)d\tau\Big)^{\frac{1}{p-1}} ds > \beta,
\end{aligned}$$

which is a contradiction, according with (2.3). So $u \ngeqslant SHu$. Now we can apply Theorem 1.1.

REMARK 2.2. 1^0) If f is nondecreasing on $[0, \max\{\alpha,\beta\}]$, the condition (2.2) can be replaced with

$$\frac{T^N f(\alpha)}{\alpha^{p-1}} \frac{1}{\inf\limits_{r\in(0,T)} \varphi(r)} < \Big[\int_0^T \big(\int_s^T \tau^{-N}\varphi(\tau)d\tau\big)^{\frac{1}{p-1}} ds\Big]^{1-p} \tag{2.4}$$

for $\inf\limits_{r\in(0,T)} \varphi(r) > 0$. Indeed, because f is nondecreasing, we have that $f(x) \leqslant f(\alpha)$, for every $|x| \leqslant \alpha$, so (2.2) becomes

$$\begin{aligned}
S_2 := \sup_{r\in(0,T)} \frac{f(\alpha) + (N-1)r^{-N} f(\alpha)\int_r^T t^{N-1}dt}{r^{-N}\varphi(r)} \\
< \Big[\int_0^T \Big(\int_s^T \tau^{-N}\varphi(\tau)d\tau\Big)^{\frac{1}{p-1}} ds\Big]^{1-p} \alpha^{p-1},
\end{aligned}$$

and we have

$$\begin{aligned}
S_2 := f(\alpha) \sup_{r\in(0,T)} \frac{1 + (N-1)r^{-N}\frac{t^N}{N}\Big|_r^T}{r^{-N}\varphi(r)} \\
= f(\alpha) \sup_{r\in(0,T)} \frac{1 + (N-1)\frac{T^N - r^N}{N} r^{-N}}{r^{-N}\varphi(r)} \\
= f(\alpha) \sup_{r\in(0,T)} \frac{\Big[1 + \frac{N-1}{N}(T^N r^{-N} - 1)\Big] r^N}{\varphi(r)} \\
= f(\alpha) \sup_{r\in(0,T)} \frac{r^N + \frac{N-1}{N}(T^N - r^N)}{\varphi(r)}.
\end{aligned}$$

Let

$$\theta(r) := r^N + \frac{N-1}{N}(T^N - r^N) = r^N(1 - \frac{N-1}{N}) + \frac{N-1}{N}T^N$$
$$= \frac{r^N}{N} + \frac{N-1}{N}T^N \leqslant \frac{T^N}{N} + \frac{N-1}{N}T^N = T^N.$$

Then

$$S_2 \leqslant f(\alpha)T^N \sup_{r\in(0,T)} \frac{1}{\varphi(r)} = f(\alpha)T^N \frac{1}{\inf\limits_{r\in(0,T)} \varphi(r)},$$

so we can replace (2.2) by (2.4).

2^0) If in addition we suppose $\varphi \equiv 1,$ the condition (2.4) becomes

$$\frac{f(\alpha)T^N}{\alpha^{p-1}} < \Big[\int_0^T \Big(\int_s^T \tau^{-N} d\tau\Big)^{\frac{1}{p-1}} ds\Big]^{1-p} \tag{2.5}$$

and if in addition $p > N,$ (2.5) can be replaced by the suficient condition

$$\frac{f(\alpha)}{\alpha^{p-1}} < \Big(\frac{p-N}{p-1}\Big)^{p-1} \frac{N-1}{T^p}. \tag{2.6}$$

Indeed, the right side from (2.5) can be written

$$\Big[\int_0^T \Big(\frac{\tau^{-N+1}}{-N+1}\Big|_s^T\Big)^{\frac{1}{p-1}} ds\Big]^{1-p} = \Big[\int_0^T \Big(\frac{T^{1-N} - s^{1-N}}{1-N}\Big)^{\frac{1}{p-1}} ds\Big]^{1-p}$$
$$= \Big[\int_0^T \Big(\frac{s^{1-N} - T^{1-N}}{N-1}\Big)^{\frac{1}{p-1}} ds\Big]^{1-p} = (N-1)\Big[\int_0^T (s^{1-N} - T^{1-N})^{\frac{1}{p-1}} ds\Big]^{1-p}. \tag{2.7}$$

Since

$$\int_0^T (s^{1-N} - T^{1-N})^{\frac{1}{p-1}} ds \leqslant \int_0^T (s^{1-N})^{\frac{1}{p-1}} ds,$$

we obtain that

$$\Big[\int_0^T (s^{1-N})^{\frac{1}{p-1}} ds\Big]^{1-p} \leqslant \Big[\int_0^T (s^{1-N} - T^{1-N})^{\frac{1}{p-1}} ds\Big]^{1-p}, \tag{2.8}$$

with

$$\Big[\int_0^T (s^{1-N})^{\frac{1}{p-1}} ds\Big]^{1-p} = \left[\frac{s^{\frac{1-N}{p-1}+1}}{\frac{1-N}{p-1}+1}\Big|_0^T\right]^{1-p}$$
$$= \Big(\frac{p-1}{p-N}\Big)^{1-p} \Big(T^{\frac{p-N}{p-1}}\Big)^{1-p} = \Big(\frac{p-N}{p-1}\Big)^{p-1} \frac{1}{T^{p-N}}$$

and (2.8) becomes

$$\Big(\frac{p-N}{p-1}\Big)^{p-1} \frac{1}{T^{p-N}} \leqslant \Big[\int_0^T (s^{1-N} - T^{1-N})^{\frac{1}{p-1}} ds\Big]^{1-p}. \tag{2.9}$$

From (2.7) and (2.9) we deduce that (2.5) can be replaced by

$$\frac{f(\alpha)T^N}{\alpha^{p-1}} < \frac{N-1}{T^{p-N}}\Big(\frac{p-N}{p-1}\Big)^{p-1},$$

which is equivalent to

$$\frac{f(\alpha)}{\alpha^{p-1}} < \frac{(N-1)T^{-N}}{T^{p-N}}\Big(\frac{p-N}{p-1}\Big)^{p-1},$$

which is (2.6).

3^0) If $\psi = 0$ for $0 \leqslant t \leqslant a < T$ and $\psi(t) = 1$ for $a \leqslant t \leqslant T$, then the condition (2.3), becomes

$$\begin{aligned} I_2 &:= \frac{\inf_{r\in(a,T),\ x\in\mathbb{R}^n_+,\ |x|\in[\frac{r}{T}\beta,\beta]} \left|f(x(r)) + (N-1)r^{-N}\int_r^T t^{N-1} f(x(t))dt\right|}{\beta^{p-1}} \\ &> \Big(\frac{p}{p-1}\Big)^{p-1}\frac{1}{(T-a)^p}. \end{aligned}$$

If, in addition, f is nondecreasing on $\big[0, \max\{\alpha,\beta\}\big]$ we obtain

$$I_2 = \frac{f(\frac{a}{T}\beta)\inf_{r\in(a,T)}\left|1+(N-1)r^{-N}\int_r^T t^{N-1}dt\right|}{\beta^{p-1}},$$

where

$$\begin{aligned} \inf_{r\in(a,T)}\left|1+(N-1)r^{-N}\int_r^T t^{N-1}dt\right| &= \inf_{r\in(a,T)}\left|1+(N-1)r^{-N}\frac{t^N}{N}\Big|_r^T\right| \\ &= \inf_{r\in(a,T)}\left|1+\frac{N-1}{N}r^{-N}(T^N - r^N\right| \\ &= \inf_{r\in(a,T)}\left|\frac{1}{N}+\frac{N-1}{N}r^{-N}T^N\right| \\ &= \frac{1}{N}+\frac{N-1}{N}T^{-N}T^N = 1, \end{aligned}$$

so $I_2 = f\big(\frac{a}{T}\beta\big)/\beta^{p-1}$ and the condition (2.3) becomes

$$\frac{f\left(\frac{a}{T}\beta\right)}{\beta^{p-1}} > \Big(\frac{p}{p-1}\Big)^{p-1}\frac{1}{(T-a)^p}. \tag{2.10}$$

For a given compact interval $[c,d]$, let λ_1 and ϕ_1 be the first eigenvalue and a corresponding positive eigenfunction of the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u, \\ u(c) = u'(d) = 0. \end{cases} \tag{2.11}$$

Notice that

$$\lambda_1 = \inf\left\{ \frac{\int_c^d |\nabla u|^p dt}{\int_c^d |u|^p dt} : u \in C^1[c,d] - \{0\}, u(c) = u'(d) = 0 \right\}, \lambda_1 > 0,$$

and there exists a function $\phi_1 \in C^1[c,d]$ with $\phi_1(c) = \phi_1'(d) = 0$ and $\phi_1(t) > 0$ on (c,d), for which the above inf is reached. In the sequel we shall always assume that $|\phi_1|_\infty = \max_{t\in[c,d]} \phi_1(t) = 1$.

THEOREM 2.2. *Let $f \in C(\mathbb{R}^n_+;\mathbb{R}^n_+)$. Assume that there exist intervals $[a,b]$ and $[A,B]$ with $[a,b] \subseteq [0,T] \subseteq [A,B]$, such that if λ, ϕ and Λ, Φ denote the first eigenvalue and the first positive eigenfunction for the interval $[a,b]$ and respectively, $[A,B]$, then the following conditions are satisfied:*

(i) *there are constants $c, C > 0$ with*

$$c\phi(t)^{p-1} \leqslant 1 \ \text{ a.e. } t \in (a,b), \tag{2.12}$$

$$1 \leqslant C\Phi(t)^{p-1} \ \text{ a.e. } t \in (0,T); \tag{2.13}$$

(ii) *there are numbers $\alpha, \beta > 0$, $\alpha \neq \beta$ such that*

$$C\frac{\max_{x\in\mathbb{R}^n_+, |x|\leqslant\alpha} |f(x)|}{\alpha^{p-1}}\left[1 + (N-1)\sup_{r\in(0,T)} \frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right] < \Lambda, \tag{2.14}$$

$$c\,\frac{\min_{x\in\mathbb{R}^n_+,\ |x|\in[\frac{r}{T}\beta,\beta],\ r\in[a,b]} |f(x(r))|}{\beta^{p-1}} \left[1 + (N-1)\inf_{r\in(0,T)} \frac{\int_r^T t^{N-1}\phi(t)^{p-1}dt}{r^N\phi(r)^{p-1}}\right] > \lambda M, \tag{2.15}$$

where $M > 1$ is such that

$$\phi(r)^{p-1} + (N-1)r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt \leqslant M\phi(r)^{p-1} \ \text{ for } r \in [0,T]. \tag{2.16}$$

Then (1.1) has at least one solution $u \in C$ with $|u|_s$ increasing, concave and

$$\min\{\alpha,\beta\} < |u|_\infty < \max\{\alpha,\beta\}.$$

Proof. We apply Theorem 2.1 to $\varphi(t) = t^N\Phi(t)^{p-1}$ for $t \in [0,T]$, $\psi(t) = \phi(t)^{p-1}$ for $t \in [a,b]$ and $\psi(t) = 0$ for $t \in [0,T]\setminus[a,b]$. First we check inequality (2.2). For $t \in (0,T)$, from (2.13) we find that

$$|f(x)| \leqslant C\Phi(t)^{p-1}|f(x)| = Ct^{-N}\varphi(t)|f(x)|.$$

Hence

$$\frac{\sup_{x\in\mathbb{R}^n_+,\ |x|\leqslant\alpha,\ r\in(0,T)}\frac{|f(x(r))|}{r^{-N}\varphi(r)}}{\alpha^{p-1}} \leqslant C\frac{\max_{x\in\mathbb{R}^n_+,\ |x|\leqslant\alpha}|f(x)|}{\alpha^{p-1}}. \tag{2.17}$$

On the other hand, because Λ and Φ are the first eigenvalue, respectively the first eigenfunction of the problem (2.11), based on the relations (1.3) and (1.5), we obtain

$$\begin{aligned}\Phi(r) &= S\big(\Lambda(\Phi(r)^{p-1} + (N-1)r^{-N}\int_r^T t^{N-1}\Phi(t)^{p-1}dt)\big)\\ &= \Lambda^{\frac{1}{p-1}} S\big(\Phi(r)^{p-1} + (N-1)r^{-N}\int_r^T t^{N-1}\Phi(t)^{p-1}dt\big),\end{aligned}$$

using Lemma 1.1, 2^0. Hence, we find

$$\begin{aligned}\Lambda^{-\frac{1}{p-1}}\Phi(r) &= S\big(\Phi(r)^{p-1} + (N-1)r^{-N}\int_r^T t^{N-1}\Phi(t)^{p-1}dt\big)\\ &\geqslant S\big(\Phi(r)^{p-1}\big),\end{aligned}$$

because $\Phi \geqslant 0$ and S is increasing, according to Lemma 1.1, 1^0. Thus, we find that

$$\Lambda^{-\frac{1}{p-1}}\Phi(r) \geqslant S\big(r^{-N}\varphi(r)\big) = \int_0^r \Big(\int_s^T \varphi(\tau)\tau^{-N}d\tau\Big)^{\frac{1}{p-1}} ds \tag{2.18}$$

basing on S definition and the fact that $\varphi \in L^1\big([0,T];R_+\big)$, so we have that $|\varphi| = \varphi$.

Since $\varphi > 0$ and $\max_{r\in[0,T]}\Phi(r) \leqslant \max_{r\in[A,B]}\Phi(r) = |\Phi|_\infty = 1$, from (2.18) we obtain

$$\Lambda^{-\frac{1}{p-1}} \geqslant \int_0^T \Big(\int_s^T \varphi(\tau)\tau^{-N}d\tau\Big)^{\frac{1}{p-1}} ds,$$

that is,

$$\Lambda \leqslant \Big[\int_0^T \Big(\int_s^T \varphi(\tau)\tau^{-N}d\tau\Big)^{\frac{1}{p-1}} ds\Big]^{1-p}. \tag{2.19}$$

We have

$$\begin{aligned}&\sup_{r\in(0,T),\ u\in\mathbb{R}^n_+,\ |u|\leqslant\alpha}\frac{\big|f(u(r)) + (N-1)r^{-N}\int_r^T t^{N-1}f(u(t))dt\big|}{r^{-N}\varphi(r)}\\ &\leqslant \sup_{r\in(0,T),\ u\in\mathbb{R}^n_+,\ |u|\leqslant\alpha}\frac{|f(u(r))|}{r^{-N}\varphi(r)}\\ &\quad + (N-1)\sup_{r\in(0,T),\ u\in\mathbb{R}^n_+,\ |u|\leqslant\alpha}\frac{r^{-N}\int_r^T t^{N-1}\big|\frac{f(u(t))}{t^{-N}\varphi(t)}\big|t^{-N}\varphi(t)dt}{r^{-N}\varphi(r)}\\ &\leqslant \sup_{r\in(0,T),\ u\in\mathbb{R}^n_+,\ |u|\leqslant\alpha}\frac{|f(u(r))|}{r^{-N}\varphi(r)}\\ &\quad + (N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{-1}\varphi(t)\sup_{t\in(0,T),\ u\in\mathbb{R}^n_+,\ |u|\leqslant\alpha}\big|\frac{f(u(t))}{t^{-N}\varphi(t)}\big|dt}{\varphi(r)}\end{aligned}$$

$$= S_3\Big[1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{-1}\varphi(t)dt}{\varphi(r)}\Big],$$

where

$$S_3 := \sup_{r\in(0,T),\ u\in\mathbb{R}_+^n,\ |u|\leqslant\alpha}\left|\frac{f(u(r))}{r^{-N}\varphi(r)}\right|$$

and $\varphi(r) = r^N\Phi(r)^{p-1}$. Now, (2.14), (2.17) and (2.19) imply (2.2).

Now we will check (2.3). The norm implied is $|.|_s$, so we obtain:

$$\inf_{r\in(0,T),\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}\frac{|f(x(r))+(N-1)r^{-N}\int_r^T t^{N-1}f(x(t))dt|}{\psi(r)}$$
$$\geqslant \Big|\inf_{r\in(0,T),\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}\frac{f(x(r))}{\psi(r)}+(N-1)\cdot$$
$$\inf_{r\in(0,T),\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}\frac{r^{-N}\int_r^T t^{N-1}\psi(t)\inf_{x\in\mathbb{R}_+^n,\ |x|\in[\frac{t}{T}\beta,\beta],\ t\in(0,T)}\frac{f(x(t))}{\psi(t)}dt}{\psi(r)}\Big|$$
$$= I_3[1+(N-1)\inf_{r\in(0,T)}\frac{r^{-N}\int_r^T t^{N-1}\psi(t)dt}{\psi(r)}], \tag{2.20}$$

where

$$I_3 := \inf_{r\in(0,T),\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}\frac{|f(x(r))|}{\psi(r)} = \min_{r\in[a,b],\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}\frac{|f(x(r))|}{\psi(r)}.$$

For $t\in(a,b)$, from (2.12) we obtain that

$$|f(x(t))| \geqslant c\phi(t)^{p-1}|f(x(t))| = c\psi(t)|f(x(t))|.$$

Therefore,

$$\frac{\inf_{r\in(0,T),\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}\frac{|f(x(r))|}{\psi(r)}}{\beta^{p-1}} \geqslant c\frac{\min_{r\in[a,b],\ x\in\mathbb{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta]}|f(x(r))|}{\beta^{p-1}}. \tag{2.21}$$

Since λ and ϕ there are the first eigenvalue, respectively eigenfunction of problem (2.11), together with (1.3) and (1.5) imply

$$\phi(r) = S\big(\lambda(\phi(r)^{p-1}+(N-1)r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt)\big)$$
$$= \lambda^{\frac{1}{p-1}}S\big(\phi(r)^{p-1}+(N-1)r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt\big),$$

using Lemma 1.1, 2^0. So

$$\lambda^{\frac{1}{1-p}}\phi(r) = S\big(\phi(r)^{p-1}+(N-1)r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt\big) \leqslant S\big(M\phi(r)^{p-1}\big),$$

basing on $\phi \geqslant 0$, S is increasing, and according with Lemma 1.1, $1^0, 2^0$ and (2.16). So, we have

$$\lambda^{\frac{1}{1-p}} \phi(r) \leqslant S\big(M\psi(r)\big) = M^{\frac{1}{p-1}} \int_0^r \Big(\int_s^T \psi(\tau)d\tau\Big)^{\frac{1}{p-1}} ds$$

and since $\max\limits_{r\in[0,T]} \phi(r) \geqslant \max\limits_{r\in[a,b]} \phi(r) = |\phi|_\infty = 1$, we obtain

$$\big(M\lambda\big)^{\frac{1}{1-p}} \leqslant \int_0^T \Big(\int_s^T \psi(\tau)d\tau\Big)^{\frac{1}{p-1}} ds,$$

that is,

$$M\lambda \geqslant \Big[\int_0^T \Big(\int_s^T \psi(\tau)d\tau\Big)^{\frac{1}{p-1}} ds\Big]^{1-p}. \tag{2.22}$$

From (2.20), (2.21), (2.15) and (2.22) we obtain (2.3). So Theorem 2.1 may be applied and the conclusion follows.

Now let

$$h_0 := \lim_{|x|\to 0,\ x\in\mathbb{R}^n_+} \frac{|f(x)|}{|x|^{p-1}} \quad \text{and} \quad h_\infty = \lim_{|x|\to \infty,\ x\in\mathbb{R}^n_+} \frac{|f(x)|}{|x|^{p-1}},$$

assuming that these limits exist in $R_+ \cup \{\infty\}$.

THEOREM 2.3. *Let be $f \in C\big(\mathbb{R}^n_+;\mathbb{R}^n_+\big)$ and suppose that there exist the intervals $[a,b]$ and $[A,B]$ with $[a,b] \subset [0,T] \subseteq [A,B]$ and $a > 0$ so that the condition (i) from Theorem 2.2 to be fulfiled ($\lambda, \phi, \Lambda, \Phi$ are from Theorem 2.2). Also, we suppose that the condition (2.16) is satisfied. In addition, we suppose that one of the following two conditions holds:*

(a)

$$Ch_0\Big(1+(N-1)\sup_{r\in(0,T)} \frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\Big) < \Lambda$$

and

$$(\frac{a}{T})^{p-1} c h_\infty \Big[1+(N-1)\inf_{r\in(0,T)} \frac{r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt}{\phi(r)^{p-1}}\Big] > \lambda M$$

or

(b)

$$(\frac{a}{T})^{p-1} c h_0 \Big[1+(N-1)\inf_{r\in(0,T)} \frac{r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt}{\phi(r)^{p-1}}\Big] > \lambda M$$

and

$$Ch_\infty\Big(1+(N-1)\sup_{r\in(0,T)} \frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\Big) < \Lambda.$$

Then (1.1) has a solution.

Proof. We apply Theorem 2.2.

Suppose (a). Then there exist $\alpha, \beta > 0,\ \alpha < \beta$ so that

$$C\frac{|f(x)|}{|x|^{p-1}}\Big(1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\Big) < \Lambda \text{ for every } x\in\mathbb{R}_+^n$$

with $0 < |x| \leqslant \alpha$ and

$$\big(\frac{a}{T}\big)^{p-1}c\frac{|f(x)|}{|x|^{p-1}}\Big[1+(N-1)\inf_{r\in(0,T)}\frac{r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt}{\phi(r)^{p-1}}\Big] > \lambda M,$$

for every $x\in\mathbb{R}_+^n$ with $|x| \geqslant \frac{a}{T}\beta$. Now it follows that the conditions (2.14) and (2.15) are satisfied.

Suppose (b). Then we can find $\alpha_0, \beta > 0$ with

$$\big(\frac{a}{T}\big)^{p-1}c\frac{|f(x)|}{|x|^{p-1}}\Big[1+(N-1)\inf_{r\in(0,T)}\frac{r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt}{\phi(r)^{p-1}}\Big] > \lambda M$$

for every $x\in\mathbb{R}_+^n$ with $0 < |x| \leqslant \beta$ and (2.15) follows. Also,

$$C\frac{|f(x)|}{|x|^{p-1}}\Big(1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\Big) < \Lambda \tag{2.23}$$

for every $x\in\mathbb{R}_+^n$ with $|x| \geqslant \alpha_0$. We choose $\alpha \neq \beta$ with $\alpha \geqslant \alpha_0$ and

$$\alpha > \left(\frac{C}{\Lambda}\max_{|x|\leqslant\alpha_0}|f(x)|\Big(1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\Big)\right)^{\frac{1}{p-1}}. \tag{2.24}$$

Then (2.24) shows that

$$C\frac{|f(x)|}{\alpha^{p-1}}\left(1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right) < \Lambda \text{ for } |x| \leqslant \alpha_0. \tag{2.25}$$

From (2.23) we obtain

$$C\frac{|f(x)|}{\alpha^{p-1}}\left(1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right) \leqslant C\frac{|f(x)|}{|x|^{p-1}}\left(1+(N-1)\sup_{r\in(0,T)}\frac{\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right) < \Lambda \tag{2.26}$$

for $\alpha_0 \leqslant |x| \leqslant \alpha$. Now, (2.25) and (2.26) guarantee (2.14).

REMARK 2.3. 1^0) The relation (2.10) from Remark 2.2, 3^0 is also found in [7] (Remark 2.3, 3^0).

2^0) For $p > N$, the relation (2.6), namely

$$\frac{f(\alpha)}{\alpha^{p-1}} < \left(\frac{p-N}{p-1}\right)^{p-1} \frac{N-1}{T^p},$$

from Remark 2.2, 2^0 is similar to the relation

$$\frac{f(\alpha)}{\alpha^{p-1}} < \left(\frac{p}{p-1}\right)^{p-1} \frac{1}{T^p}$$

from [7], Remark 2.3, 3^0.

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