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EXISTENCE OF POSITIVE ENTIRE SOLUTIONS OF A SEMILINEAR p–LAPLACIAN PROBLEM WITH A GRADIENT TERM

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Abstract. In this paper, we study a semilinear p-Laplacian problem

$$-\Delta_p u + h(x)|\nabla u|^q = b(x)g(u),\ u>0,\ x\in\mathbb{R}^N,\ \lim_{|x|\to\infty} u(x)=0,$$

where $q\in(p-1,p]$, $b,\ h\in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha\in(0,1)$, $h(x)\geqslant 0$, $b(x)>0, \forall x\in\mathbb{R}^N$, and $g\in C^1((0,\infty),(0,\infty))$ which may be singular at 0. Using a sub-supersolution argument and a perturbed argument, we obtain the existence of entire solutions to the problem. No monotonicity condition is imposed on the functions $g(s)$ and $\frac{g(s)}{s^{p-1}}$.

1. Introduction

In this paper, we consider the existence of entire solutions of the following semilinear p-Laplacian problem

$$\begin{cases} -\Delta_p u + h(x)|\nabla u|^q = b(x)g(u) \ \text{ in } \mathbb{R}^N, \\ u>0,\ \lim_{|x|\to\infty} u(x)=0, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $q\in(p-1,p], h\in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha\in(0,1)$ is non-negative in $\mathbb{R}^N$, g satisfies the following conditions:

(g_1) $g\in C^1((0,\infty),(0,\infty))$,

(g_2) $\lim_{s\to 0^+}\frac{g(s)}{s^{p-1}}=\infty$,

(g_3) $\lim_{s\to\infty}\frac{g(s)}{s^{p-1}}=0$,

and b satisfies

(b_1) $b\in C^{\alpha}_{loc}(\mathbb{R}^N)$ and $b(x)>0, \forall x\in\mathbb{R}^N$,

(b_2) the p-Laplacian problem

$$\begin{cases} -\Delta_p u = b(x) \ \text{ in } \mathbb{R}^N, \\ u>0,\ \lim_{|x|\to\infty} u(x)=0, \end{cases} \tag{1.2}$$

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has a unique solution $w \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$.

In recent years, more and more attention has been paid to the existence of positive entire solutions for semilinear elliptic problems. Results relating to these problems can be found in [1-3, 6-8, 10, 11-14, 16, 18, 19]. For the equation considered over a bounded smooth domain Ω instead of $\mathbb{R}^N$, the corresponding problem was studied, for instance, in [15, 20] and the references cited therein.

The following model

$$\begin{cases} -\Delta u = b(x)g(u) \ \text{ in } \ \mathbb{R}^N, \\ u > 0, \ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

arises from many branches of mathematics and applied mathematics.

For $g(u) = u^{-\gamma}$ with $\gamma > 0$, if b satisfies (b_1) and the following condition

(b_3) $\int_0^\infty r\phi(r)dr < \infty$, where $\phi(r) = \max_{|x|=r} b(x)$,

Lair and Shaker [13] showed that problem (1.3) has a unique solution $u \in C^{2,\alpha}_{loc}(\mathbb{R}^N)$.

Later, Lair and Shaker [14] and Zhang [19] extended the above results to the more general g which satisfies (g_1) and

(g_4) g is non-increasing on $(0,\infty)$ and $\lim_{s\to 0^+} g(s) = \infty$.

Cîrstea and Rădulescu [3] proved the above results if g satisfies (g_1) and:

(g_5) $\lim_{s\to 0^+} \frac{g(s)}{s} = \infty$;

(g_6) $\frac{g(s)}{s+s_0}$ is decreasing on $(0,\infty)$ for some $s_0 > 0$;

(g_7) g is bounded in a neighborhood of ∞.

Recently, Goncalves and Santos [11] also generalized the above results to the case that g satisfies (g_1), (g_5) and

(g_8) $\lim_{s\to\infty} \frac{g(s)}{s} = 0$;

(g_9) $\frac{g(s)}{s}$ is decreasing on $(0,\infty)$.

Very recently, Covei [4] generalized the problem (1.3) to the following p-laplacian equation

$$\begin{cases} -\Delta_p u = b(x)g(u) \ \text{ in } \ \mathbb{R}^N, \\ u > 0, \ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.3}$$

and showed that the problem (1.4) has a positive solution $u \in C^{1,\alpha}(\mathbb{R}^N)$ if g satisfies $(g_2), (g_3)$ and

(g_{10}) $\frac{g(s)}{s^{p-1}}$ is decreasing on $(0,\infty)$;

(g_{11}) $g \in \mathrm{Lip}_{loc}((0,\infty),(0,\infty))$ with g singular at 0;

and b satisfies (b_1) and

(b_4) $0 < \int_1^\infty r^{\frac{1}{p-1}}\phi^{\frac{1}{p-1}}dr < \infty$ if $1 < p \leqslant 2$; $0 < \int_1^\infty r^{\frac{(p-2)N+1}{p-1}}\phi \, dr < \infty$ if $2 \leqslant p < \infty$.

Motivated by the results [4] and [17], in this paper, we will extend the results of [17] and continue to consider the existence of entire solutions to problem (1.1) for the functions $g(s)$ and $\frac{g(s)}{s^{p-1}}$ which do not have monotonicity.

Now we state our main result.

THEOREM 1.1. *Let $q \in (p-1,p], h \in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ is non negative in $\mathbb{R}^N$, and b satisfies (b_1) and (b_2). If g satisfies (g_1)-(g_3), then problem (1.1) has at least one solution $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$.*

The paper is organized as follows. In Section 2, we give some preliminaries that which are going to be used later. In Section 3, we prove Theorem 1.1.

2. Preliminaries

We first consider the following problem

$$\begin{cases} -\Delta_p u + h(x)|\nabla u|^q = b(x)g(u) \ \text{ in } \ \Omega, \\ u > 0, \ \ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N \geqslant 1)$.

For the convenience, we set $f(x,u,\nabla u) = b(x)g(u) - h(x)|\nabla u|^q$. Now we introduce a sub-supersolution method with the boundary restriction.

DEFINITION 2.1. A function $\underline{u} \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ is called a subsolution of problem (2.1) if

$$\begin{cases} -\Delta_p \underline{u} \leqslant f(x,\underline{u},\nabla\underline{u}) \ \text{ in } \ \Omega, \\ \underline{u} > 0, \ \ \underline{u}|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

DEFINITION 2.2. A function $\overline{u} \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ is called a supersolution of problem (2.1) if

$$\begin{cases} -\Delta_p \overline{u} \geqslant f(x,\overline{u},\nabla\overline{u}) \ \text{ in } \ \Omega, \\ \overline{u} > 0, \ \ \overline{u}|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

LEMMA 2.1. ([15, Lemma 2.4]) *Let $f(x,u,\xi)$ satisfies the following two basic conditions:*

(D_1) $f(x,u,\xi)$ is locally Hölder continuous in $\Omega \times (0,\infty) \times \mathbb{R}^N$ and continuously differentiable with respect to the variables u and ξ;

(D_2) for any $\Omega_1 \subset\subset \Omega$ and any $a,b \in (0,\infty) (a < b)$, there exists a corresponding constant $C = C(\Omega_1,a,b) > 0$ such that $|f(x,u,\xi)| \leqslant C(1+|\xi|^p), \forall x \in \overline{\Omega}_1, \forall u \in [a,b], \forall \xi \in \mathbb{R}^N$.

If problem (2.1) has a supersolution $\overline{u}$ and a subsolution $\underline{u}$ such that $\underline{u} \leqslant \overline{u}$ in Ω, then problem (2.1) has at least one solution $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ in the ordered interval $[\underline{u},\overline{u}]$.

LEMMA 2.2. ([4, Theorem 1.2]) *Let b satisfies (b_1). If g satisfies (g_1)-(g_3) and (g_{10}), then the following problem*

$$\begin{cases} -\Delta_p u = b(x)g(u) \ \ in \ \ \Omega, \\ u > 0, \ \ u|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

has a solution $u \in C(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$.

LEMMA 2.3. *If g satisfies (g_1)-(g_3), then there exists a function $\overline{f}_1$ such that*
(i) $\overline{f}_1 \in C^1((0,\infty),(0,\infty))$,
(ii) $\frac{g(s)}{s^{p-1}} \leqslant \overline{f}_1(s), \forall s > 0$ *and* $\lim_{s\to 0^+} \overline{f}_1(s) = \infty$,
(iii) $\overline{f}_1$ *is non-increasing on* $(0,\infty)$,
(iv) $\lim_{s\to\infty} \overline{f}_1(s) = 0$.

Proof. By (g_1)-(g_3), we can denote

$$\overline{f}(s) = \sup_{t \geqslant s > 0} \frac{g(t)}{t^{p-1}}. \tag{2.5}$$

Observe that

$$\overline{f}(s) \geqslant \frac{g(t)}{t^{p-1}}, \ \ \forall s > 0 \ \text{ and } \ t \geqslant s,$$

and $\overline{f}$ is non-increasing on $(0,\infty)$. Moreover,

$$\lim_{s\to 0^+} \overline{f}(s) = \infty \quad \text{and} \quad \lim_{s\to\infty} \overline{f}(s) = 0.$$

Now we can assume $\overline{f} \in C^1(0,\infty)$. If not, we can replace it by

$$\overline{f}_1(s) = \frac{2}{s}\int_{\frac{s}{2}}^{s} \overline{f}(t)dt, \ \ s > 0.$$

Obviously,

$$\overline{f}(s) \leqslant \overline{f}_1(s) \leqslant \overline{f}(\frac{s}{2}), \ \ \forall s > 0.$$

And, for $s > 0$,

$$\begin{aligned} \overline{f}_1'(s) &= \frac{2}{s}(\overline{f}(s) - \frac{1}{2}\overline{f}(\frac{s}{2})) - \frac{2}{s^2}\int_{\frac{s}{2}}^{s} \overline{f}(t)dt \\ &\leqslant \frac{2}{s}(\overline{f}(s) - \frac{1}{2}\overline{f}(\frac{s}{2})) - \frac{2}{s^2}\frac{s}{2}\overline{f}(s) = \frac{1}{s}(\overline{f}(s) - \overline{f}(\frac{s}{2})) \leqslant 0, \end{aligned}$$

i.e., $\overline{f}_1 \in C^1((0,\infty),(0,\infty))$. Hence Lemma 2.3 holds. $\square$

LEMMA 2.4. *Let $q \in (p-1,p]$, $b,h \in C^{\alpha}(\overline{\Omega})$, $h(x) \geqslant 0$, $b(x) > 0$, $\forall x \in \overline{\Omega}$. If g satisfies (g_1)-(g_3), then problem (2.1) has at least one solution $u \in C(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$.*

Proof. Let $\psi_1 \in C^1(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$ be the first eigenfunction corresponding to the first eigenvalue λ_1 of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u \ \text{ in } \ \Omega, \\ u > 0, \ \ u|_{\partial\Omega} = 0. \end{cases} \tag{2.6}$$

Let $\beta = \frac{q}{q-p+1}$. It follows from (g_2) that there exists a positive constant $\delta_1 \in (0,1)$ such that

$$\frac{g(s)}{s^{p-1}} \geqslant \frac{\lambda_1 \beta^{p-1} + |h|_\infty \beta^q |\nabla \psi_1|_\infty^q}{\min_{x\in\overline{\Omega}} b(x)}, \ \ \forall s \in (0,\delta_1).$$

Let $\underline{u} = c_1 \psi_1^\beta$ with $c_1 \in (0, \min\{1, \frac{\delta_1}{|\psi_1|_\infty^\beta}\})$. Since $c_1^{q-p+1} < 1$, we have

$$\begin{aligned} -\Delta_p \underline{u} + h(x)|\nabla \underline{u}|^q = {} & \beta^{p-1}\lambda_1 (c_1 \psi_1^\beta)^{p-1} \\ & - (p-1)\beta(\beta-1)c_1^{p-1}|\beta\psi_1^{\beta-1}|^{p-2}\psi_1^{\beta-2}|\nabla\psi_1|^p \\ & + h(x)\beta^q c_1^q \psi_1^{q(\beta-1)}|\nabla\psi_1|^q \\ \leqslant {} & \min_{x\in\overline{\Omega}} b(x) g(c_1\psi_1^\beta) \leqslant b(x) g(c_1\psi_1^\beta) = b(x) g(\underline{u}), \ \ x \in \Omega. \end{aligned}$$

So $\underline{u} = c_1\psi_1^\beta$ is a subsolution to problem (2.1).

To construct a supersolution, by Lemma 2.2, we obtain that the following problem

$$\begin{cases} -\Delta_p u = b(x) u^{p-1}(\overline{f}_1(u) + \frac{1}{u}) \ \text{ in } \ \Omega, \\ u > 0, \ \ u|_{\partial\Omega} = 0, \end{cases} \tag{2.7}$$

has a solution $u \in C(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$, which is a supersolution to problem (2.1).

Using the same maximum principle argument as the following proof of (3.2) in Section 3, we can get that $\underline{u}(x) \leqslant \overline{u}(x), x \in \Omega$. It follows from Lemma 2.1 that problem (2.1) has at least one solution $u \in C(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$ in the ordered interval $[\underline{u}, \overline{u}]$. □

LEMMA 2.5. *Let b satisfies (b_1) and (b_2). If g satisfies (g_1)-(g_3) and $\overline{f}_1$ is in Lemma 2.3, then there exists a function $v \in C^1_{loc}(\mathbb{R}^N)$ satisfying*

$$\begin{cases} -\Delta_p v \geqslant b(x) v^{p-1}(\overline{f}_1(v) + \frac{1}{v}) \ \textit{ in } \ \mathbb{R}^N, \\ v > 0, \ \ \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{2.8}$$

Proof. By (g_1)-(g_3), we define

$$\Gamma(t) = \int_0^t \Big(\frac{s}{s\overline{f}_1(s) + 1}\Big)^{\frac{1}{p-1}} ds, \ \ t \geqslant 0.$$

It follows from L'Hôspital's rule that

$$\lim_{t\to\infty}\frac{\Gamma(t)}{t}=\lim_{t\to\infty}\Big(\frac{t}{t\overline{f}_1(t)+1}\Big)^{\frac{1}{p-1}}=\lim_{t\to\infty}\Big(\frac{1}{\overline{f}_1(t)+t^{-1}}\Big)^{\frac{1}{p-1}}=\infty.$$

Let w be the solution of problem (1.2) and $c_0=\max_{\mathbb{R}^N}w(x)$. Therefore, there exists a positive constant c_2 (large enough) such that

$$c_0c_2\leqslant\Gamma(c_2)=\int_0^{c_2}\Big(\frac{s}{s\overline{f}_1(s)+1}\Big)^{\frac{1}{p-1}}ds. \tag{2.9}$$

Now we define a function v by

$$w(x)=\frac{1}{c_2}\int_0^{v(x)}\Big(\frac{s}{s\overline{f}_1(s)+1}\Big)^{\frac{1}{p-1}}ds,\ \ \forall x\in\mathbb{R}^N. \tag{2.10}$$

Hence combining (2.9) with (2.10), we have $0<v(x)\leqslant c_2$. Obviously, we obtain $v^{p-1}(x)\leqslant c_2^{p-1}$. It follows from $\lim_{|x|\to\infty}w(x)=0$ that $\lim_{|x|\to\infty}v(x)=0$.

Moreover, by Lemma 2.3, we obtain

$$\begin{aligned}c_2^{p-1}b(x)&=-c_2^{p-1}\Delta_p w\\&=\frac{-\Delta_p v}{\overline{f}_1(v(x))+(v(x))^{-1}}-\frac{d}{dv}\Big(\frac{1}{\overline{f}_1(v(x))+(v(x))^{-1}}\Big)|\nabla v(x)|^p\\&\leqslant\frac{-\Delta_p v}{\overline{f}_1(v(x))+(v(x))^{-1}},\ \ x\in\mathbb{R}^N,\end{aligned}$$

i.e.,

$$\begin{aligned}-\Delta_p v&\geqslant c_2^{p-1}b(x)\big(\overline{f}_1(v(x))+(v(x))^{-1}\big)\\&\geqslant b(x)v^{p-1}(x)\big(\overline{f}_1(v(x))+(v(x))^{-1}\big),\ \ x\in\mathbb{R}^N.\end{aligned}$$

This completes the proof. □

3. Proof of Theorem 1.1

Now consider the perturbed problem

$$\begin{cases}-\Delta_p u_k+h(x)|\nabla u_k|^q=b(x)g(u_k)\ \text{ in }\ B(0,k),\\ u_k>0,\ \ u_k|_{\partial B(0,k)}=0,\end{cases} \tag{3.1}$$

where $B(0,k)=\{x\in\mathbb{R}^N:|x|<k\},k=1,2,3,\dots$

It follows from Lemma 2.4 that problem (3.1) has one solution $u_k\in C(\overline{B}(0,k))\cap C^{1,\alpha}(B(0,k))$.

Set $u_k(x)=0\ \forall\,|x|>k$. Let v be as in Lemma 2.5, we assert that

$$u_k(x)\leqslant v(x),\ \ x\in\mathbb{R}^N,k=1,2,3,\dots\,. \tag{3.2}$$

To show (3.2) we need Diaz-Saà's inequality (in [5]).

LEMMA 3.1. *For $i=1,2$ let $w_i \in L^\infty(\Omega)$ such that:*

$$\begin{cases} w_i > 0 \ \text{a.e. in } \Omega \ \text{ and } \ w_1 = w_2 \ \text{ on } \partial\Omega, \\ w_i \in W^{1,p}(\Omega) \ \text{ and } \ \Delta_p w_i^{\frac{1}{p}} \in L^\infty(\Omega). \end{cases}$$

Then,

$$\int_\Omega \left(\frac{-\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} - \frac{-\Delta_p w_2^{\frac{1}{p}}}{w_2^{\frac{p-1}{p}}} \right)(w_1 - w_2)dx \geqslant 0,$$

provided $\frac{w_i}{w_j} \in L^\infty(\Omega)$ for $i \neq j$, where $i,j = 1,2$.

Verification of (3.2). Consider the open subset of $\mathbb{R}^N$, namely

$$B_{k,v} = \{x \in \mathbb{R}^N | u_k(x) > v(x)\} \subset\subset B(0,k).$$

Setting $w_1 := (u_k)^p$ and $w_1 := v^p$ we get,

$$\begin{aligned} 0 &\leqslant \int_{B_{k,v}} \left(\frac{-\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} - \frac{-\Delta_p w_2^{\frac{1}{p}}}{w_2^{\frac{p-1}{p}}} \right)(w_1 - w_2)dx \\ &= \int_{B_{k,v}} \left(\frac{-\Delta_p u_k}{u_k^{p-1}} - \frac{-\Delta_p v}{v^{p-1}} \right)(u_k^p - v^p)dx \\ &\leqslant \int_{B_{k,v}} \left(\frac{b(x)g(u_k) - h(x)|\nabla u_k|^q}{u_k^{p-1}} - \frac{b(x)v^{p-1}(\overline{f}_1(v) + \frac{1}{v})}{v^{p-1}} \right)(u_k^p - v^p)dx \\ &= \int_{B_{k,v}} (b(x)\left(\frac{g(u_k)}{u_k^{p-1}} - (\overline{f}_1(v) + \frac{1}{v})) - \frac{h(x)|\nabla u_k|^q}{u_k^{p-1}} \right)(u_k^p - v^p)dx \\ &\leqslant \int_{B_{k,v}} b(x)\left(\frac{g(u_k)}{u_k^{p-1}} - (\overline{f}_1(v) + \frac{1}{v}) \right)(u_k^p - v^p)dx < 0, \end{aligned}$$

which is impossible. So $B_{k,v} = \emptyset$ and (3.2) holds.

Now, we need to estimate $\{u_k\}$. For any bounded $C^{1,\alpha}$- smooth domain $\Omega' \subset \mathbb{R}^N$, take Ω_1 and Ω_2 with $C^{1,\alpha}$- smooth boundaries, and K_1 large enough, such that

$$\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset B(0,k), \ \ k \geqslant K_1.$$

Note that

$$u_k(x) \geqslant \underline{u}(x) > 0, \ \ \forall x \in B(0,K_1), \tag{3.3}$$

when $B(0,K_1)$ is the substitution for Ω in the proof of Lemma 2.4.

Let

$$\rho_k(x) = b(x)g(u_k) - h(x)|\nabla u_k|^q, \ \ x \in \overline{B}(0,K_1).$$

Since $-\Delta_p u_k(x) = \rho_k(x), x \in B(0,K_1)$, by the interior estimate theorem of Ladyzenskaja and Ural'tseva [12, Theorem 3.1, p.266], we get a positive constant C_1 independent of k such that

$$\max_{x\in\overline{\Omega}_2} |\nabla u_k(x)| \leqslant C_1 \max_{x\in\overline{B}(0,K_1)} u_k(x) \leqslant C_1 \max_{x\in\overline{B}(0,K_1)} v(x), \ \ \forall x \in B(0,K_1), \tag{3.4}$$

i.e., $|\nabla u_k(x)|$ is uniformly bounded on $\overline{\Omega}_2$. It follows that $\{\rho_k\}_{K_1}^{\infty}$ is uniformly bounded on $\overline{\Omega}_2$ and hence $\rho_k \in L^{p_1}(\Omega_2)$ for any $p_1 > 1$. Since

$$-\Delta_p u_k(x) = \rho_k(x), \ x \in \Omega_2,$$

we see by [9, Theorem 9.11] that there exists a positive constant C_2 independent of k such that

$$\|u_k\|_{W^{1,p_1}(\Omega_1)} \leqslant C_2(\|\rho_k\|_{L^{p_1}(\Omega_2)} + \|u_k\|_{L^{p_1}(\Omega_2)}), \ \ \forall k \geqslant K_1. \tag{3.5}$$

Taking $p_1 > N$ such that $\alpha < 1 - \frac{N}{p_1}$ and applying Sobolev's embedding inequality, we see that $\{\|u_k\|_{C^{1,\alpha}(\overline{\Omega}_1)}\}_{K_1}^{\infty}$ is uniformly bounded. Therefore $\rho_k \in C^{\alpha}(\overline{\Omega}_1)$ and $\{\|\rho_k\|_{C^{\alpha}(\overline{\Omega}_1)}\}_{K_1}^{\infty}$ is uniformly bounded. It follows from Schauder's interior estimate theorem (see [19, Chapter 1, p.2]) that there exists a positive constant C_3 independent of k such that

$$\|u_k\|_{C^{1,\alpha}(\overline{\Omega}')} \leqslant C_3(\|\rho_k\|_{C^{\alpha}(\overline{\Omega}_1)} + \|u_k\|_{C(\overline{\Omega}_1)}), \ \ \forall k \geqslant K_1, \tag{3.6}$$

i.e., $\{\|u_k\|_{C^{1,\alpha}(\overline{\Omega}')}\}_{K_1}^{\infty}$ is uniformly bounded. Using Ascoli-Arzela's theorem and the diagonal sequential process, we get that $\{u_k\}_{K_1}^{\infty}$ has a subsequence that converges uniformly in the $C^1(\overline{\Omega}')$ norm to a function $u \in C^1(\overline{\Omega}')$ and u satisfies

$$-\Delta_p u + h(x)|\nabla u|^q = b(x)g(u), \ \ x \in \Omega'.$$

By (3.3), we obtain that

$$u > 0, \ \ \forall x \in \Omega'.$$

Applying Schauder's regularity theorem we have that $u \in C^{1,\alpha}(\Omega')$. Since Ω' is arbitrary, we also have that $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$. It follows from (3.2) that $\lim_{|x|\to\infty} u(x) = 0$. Hence Theorem 1.1 is proved. □

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