Differential
Equations
& Applications
Volume 3, Number 2 (2011), 235–246

EXISTENCE OF POSITIVE SOLUTIONS TO A QUASILINEAR ELLIPTIC SINGULAR NEUMANN PROBLEM

QING MIAO AND ZUODONG YANG

(*Communicated by J. -M. Rakotoson*)

Abstract. We show the existence of positive solution for the following singular Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u)+\frac{a(x)}{u^{\beta}}=\lambda h(x)u^{p} \ \text{ in } \ B_R, \\ \frac{\partial u}{\partial \nu}=0 \ \text{ on } \ \partial B_R, \end{cases}$$

where $R>0, \lambda>0$ is a positive parameter, $\beta>0$, $p\in[0,m-1)$. By means of double perturbation argument and variational methods, we obtain a positive solution $u\in C^1(\overline{B}_R\backslash\{0\})\cap C(\overline{B}_R)$.

1. Introduction

In this paper,we are concerned with the existence of a positive solution to the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u)+\frac{a(x)}{u^{\beta}}=\lambda h(x)u^{p} \ \text{ in } \ B_R, \\ u>0 \ \text{ in } \ B_R, \\ \frac{\partial u}{\partial \nu}=0 \ \text{ on } \ \partial B_R, \end{cases} \tag{P}$$

where $B_R=B_R(0)\subset\mathbb{R}^N(N\geqslant 1)$, $m\geqslant 2$, $0\leqslant p<m-1$, $\beta>0$ is a constant, $\lambda>0$ is a positive parameter. Throughout this paper, we assume that $h(x)=h(r)$ and $a(x)=a(r)$, $r=\|x\|$, are two nonnegative C^1-functions with $a,h\not\equiv 0$.

The problem of the above form are mathematical models occuring in studies of the m-Laplace system, generalized reaction-diffusion theory, non-Newtonian fluid theory [2,19], non-Newtonian filtration [14] and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity m is characteristic of the medium. Media with $m>2$ are called dilatant fluids and those with $m<2$ are called pseudoplastics. If $m=2$, they are Newtonian fluids. When $m\neq 2$, the problem becomes more complicated since certain nice properties in herent to the case $m=2$ seem to be lost or at least difficult to verify. The main differences between $m=2$ and $m\neq 2$ can be founded in [8,12].

Mathematics subject classification (2010): 35J25, 35J65.

Keywords and phrases: quasilinear elliptic equation, sub-supersolution, variational method, Neumann conditions.

Project Supported by the National Natural Science Foundation of China (Grant No.10871060) and by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB110005).

Paper DEA-03-14

In recent years, the existence and nonexistence of the positive solutions for the quasilinear elliptic equations

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u)+f(x,u)=0,\ \ x\in \mathbb{R}^N,$$

with $m>1$ have been studied by many authors, see [9-11, 25, 27-28].

In [24], the author concerned the entire radially symmetric solutions of the problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u)=q(x)f(u),\ \ x\in\mathbb{R}^N.$$

In [13], the authors considerer the existence of positive solutions of the quasilinear eigenvalue problem

$$-\Delta_m u=\lambda f(u)\ \text{ in }\Omega,\ \ u=0\ \text{ on }\partial\Omega, \tag{E}$$

where $1<m<\infty$, $\lambda>0$, and $\Omega\subset\mathbb{R}^N(N\geqslant 2)$ is a bounded, connected, smooth domain, under appropriate smoothness conditions on f.

By a positive solution of Eq.(E) we mean a pair (λ,u) in $\mathbb{R}^+\times C_0^1(\overline{\Omega})$ satisfying Eq.(E) in the weak sense and with $u>0$ in Ω.

It was shown in [26] that problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u)+\lambda q(x)u^{-\gamma}=0,\ \ x\in \mathbb{R}^N,$$

has a positive decaying entire solution for all $\lambda>0$ if $1<m<N$, $q(r)=q(|x|)\in C(\mathbb{R}^+)$, $q>0$ for $r>0$, and $0\leqslant\gamma<(m-1)$ for any

$$0<\varepsilon<(N-m)(m-1-|\gamma|)/(m-1)$$

such that

$$\int_1^{+\infty} r^{m+\varepsilon-1+[(N-m)|\gamma|/(m-1)]}q(r)dr<\infty,$$

and for $r\in(0,1)$, $\delta<1$, $q(r)=O(r^{-\delta})$.

In [20], the author investigated singular p-Laplacian equations of the form

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u)+f(x,u,\nabla u)u^{-\beta}=0,\ \ x\in\mathbb{R}^N,$$

where $1<m\leqslant N$, $0\leqslant\beta<m-1$, and f satisfy:

(f1) $f(x,u,q)\in C^{\alpha}_{loc}(\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N)$, i.e., $f:\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}$ is locally Hölder continuous with index $\alpha\in(0,1)$, here $\mathbb{R}_+=(0,\infty)$ and $\overline{\mathbb{R}}_+=[0,\infty)$;

(f2) for every bounded domain $D\subset\mathbb{R}^N$, for any $M>0$, $\exists\,\rho(D,M)>0$ such that

$$|f(x,u,q)|\leqslant\rho(D,M)(1+|q|^m),\ x\in D,\ \ 0\leqslant u\leqslant M,\ q\in\mathbb{R}^N,$$

and that there exist functions F,ϕ such that the conditions (F1)-(F4) below are valid. Then for any $\beta\in[0,m-1)$, the problem has a positive entire solution u satisfying

$$\varepsilon^{-1}\phi(|x|)\leqslant u(x)\leqslant\varepsilon,\ \ x\in\mathbb{R}^N,\ \ \lim_{|x|\to\infty}u(x)=0,$$

where $\varepsilon > 1$ is a constant.

(F1) $F \in C^{\alpha}_{loc}(\overline{\mathbb{R}}_+ \times \mathbb{R}_+ \times \overline{\mathbb{R}}_+), \phi \in C^{\alpha}_{loc}(\overline{\mathbb{R}}_+ \times \mathbb{R}_+)$ and

$$0 < \phi(|x|,u) \leqslant f(x,u,p) \leqslant F(|x|,u,|p|) \text{ for all } (x,u,p) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N;$$

(F2) $\phi(r,u)$ is non-decreasing in $u \in \mathbb{R}_+$; $F(r,u,q)$ is non-decreasing both in $u \in \mathbb{R}_+$ and in $q \in \overline{\mathbb{R}}_+$;

(F3) for any $r \in I$, there is some subinterval I of $\mathbb{R}_+$ such that

$$\lim_{\xi \to \infty} \xi^{1-\frac{\beta}{m-1}} \phi(r, \xi^{-1}\varphi(r)) = \infty;$$

(F4) for any $r \in \mathbb{R}_+$,

$$\lim_{\xi \to \infty} \xi^{-1+\frac{\beta}{m-1}} F(r,\xi,\xi) = 0$$

and there exists a nonnegative measurable function $M(t) \in \Lambda_1$, with a constant $\xi_0 > 1$ such that

$$\xi^{-1+\frac{\beta}{m-1}} F(r,\xi,\xi)(\varphi(r))^{-\beta} \leqslant M(r), \;\; r \in R_+ \text{ for all } \xi \geqslant \xi_0.$$

For $m = 2$, there is a lot of papers in the literature dealing with singular problem with Dirichlet or Neumann boundary conditions, we can cite for example, the papers of Cirstea and Radulescu [4], Crandall, Rabinowitz and Tartar [5], del Pino and Hernandez [21] and reference therein.

The related results to a singular semilinear elliptic the boundary value problem

$$\begin{cases} \Delta u + \lambda q(x) u^{\gamma} = 0, \;\; x \in \Omega, \\ u = 0, \;\; x \in \partial\Omega, \end{cases}$$

have been extensively studied when $\Omega \subset \mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, see [3, 6-7, 15-17, 23].

The results to a semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u + a(x) u^{-\beta} = \lambda h(x) u^p, \;\; x \in B_R, \\ u > 0, \;\; x \in B_R, \\ \frac{\partial u}{\partial \nu} = 0, \;\; x \in \partial B_R, \end{cases}$$

and

$$\begin{cases} -\Delta u = \log u + h(x) u^q, \;\; x \in B_R, \\ u > 0, \;\; x \in B_R, \\ \frac{\partial u}{\partial \nu} = 0, \;\; x \in \partial B_R, \end{cases}$$

have been studied, see [1, 22].

The class of problems considered in this paper was motivated by the semilinear problem with Neumann problem. When $p = 2$, the main point of interest of the author with respect to the problem is the existence of positive solution, i.e. $u \in C^2(\overline{B}_R \backslash \{0\}) \cap C(\overline{B}_R)$, $u > 0$ in B_R (see [1]). To obtain their results, the authors use the nice properties of the operator $-\Delta$, for example, strong maximum principle, strong comparison principle and so on. Moreover, the authors obtained the estimate $||U_r||_{C^3[\rho,R]}$. This paper, we are interested in the existence of positive weak solution of (P), i.e. $u \in C^1(\overline{B}_R\backslash\{0\}) \cap C(\overline{B}_R)$, $u > 0$ in $\overline{B}_R$. Unfortunately, as we know, for the operator $-\Delta_m u$ with $m \neq 2$, many nice properties inherent to $-\Delta$ are lost or difficult to verify. For example, the strong maximum principle is lost. In this paper, we will use the weak comparison principle. Because of the weak regularity, we only consider the estimate $||\phi_m(u_r')||_{C^1[\rho,R]}$.

We modify the method developed in [1], and give the following theorem.

THEOREM 1.1. *The problem* (P) *has a positive solution* $u \in C^1(\overline{B}_R\backslash\{0\}) \cap C(\overline{B}_R)$, *if one of the following conditions holds.*
(1) *For* $\lambda \geqslant ||a||_\infty$, $\beta > 0$ *and* $h(x) \geqslant 1$ *for all* $x \in B_R$.
(2) *For* $\lambda > 0$ *large enough and* $0 < \beta < 1$.
(3) *For* $\lambda > 0$ *large enough,* $\frac{a(x)}{d(x,\partial B_R)^{m\beta/(m-1)}}$ *bounded in* $\overline{B}_R$ *and* $0 < \beta < 1$.

2. The perturbed problems

In this section, we study the perturbed problems used to get a solution to (P). In the first perturbation for each $\varepsilon > 0$, we consider a family of approximate problems

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + \frac{a(x)u}{(u+\varepsilon)^{\beta+1}} = \lambda h(x)u^p \ \text{ in } \ B_R, \\ u > 0 \ \text{ in } \ B_R, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial B_R. \end{cases} \tag{P_ε}$$

Our goal is to get a solution to the above problem for each $\varepsilon > 0$ and let $\varepsilon \to 0$ in order to find a solution to (P).

For each $0 < r < R$, define $A := B_R\backslash\overline{B}_r$. Consider the second family of problems

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) + \frac{a(x)u}{(u+\varepsilon)^{\beta+1}} = \lambda h(x)u^p \ \text{ in } \ B_R\backslash\overline{B}_r, \\ u > 0 \ \text{ in } \ B_R\backslash\overline{B}_r, \\ u = \sigma \ \text{ on } \ \partial B_r, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial B_R, \end{cases} \tag{$P_{\varepsilon,r}$}$$

where $\sigma > 0$ is an appropriate fixed constant. When $r \to 0^+$, we get a solution to (P_ε).

2.1. Existence of subsolution to $(P_{\varepsilon,r})$

In this subsection, we will show the existence of subsolution $\underline{u}$ to $(P_{\varepsilon,r})$, which does not depend on r and ε. To this end, we will drive our study in three cases.

CASE 1. $\left(\lambda \geqslant ||a||_\infty,\ \beta > 0 \text{ and } h(x) \geqslant 1 \text{ for all } x \in B_R\right)$

Since $\frac{a(x)}{(1+\varepsilon)^{\beta+1}} \leqslant \lambda h(x)$, in this case, $\underline{u} = 1$ is a subsolution to $(P_{\varepsilon,r})$, provided that $\sigma > 1$.

CASE 2. $(\lambda > 0$ large enough and $0<\beta<1)$
Let ξ be a positive solution of the below problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) = h(x)(d(x,\partial B_R))^{p\gamma} \text{ in } B_R, \\ u=0 \text{ on } \partial B_R, \end{cases}$$

where $\gamma = \frac{m(m-1)}{(m-1)+\beta}$.

Define $\underline{u} = k\xi^{\gamma/(m-1)}$. Computing the equation

$$\begin{aligned} &-\mathrm{div}(|\nabla \underline{u}|^{m-2}\nabla \underline{u}) + \frac{a(x)}{\underline{u}^\beta} \\ &\quad = -\Big(\frac{k\gamma}{m-1}\Big)^{m-1}(\gamma-m+1)\xi^{\gamma-m}|\nabla\xi|^m \\ &\qquad + \Big(\frac{k\gamma}{m-1}\Big)^{m-1}\xi^{\gamma-m+1}h(x)(d(x,\partial B_R))^{p\gamma} \\ &\qquad + \frac{a(x)k^{-\beta}}{\xi^{\beta\gamma/(m-1)}}. \end{aligned}$$

Since $\beta\in(0,1)$, $\gamma>m-1$ and consequent for $k>0$ large, in the interior and near the boundary ∂B_R we have

$$a(x)k^{-\beta} - \Big(\frac{k\gamma}{m-1}\Big)^{m-1}(\gamma-m+1)|\nabla\xi|^m < 0.$$

Now take $\lambda>0$ large enough such that

$$\Big(\frac{k\gamma}{m-1}\Big)^{m-1}\xi^{\gamma-m+1}h(x)(d(x,\partial B_R))^{p\gamma} \leqslant \lambda h(x)\underline{u}^p,$$

hence

$$-\mathrm{div}(|\nabla \underline{u}|^{m-2}\nabla \underline{u}) + \frac{a(x)}{\underline{u}^\beta} \leqslant \lambda h(x)\underline{u}^p \text{ in } B_R.$$

Finally we choose σ such that $k\xi^{\gamma/(m-1)} < \sigma$.

CASE 3. $(\lambda>0$ large enough and $\frac{a(x)}{d(x,\partial B_R)^{m\beta/(m-1)}}$ bounded in $\overline{B}_R$ and $0<\beta<1)$
Fixing $\gamma = m$, $\underline{u} = k\xi^{m/(m-1)}$ and repeating the same argument of Case 2, we get

$$\begin{aligned} &-\mathrm{div}(|\nabla \underline{u}|^{m-2}\nabla \underline{u}) + \frac{a(x)}{\underline{u}^\beta} \\ &\quad = -\Big(\frac{km}{m-1}\Big)^{m-1}|\nabla\xi|^m + \Big(\frac{km}{m-1}\Big)^{m-1}\xi h(x)(d(x,\partial B_R))^{pm} + \frac{a(x)k^{-\beta}}{\xi^{\beta m/(m-1)}}, \end{aligned}$$

since $\frac{a(x)}{d(x,\partial B_R)^{m\beta/(m-1)}}$ bounded in $\overline{B}_R$, there exists $N>0$ such that

$$-\operatorname{div}(|\nabla\underline{u}|^{m-2}\nabla\underline{u})+\frac{a(x)}{\underline{u}^{\beta}}$$
$$\leqslant-\Big(\frac{km}{m-1}\Big)^{m-1}|\nabla\xi|^{m}+\Big(\frac{km}{m-1}\Big)^{m-1}\xi h(x)(d(x,\partial B_R))^{pm}+Nk^{-\beta},$$

thus, using similar argument to Case 2, for k and λ large, we arrive at the same conclusion.

2.2. Existence of supersolution to $(P_{\varepsilon,r})$

In this subsection, we will use variational method to get a supersolution to $(P_{\varepsilon,r})$. Hereafter, $\sigma>||\underline{u}||_\infty$ and X denotes the subspace of $H^1(A)$, $A=B_R\backslash\overline{B}_r$, given by

$$X=\{v\in H^1(A):v(r)=0,v\text{ is radially symmetic}\}$$

endowed with the norm

$$||v||=\Big(\int_A|\nabla v|^m ds\Big)^{\frac{1}{m}}.$$

We stress that $||.||$ is a norm in X, since Poincare inequality holds in X, that is, there exist $\eta>0$ such that

$$\int_A|v|^m dx\leqslant\eta\int_A|\nabla v|^m dx,\ \ \forall\, v\in X. \tag{2.1}$$

Note that $v\in X$ is a solution of the problem below

$$\begin{cases}-(s^{N-1}\phi_m(v'))'=\lambda s^{N-1}h(s)(v+\sigma)^p \ \text{ in } \ (r,R),\\ v>0 \ \text{ in } \ (r,R),\\ v(r)=0,\\ v'(R)=0,\end{cases} \tag{2.2}$$

where $\phi_m(s)=|s|^{m-2}s$, the function $u=v+\sigma$ is a supersolution to $(P_{\varepsilon,r})$, because it is easy to check that

$$\begin{cases}-(s^{N-1}\phi_m(u'))'+\frac{s^{N-1}a(x)u}{(u+\varepsilon)^{\beta+1}}\geqslant\lambda s^{N-1}h(s)u^p \ \text{ in } \ (r,R),\\ u>0 \ \text{ in } \ (r,R),\\ u(r)=\sigma,\\ u'(R)=0.\end{cases}$$

To get a solution to (2.2), we will apply variational methods to the functional $I:X\to R$ given by

$$I(v)=\frac{1}{m}\int_r^R s^{N-1}|v'|^m ds-\lambda\int_r^R s^{N-1}h(s)F(v(s))ds,$$

where $F(t)=\int_0^t((z+\sigma)_+)^p dz$ and $z_+=\max\{z,0\}$.

LEMMA 2.1. *The functional I is C^1, weakly lower semicontinuous and coercive so that there exists $v \in X$ such that*

$$I(v) = \min_{u \in X} I(u) \quad \textit{and} \quad I'(v) = 0.$$

The proof is standard by (2.1).

From the above details, we conclude that v is a solution to (2.2). Also, since v is a weak solution of (2.2), we have

$$v(s) = \int_r^s \phi_m^{-1}\left[t^{1-N}\int_t^R \lambda z^{N-1} h(z)(v(z)+\sigma)^p dz\right] dt,$$

in which

$$\phi_m^{-1}(u) = \begin{cases} u^{\frac{1}{m-1}}, & \text{if } u \geqslant 0, \\ -(-u)^{\frac{1}{m-1}}, & \text{if } u < 0. \end{cases}$$

Then we define $u_r := v + \sigma$.

LEMMA 2.2. *The function $u_r = v + \sigma$ is a supersolution to $(P_{\varepsilon,r})$ with $u_r(s) \geqslant \sigma$ and $u_r(s) \geqslant \underline{u}(s)$ for every $s \in [r,R]$. Moreover,*

$$u_r(s) = \sigma + \int_r^s \phi_m^{-1}\left[t^{1-N}\int_t^R \lambda z^{N-1} h(z) u_r^p(z) dz\right] dt, \tag{2.3}$$

then $u_r \in C^1[r,R]$.

LEMMA 2.3. *There exists a constant $M > 0$ such that $||u_r||_{C[r,R]} \leqslant M$ for all $r \in (0,R)$. Moreover, for each $\rho \in (0,R)$, there exists $C_\rho > 0$ and $r_\rho \in (0,R)$ such that we have the following estimates:*

$$||u_r||_{C[\rho,R]}, \quad ||u_r||_{C^1[\rho,R]}, \quad ||\phi_m(u_r')||_{C^1[\rho,R]} \ \leqslant C_\rho.$$

Proof. From (2.3), we have that

$$\begin{aligned} u_r(s) &= \sigma + \int_r^s \phi_m^{-1}\left[t^{1-N}\int_t^R \lambda z^{N-1} h(z) u_r^p(z) dz\right] dt \\ &\leqslant \sigma + \int_r^R \phi_m^{-1}\left[t^{1-N}\int_t^R \lambda z^{N-1} h(z) u_r^p(z) dz\right] dt \\ &\leqslant \sigma + ||u_r||_{C[r,R]}^{\frac{p}{m-1}} \int_r^R \phi_m^{-1}\left[t^{1-N}\int_t^R \lambda z^{N-1} h(z) dz\right] dt. \end{aligned}$$

But

$$\int_r^R \phi_m^{-1}\left[t^{1-N}\int_t^R \lambda z^{N-1} h(z) dz\right] dt \leqslant C_1$$

for some constant independent of r.

Using the hypothesis $p \in (0, m-1)$, there exist positive constants C_2, C_3 independents of r verifying the inequality

$$u_r(s) \leqslant C_2 + C_3 ||u_r||_{C[r,R]}^{\frac{p}{m-1}},$$

which implies that there exists $M > 0$ independent of r, such that

$$||u_r||_{C[r,R]} \leqslant M.$$

In fact, since

$$||u_r||_{C[r,R]}^{\frac{m-1-p}{m-1}} \leqslant C_2 ||u_r||_{C[r,R]}^{-\frac{p}{m-1}} + C_3,$$

if $||u_r||_{C[r,R]} \to \infty$ as $r \to 0$, we get a contradiction. The estimates involving the norms $||u_r||_{C^1[\rho,R]}$, $||\phi_m(u_r')||_{C^1[\rho,R]}$ follows easily using the estimative found for $||u_r||_{C[\rho,R]}$. $\square$

2.3. Existence of solution to $(P_{\varepsilon,r})$

In this subsection, we use the sub-supersolution ($\underline{u}$ and u_r respectively) to obtain a solution for the problem $(P_{\varepsilon,r})$.

We use an interaction method starting from $u_0 = \underline{u}$ and define the sequence u_n, $n \in \mathbb{N}$, by solving the problems

$$\begin{cases} -(s^{N-1}\phi_m(u_{n+1}'))' + \frac{s^{N-1}a(s)u_{n+1}}{(u_n+\varepsilon)^{\beta+1}} = \lambda s^{N-1}h(s)u_n^p \ \text{ in } \ (r,R), \\ u_{n+1} > 0 \ \text{ in } \ (r,R), \\ u_{n+1}(r) = \sigma, \\ u_{n+1}'(R) = 0. \end{cases}$$

From [18], we first give the following Lemma.

LEMMA 2.4. (Weak Comparison Principle) *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0,\infty) \to (0,\infty)$ is continuous and nondecreasing. Let $u_1, u_2 \in W^{1,m}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{m-2}\nabla u_1 \nabla\psi dx + \int_\Omega \theta(u_1)\psi dx \leqslant \int_\Omega |\nabla u_2|^{m-2}\nabla u_2 \nabla\psi dx + \int_\Omega \theta(u_2)\psi dx$$

for all nonnegative $\psi \in W_0^{1,m}(\Omega)$. Then the inequality

$$u_1 \leqslant u_2 \ \ on \ \ \partial\Omega,$$

implies that

$$u_1 \leqslant u_2 \ \ in \ \ \Omega.$$

LEMMA 2.5. *The sequence u_n is nondecreasing and satisfies*

$$u_0(s) \leqslant u_n(s) \leqslant u_{n+1}(s) \leqslant u_r(s) \ \ for\ all\ s \in [r,R] \ \ and\ all \ \ n \in \mathbb{N}.$$

Proof. We just need to see that $u_0 \leqslant u_1 \leqslant u_r$. From the definition of u_1, we have

$$\begin{cases} -(s^{N-1}\phi_m(u_0'))' + g(s)u_0 \leqslant -(s^{N-1}\phi_m(u_1'))' + g(s)u_1, \ \ \forall s \in (r,R), \\ (u_0 - u_1)(r) \leqslant 0, \quad (u_0 - u_1)'(R) = 0, \end{cases}$$

where $g(s) = \frac{s^{N-1}a(s)}{(u_0+\varepsilon)^{\beta+1}}$. From Lemma 2.4, we obtain that

$$u_0 \leqslant u_1(s), \ \ \forall s \in [r,R].$$

On the other hand,

$$\begin{cases} -(s^{N-1}\phi_m(u_1'))' + g(s)u_1 \leqslant -(s^{N-1}\phi_m(u_r'))' + g(s)u_r, \ \ \forall s \in (r,R), \\ (u_1 - u_r)(r) \leqslant 0, \quad (u_1 - u_r)'(R) = 0, \end{cases}$$

and again apply Lemma 2.4, we get

$$u_1 \leqslant u_r(s), \ \ \forall s \in [r,R].$$

Thus,

$$u_0 \leqslant u_1(s) \leqslant u_r(s), \ \ \forall s \in [r,R].$$

Now, the proof follows by induction at n. □

By Lemma 2.5, we define the pointwise limit

$$u_r^\varepsilon(s) := \lim_{n\to\infty} u_n(s), \ \ \forall s \in [r,R],$$

and we see that

$$1 \leqslant u_r^\varepsilon(s) \leqslant u_r(s), \ \ \forall s \in [r,R]. \tag{2.4}$$

LEMMA 2.6. *The function* u_r^ε *belongs to* $C^1[r,R]$ *and verifies* $(P_{\varepsilon,r})$.

Proof. For each $n \in \mathbb{N}$, we have

$$u_n(s) = \sigma + \int_r^s \phi_m^{-1}\Big[t^{1-N}\int_t^R z^{N-1}(\lambda h(z)u_{n-1}^p - \frac{a(z)u_n}{(u_{n-1}+\varepsilon)^{\beta+1}})dz\Big]dt,$$

since

$$\underline{u}(s) \leqslant u_n(s) \leqslant u_r(s) \leqslant M, \ \ \forall n \in \mathbb{N}.$$

From Lemma 2.3, we obtain that $||\phi_m(u_r')||_{C^1[\rho,R]}$ is bounded. So for some subsequence, still denote by u_n, we have that

$$u_n \to u_r^\varepsilon \ \text{ in } \ C^1[r,R].$$

Therefore, u_r^ε is a solution of the problem

$$\begin{cases} -(s^{N-1}\phi_m(u_r^{\varepsilon\prime}))' + \frac{s^{N-1}a(s)u_r^\varepsilon}{(u_r^\varepsilon+\varepsilon)^{\beta+1}} = \lambda s^{N-1}h(s)(u_r^\varepsilon)^p \ \text{ in } \ (r,R), \\ u_r^\varepsilon > \sigma \ \text{ in } \ (r,R), \\ u_r^\varepsilon(r) = \sigma, \\ (u_r^\varepsilon)'(R) = 0. \end{cases}$$

2.4. Existence of solution to (P_ε)

From Lemma 2.6, u_r^ε is given by

$$u_r^\varepsilon = \sigma + \int_t^s \phi_m^{-1}\Big[t^{1-N}\int_t^R z^{N-1}(\lambda h(z)(u_r^\varepsilon)^p - \frac{a(z)u_r^\varepsilon}{(u_r^\varepsilon+\varepsilon)^{\beta+1}})dz\Big]dt, \tag{2.5}$$

and

$$\underline{u}(s) \leqslant u_r^\varepsilon(s) \leqslant u_r(s) \leqslant M, \ \ \forall s \in [r,R]. \tag{2.6}$$

Using (2.5) and (2.6), as in Lemma 2.3, we can prove that for each $\rho \in (0,R)$ there exist $C_\rho > 0$ and $r_\rho \in (r,R)$, such that

$$||u_r^\varepsilon||_{C[\rho,R]}, \quad ||u_r^\varepsilon||_{C^1[\rho,R]}, \ ||\phi_m({u_r^\varepsilon}')||_{C^1[\rho,R]} \ \leqslant C_\rho.$$

Then from the compact imbedding $C^1[\rho,R] \hookrightarrow C[\rho,R]$, we see that there exist a sequence $r_n \in (0,R)$ with $r_n \to 0$ and $u_\varepsilon : (0,R] \to R$, such that the sequence $w_n = u_{r_n}^\varepsilon$ satisfies

$$w_n \to u_\varepsilon \ \text{ in } \ C^1_{loc}(0,R),$$

and

$$w_n \to u_\varepsilon \ \text{ in } \ C[\rho,R] \ \ \forall \rho \in (0,R).$$

The above limits imply that u_ε is a solution of the problem (P_ε) written in radial form:

$$\begin{cases} -(s^{N-1}\phi_m(u_\varepsilon'))' + \frac{s^{N-1}a(s)u_\varepsilon}{(u_\varepsilon+\varepsilon)^{\beta+1}} = \lambda s^{N-1}h(s)(u_\varepsilon)^p \ \text{ in } \ (0,R), \\ u_\varepsilon > 0 \ \text{ in } \ (0,R), \\ u_\varepsilon'(R) = 0. \end{cases}$$

2.5. Existence of solution to (P)

Now we would like to pass the limit in the family u_ε obtained in above subsection and get a solution to (P). In order to do that, we need some estimates like the ones in Lemma 2.3.

First, we observe that the following estimate holds in $(0,R)$,

$$\underline{u}(s) \leqslant u_\varepsilon(s) \leqslant M, \ \ \forall s \in (0,R). \tag{2.7}$$

Note that the family u_ε verifies

$$u_\varepsilon(s) = u_\varepsilon(R/2) + \int_{R/2}^s \phi_m^{-1}\Big[t^{1-N}\int_t^R z^{N-1}(\lambda h(z)u_\varepsilon^p - \frac{a(z)u_\varepsilon}{(u_\varepsilon+\varepsilon)^{\beta+1}})dz\Big]dt, \tag{2.8}$$

if $s \in [\frac{R}{2},R]$,

$$u_\varepsilon(s) = u_\varepsilon(R/2) - \int_s^{R/2} \phi_m^{-1}\Big[t^{1-N}\int_t^R z^{N-1}(\lambda h(z)u_\varepsilon^p - \frac{a(z)u_\varepsilon}{(u_\varepsilon+\varepsilon)^{\beta+1}})dz\Big]dt, \tag{2.9}$$

if $s \in (0,\frac{R}{2}]$.

Using (2.7)-(2.9), for each $\rho \in (0,R)$ fixed, there exist $C_\rho > 0$ and $\varepsilon_\rho \in (r,R)$, such that

$$||u_\varepsilon||_{C[\rho,R]}, \quad ||u_\varepsilon||_{C^1[\rho,R]}, \quad ||\phi_m(u'_\varepsilon)||_{C^1[\rho,R]} \leqslant C_\rho, \quad \forall \varepsilon \in (0,\varepsilon_\rho).$$

Hence, there exist a sequence $l_n \in (0,R)$ with $l_n \to 0$ and $u : (0,R] \to R$, such that the sequence $z_n = u_{l_n}$ satisfies

$$z_n \to u \ \text{ in } C^1_{loc}(0,R),$$

and

$$z_n \to u \ \text{ in } \ C[\rho,R], \ \forall \rho \in (0,R).$$

Using the above information, $u(s) \geqslant \underline{u}(s)$, $\forall s \in (0,R]$ and u is a solution of the problem

$$\begin{cases} -(s^{N-1}\phi_m(u'))' + \frac{s^{N-1}a(s)}{u^\beta} = \lambda s^{N-1}h(s)u^p \ \text{ in } \ (0,R), \\ u > 0 \ \text{ in } \ (0,R), \\ u'(R) = 0. \end{cases} \tag{2.10}$$

We see that $u \in C^1(0,R) \cap C(0,R]$. Now, the next Lemma shows that we can extend u to the interval $[0,R]$ in such a way that u is continuous on $[0,R]$.

We use the same arguments with Lemma 2.6 in [1], and give the following lemma.

LEMMA 2.7. *If u is the solution found for (2.10), we have that $\lim_{r\to 0} u(r)$ exists. Hence, we can extend u to the interval $[0,R]$, assuming $u(0) = \lim_{r\to 0} u(r)$. Thus getting the regularity $u \in C^1(\overline{B}_R \backslash \{0\}) \cap C(\overline{B}_R)$.*

Using the above information, we give the proof of Theorem 1.1.

Acknowledgements. The authors would like to thank the referee for his/her suggestions in the revision of this work.

REFERENCES

[1] C. O. ALVES, M. MONTENEGRO, *Positive solutions to a singular Neumann problem*, J. Math. Anal. Appl., **352** (2009), 112–119.
[2] G. ASTRITA, G. MARRUCCI, *Principles of non-Newtonian fluid mechanics*, McGraw-Hill, 1974.
[3] H. BREZIS, L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), 437–477.
[4] F. CIRSTEA, V. RADULESCU, *Existence and uniqueness of positive solutions to a semilinear elliptic problem in $\mathbb{R}^N$*, J. Math. Anal. Appl., **229** (1999), 417–425.
[5] M. G. CRANDALL, P. H. RABINOWITZ, L. TARTAR, *On a Dirichlet problem with singular nonlinearity*, Comm. Partial Differential Equations, **2** (1997), 193–222.
[6] J. I. DIAZ, J. M. MOREL, L. OSWALD, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations, **12**, 12 (1987), 1333–1344.
[7] A. EDELSON, *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl., **139** (1989), 523–532.
[8] Z. M. GUO, *Some existence and multiplicity results for a class of quasilinear elliptic eigenvalue problems*, Nonlinear Anal., **18** (1992), 957–971.
[9] Z. M. GUO, *Existence and uniqueness of positive radial solutions for a class of quasilinear elliptic equations*, Appl. Anal., **47** (1992), 173–190.

[10] Z. M. GUO, *Boundary value problems of a class of quasilinear ordinary differential equations*, Differential Integral Equations, **6**, 3 (1993), 705–719.

[11] Z. M. GUO, *On the number of positive solutions for quasilinear elliptic eigenvalue problems*, Nonlinear Anal., **27**, 2 (1996), 229–247.

[12] Z. M. GUO, J. R. L. WEBB, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large*, Proc. Roy. Soc. Edinburgh. Sect. A, **124** (1994), 189–198.

[13] Z. M. GUO, J. R. L. WEBB, *Large and small solutions of a class of quasilinear elliptic eigenvalue problems*, J. Differential Equations, **180** (2002), 1–50.

[14] A. S. KALASHNIKOV, *On a nonlinaer equation appearing in the theory of non-stationary filtration*, Trudy Sem. Petrovsk., **4** (1978), 137–146.

[15] T. KUSANO, C. A. SWANSON, *Entire positive solutions of singular semilinear elliptic equations*, Japan J. Math., **11** (1985), 145–155.

[16] A. C. LAZER, P. J. MCKENNA, *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc., **111**, 3 (1991), 721–730.

[17] A. V. LAIR, A. W. SHAKER, *Entire solution of a singular semilinear elliptic problem*, J. Math. Anal. Appl., **200** (1996), 498–505.

[18] Q. S. LU, Z. D. YANG, E. H. TWIZELL, *Existence of entire explosive solutions of quasilinear elliptic equations*, Appl. Math. Comput., **148** (2004), 359–372.

[19] L. K. MARTINSON, K. B. PAVLOV, *Unsteady shear flows of a conducting fluid with a rheological power law*, Magnitnaya Gidrodinamika, **2** (1971), 50–58.

[20] Q. MIAO, Z. D. YANG, *Bounded Positive Entire Solutions of Singular p-Laplacian Equations*, Nonlinear Anal., **69** (2008), 3749–3760.

[21] M. DEL PINO, G. HERNANDEZ, *Solvability of the Neumann problem in a ball for $-\Delta u + u^{-\nu} = h(|x|), \nu > 1$*, J. Differential Equations, **124** (1996), 108–131.

[22] O. S. DE QUEIROZ, *A Neumann problem with logarithmic nonlinearity in a ball*, Nonlinear Anal., **70** (2009), 1656–1662.

[23] B. J. XUAN, Z. C. CHEN, *Solvability of singular quasilinear elliptic equation*, Chinese Ann. Math. Ser. A, **20**, 1 (1999), 117–128.

[24] Z. D. YANG, *Non-existence of positive entire solutions for elliptic inequalities of p-laplacian*, Appl. Math. J. Chinese Univ. Ser. B, **12**, 4 (1997), 399–410.

[25] Z. D. YANG, *Existence of entire explosive positive radial solutions for a class of quasilinear elliptic systems*, J. Math. Anal. Appl., **288**, 2 (2003), 768–783.

[26] Z. D. YANG, *Existence of positive entire solutions for singular and non-singular quasi-linear elliptic equation*, J. Comput. Appl. Math., **197**, 2 (2006), 355–364.

[27] Z. D. YANG, Q. S. LU, *Blow-up estimates for a quasilinear reaction-diffusion system*, Math. Methods Appl. Sci., **26** (2003), 1005–1023.

[28] Z. D. YANG, Q. S. LU, *Nonexistence of positive solutions to a quasilinear elliptic system and blow-up estimates for a quasilinear reaction-diffusion system*, J. Comput. Appl. Math., **50** (2003), 37–56.

(Received June 19, 2010)
(Revised September 4, 2010)

Qing Miao[a] *and Zuodong Yang*[ab]
[a] *Institute of Mathematics, School of Mathematical Science*
Nanjing Normal University
Jiangsu Nanjing 210046
China;

[b] *College of Zhongbei*
Nanjing Normal University
Jiangsu Nanjing 210046
China
e-mail: zdyang_jin@263.net