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EXISTENCE OF BOUNDED SOLUTIONS FOR A CLASS OF NONLINEAR FOURTH–ORDER EQUATIONS

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Abstract. In this article, we consider a class of nonlinear elliptic fourth-order equations with the principal part satisfying a strengthened coercivity condition. It is supposed that the lower-order term of the equations admits an arbitrary growth with respect to unknown function and the growth rates of derivatives of this function coinciding with the exponents of the corresponding energy space. We prove a theorem on existence of bounded generalized solutions of the Dirichlet problem for equations of the given class.

1. Introduction

In [18] a class of nonlinear elliptic equations of the form

$$\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^{\alpha}\mathscr{A}_{\alpha}(x,u,\ldots,D^{m}u)=0 \text{ in } \Omega,\ m>1, \tag{1.1}$$

was introduced, all generalized solutions of which are bounded and Hölder continuous. This class is characterized by a strengthened coercivity condition on leading coefficients $\mathscr{A}_{\alpha}$, $1\leqslant|\alpha|\leqslant m$. In a model case this condition means that for every $x\in\Omega$ and for every $\xi=\{\xi_{\alpha}\in\mathbb{R}:|\alpha|\leqslant m\}$ the following inequality holds:

$$\sum_{1\leqslant|\alpha|\leqslant m}\mathscr{A}_{\alpha}(x,\xi)\xi_{\alpha}\geqslant C\left(\sum_{|\alpha|=1}|\xi_{\alpha}|^{q}+\sum_{|\alpha|=m}|\xi_{\alpha}|^{p}\right).$$

Here Ω is a bounded domain of $\mathbb{R}^{n}$, $p\geqslant 2$, $mp<q<n$ and $C>0$. At the same time, in [18] it was supposed that the lower-order term $\mathscr{A}_{0}$ may have the growth of a rate less than $nq/(n-q)-1$ with respect to the function u and the growth of rates definitely less than q and p with respect to the derivatives $D^{\alpha}u$, $|\alpha|=1$, and the derivatives $D^{\alpha}u$, $|\alpha|=m$, accordingly.

We observe that the proof of boundedness of generalized solutions in [18] uses a modification of Moser's method [15]. The approach of [18] was developed in [9, 10, 16], where the boundedness and regularity of solutions were studied for high-order

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equations and variational inequalities with degenerate nonlinear elliptic operator satisfying a strengthened coercivity condition. A system of two degenerate nonlinear elliptic fourth-order equations with a strengthened coercivity condition was considered in [11], and following the approach of [9, 10, 16] results on the boundedness and Hölder continuity of generalized solutions of the Dirichlet problem for this system were obtained.

Using an analogue of Stampacchia's method [5, 19, 20], a weaker condition on integrability of data was established in [12] to guarantee the boundedness of generalized solutions of nonlinear fourth-order equations with a strengthened coercivity. Moreover, a dependence of summability of generalized solutions of these equations on integrability of data was described in [12]. Analogous results for nonlinear high-order equations with a strengthened coercivity were obtained in [21].

Let us give the precise description of the main results of [21].

Let m, $n \in \mathbb{N}$ be numbers such that $m \geqslant 3$, $n > 2(m-1)$. Let $p \in \mathbb{R}$ be a number such that $2n(m-2)/[n(m-1)-2] < p < n/m$. We set $\overline{p} = 2p/[p(m-1)-2(m-2)]$, and let $q \in \mathbb{R}$ be a number such that $\max(\overline{p}, mp) < q < n$.

Let Ω be a bounded open set of $\mathbb{R}^n$. We denote by $W^{1,q}_{m,p}(\Omega)$ the set of all functions $u \in W^{1,q}(\Omega)$ having for every n-dimensional multiindex α, $|\alpha| = m$, the weak derivative $D^\alpha u \in L^p(\Omega)$. $W^{1,q}_{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}.$$

We denote by $\overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,q}_{m,p}(\Omega)$.

We set $q^* = nq/(n-q)$. As is known (see for instance [4, Chapter 7]),

$$\overset{\circ}{W}{}^{1,q}(\Omega) \subset L^{q^*}(\Omega), \tag{1.2}$$

and there exists a positive constant c depending only on n and q such that for every function $u \in \overset{\circ}{W}{}^{1,q}(\Omega)$,

$$\left(\int_\Omega |u|^{q^*} dx \right)^{1/q^*} \leqslant c \left(\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^q dx \right)^{1/q}. \tag{1.3}$$

We shall use the following notation: Λ_m is the set of all n-dimensional multi-indices α such that $1 \leqslant |\alpha| \leqslant m$; $\mathbb{R}^{n,m}$ is the space of all functions $\xi : \Lambda_m \to \mathbb{R}$; if $u \in W^{m,1}(\Omega)$, then $\nabla_m u : \Omega \to \mathbb{R}^{n,m}$ is the mapping such that for every $x \in \Omega$ and $\alpha \in \Lambda_m$, $(\nabla_m u(x))_\alpha = D^\alpha u(x)$. For every measurable set $E \subset \Omega$ we denote by $\operatorname{meas} E$ Lebesgue measure of the set E.

Next, let $c_1, c_2, c_3 > 0$, $\max(\overline{p}, mp) < q_1 < q$ and let the numbers p_α be defined by $p_\alpha = q$ if $|\alpha| = 1$, and

$$\frac{1}{p_\alpha} = \frac{|\alpha|-1}{p(m-1)} + \frac{m-|\alpha|}{q_1(m-1)} \text{ if } 2 \leqslant |\alpha| \leqslant m.$$

Let $g \in L^1(\Omega)$, $g \geqslant 0$ in Ω, and let for every $\alpha \in \Lambda_m$, $A_\alpha : \Omega \times \mathbb{R}^{n,m} \to \mathbb{R}$ be a Carathéodory function. Assume that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{n,m}$ the following inequalities hold:

$$\sum_{\alpha \in \Lambda_m} |A_\alpha(x,\xi)|^{p_\alpha/(p_\alpha - 1)} \leqslant c_1 \sum_{\beta \in \Lambda_m} |\xi_\beta|^{p_\beta} + g(x), \tag{1.4}$$

$$\sum_{\alpha \in \Lambda_m} A_\alpha(x,\xi)\xi_\alpha \geqslant c_2 \left(\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=m} |\xi_\alpha|^p \right) - c_3 \sum_{2 \leqslant |\alpha| \leqslant m-1} |\xi_\alpha|^{p_\alpha} - g(x). \tag{1.5}$$

Let

$$F \in L^{q^*/(q^*-1)}(\Omega). \tag{1.6}$$

The following Dirichlet problem is considered:

$$\sum_{\alpha \in \Lambda_m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_m u) = F \quad \text{in} \quad \Omega, \tag{1.7}$$

$$D^\alpha u = 0, \quad |\alpha| \leqslant m-1, \quad \text{on} \quad \partial\Omega. \tag{1.8}$$

DEFINITION 1.1. A generalized solution of problem (1.7), (1.8) is a function $u \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ such that for every function $v \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$,

$$\int_\Omega \left(\sum_{\alpha \in \Lambda_m} A_\alpha(x, \nabla_m u) D^\alpha v \right) dx = \int_\Omega F v dx. \quad \square \tag{1.9}$$

REMARK 1.1. If $v \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$, by virtue of Nirenberg-Gagliardo interpolation inequality [17], we have $D^\alpha v \in L^{p_\alpha}(\Omega)$, $1 < |\alpha| < m$. Then condition (1.4) ensures the existence of the integral in the left-hand side of (1.9). Moreover, it follows from (1.2) and (1.6) that for every function $v \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ the function Fv is summable on Ω. $\square$

By virtue of (1.2), every generalized solution of problem (1.7), (1.8) belongs to the space $L^{q^*}(\Omega)$. However, if the functions F and g have an improved summability, then the summability of any generalized solution of the problem under consideration is higher than the summability characterized by the exponent q^*. The corresponding dependence is described by the following theorem which is the main result of [21].

THEOREM 1.1. *Suppose that $r > q^*/(q^*-1)$, the functions F and g belong to $L^r(\Omega)$, and M is a majorant for the norms $\|g\|_{L^r(\Omega)}$ and $\|F\|_{L^r(\Omega)}$. Assume that u is a generalized solution of problem (1.7), (1.8). Then the following assertions hold:*

(i) *if $r < n/q$ and $q^* < \lambda < nr(q-1)/(n-qr)$, then $u \in L^\lambda(\Omega)$ and $\|u\|_{L^\lambda(\Omega)} \leqslant C_1$, where C_1 is a positive number depending only on n, m, p, q, q_1, $\operatorname{meas}\Omega$, c_1, c_2, c_3, r, M and λ;*

(ii) *if* $r = n/q$, *then* $\int_\Omega \exp(b|u|^{1/\sigma})\,dx \leqslant C_2$, *where* $\sigma > 1$ *depends only on* n, m, p, q, q_1, *and* b *and* C_2 *are positive numbers depending only on* n, m, p, q, q_1, meas Ω, c_1, c_2, c_3 *and* M*;*

(iii) *if* $r > n/q$, *then* $u \in L^\infty(\Omega)$ *and* $\|u\|_{L^\infty(\Omega)} \leqslant C_3$, *where* C_3 *is a positive number depending only on* n, m, p, q, q_1, meas Ω, c_1, c_2, c_3, r *and* M.

REMARK 1.2. If $t > n/q$, $r > n^2/(nq-n+q)$, $g \in L^t(\Omega)$ and $F \in L^r(\Omega)$, then the boundedness of generalized solutions of problem (1.7), (1.8) follows from [18]. Since $n^2/(nq-n+q) > n/q$, assertion (iii) of Theorem 1.1 gives a weaker (as compared with [18]) condition on the summability of the right-hand side of equation (1.7) under which generalized solutions of problem (1.7), (1.8) are bounded. This condition $(r > n/q)$ coincides with a condition for the boundedness of generalized solutions of equations of the second order [13]. □

In the present article, we consider a class of nonlinear fourth-order equations of type (1.1) ($m = 2$) with the principal part satisfying a strengthened coercivity condition and the lower-order term admitting, unlike [12, 18, 21], an arbitrary growth with respect to the function u, the growth of the rate q with respect to the derivatives $D^\alpha u$, $|\alpha| = 1$, and the growth of the rate p with respect to the derivatives $D^\alpha u$, $|\alpha| = 2$. At the same time, it is supposed that the lower-order term satisfies a sign condition. The main result of the article is a theorem on existence of bounded generalized solutions of the Dirichlet problem for equations investigated. We note that in the case under consideration q and p are the exponents of the energy space corresponding to the given problem.

Similar results for nonlinear elliptic second-order equations with natural growth lower-order terms were established for instance in [1-3].

Finally, we remark that a theory of existence and properties of solutions of nonlinear elliptic fourth-order equations with coefficients satisfying a strengthened coercivity condition and L^1-right-hand sides was developed in [6, 8].

2. Preliminaries and statement of the main result

Let $n \in \mathbb{N}$, $n > 2$, and let Ω be a bounded open set of $\mathbb{R}^n$.

We shall use the following notation: Λ is the set of all n-dimensional multi-indices α such that $|\alpha| = 1$ or $|\alpha| = 2$; $\mathbb{R}^{n,2}$ is the space of all mappings $\xi : \Lambda \to \mathbb{R}$; if $u \in W^{2,1}(\Omega)$, then $\nabla_2 u : \Omega \to \mathbb{R}^{n,2}$, and for every $x \in \Omega$ and for every $\alpha \in \Lambda$, $(\nabla_2 u(x))_\alpha = D^\alpha u(x)$.

Let $p \in (1, n/2)$ and $q \in (2p, n)$. We denote by $W^{1,q}_{2,p}(\Omega)$ the set of all functions in $W^{1,q}(\Omega)$ that have the second-order generalized derivatives in $L^p(\Omega)$. The set $W^{1,q}_{2,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \left(\sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}.$$

We denote by $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\Omega)$.

Next, let $c_1, c_2, c_3 > 0$, let g_1, g_2, g_3 be nonnegative summable functions on Ω, and let for every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ be a Carathéodory function. We assume that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{n,2}$ the following inequalities hold:

$$\sum_{|\alpha|=1} |A_\alpha(x,\xi)|^{q/(q-1)} \leqslant c_1 \bigg(\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \bigg) + g_1(x), \tag{2.1}$$

$$\sum_{|\alpha|=2} |A_\alpha(x,\xi)|^{p/(p-1)} \leqslant c_2 \bigg(\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \bigg) + g_2(x), \tag{2.2}$$

$$\sum_{\alpha\in\Lambda} A_\alpha(x,\xi)\xi_\alpha \geqslant c_3 \bigg(\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \bigg) - g_3(x). \tag{2.3}$$

We also assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^{n,2}$, $\xi \neq \xi'$,

$$\sum_{\alpha\in\Lambda} [A_\alpha(x,\xi) - A_\alpha(x,\xi')](\xi_\alpha - \xi'_\alpha) > 0. \tag{2.4}$$

Let g_4 and g_5 be nonnegative summable functions on Ω, let b be a nonnegative continuous function on $\mathbb{R}_+$, and let $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^{n,2}$ the following inequalities hold:

$$|B(x,s,\xi)| \leqslant b(|s|) \bigg(\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \bigg) + g_4(x), \tag{2.5}$$

$$B(x,s,\xi)s \geqslant -g_5(x). \tag{2.6}$$

Further, let

$$f \in L^{q^*/(q^*-1)}(\Omega). \tag{2.7}$$

We consider the following Dirichlet problem:

$$\sum_{\alpha\in\Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u) + B(x, u, \nabla_2 u) = f \quad \text{in} \quad \Omega, \tag{2.8}$$

$$D^\alpha u = 0, \quad |\alpha| = 0, 1, \quad \text{on} \quad \partial\Omega. \tag{2.9}$$

Observe that, by virtue of (2.1) and (2.2), for every $u, v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ and $\alpha \in \Lambda$ the function $A_\alpha(x, \nabla_2 u) D^\alpha v$ is summable on Ω, and, by (2.5), for every $u, v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ the function $B(x, u, \nabla_2 u)v$ is summable on Ω. Moreover, it follows from (1.2) and (2.7) that for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ the function fv is summable on Ω.

DEFINITION 2.1. A generalized solution of problem (2.8), (2.9) is a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ such that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \bigg(\sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v + B(x, u, \nabla_2 u) v \bigg) dx = \int_\Omega f v dx. \quad \square \tag{2.10}$$

The following theorem is the main result of the present article.

THEOREM 2.1. *Let* $r > n/q$, *let the functions* g_2, g_3, g_5 *and* f *belong to* $L^r(\Omega)$, *and let* M *be a majorant for* $L^r(\Omega)$*-norms of the functions* g_2, g_3, g_5 *and* f. *Then there exists a generalized solution* u *of problem* (2.8), (2.9) *such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1, \tag{2.11}$$

where C_1 *is a positive constant depending only on* n, p, q, c, c_2, c_3, r, M *and* meas Ω.

REMARK 2.1. Condition $r > n/q$ in the statement of Theorem 2.1 coincides with the condition of boundedness of generalized solutions of the Dirichlet problem considered in [12] for equation (2.8) with $B \equiv 0$. □

REMARK 2.2. The proof of Theorem 2.1 is based on the consideration of a sequence of approximate problems for equations with bounded lower-order terms, obtaining the uniform boundedness of their solutions and the subsequent limit passage. At the same time, the proof of the uniform boundedness of solutions of the approximate problems uses the approach of [12] analogous to Stampacchia's method. The limit passage in the approximate problems is justified using ideas of [3, 6, 7]. □

EXAMPLE 2.1. Let for every n-dimensional multiindex α, $|\alpha| = 1$, $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ be the function defined by

$$A_\alpha(x,\xi) = \Big(\sum_{|\beta|=1} \xi_\beta^2\Big)^{(q-2)/2} \xi_\alpha, \quad (x,\xi) \in \Omega \times \mathbb{R}^{n,2},$$

and let for every n-dimensional multiindex α, $|\alpha| = 2$, $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ be the function defined by

$$A_\alpha(x,\xi) = \Big(\sum_{|\beta|=2} \xi_\beta^2\Big)^{(p-2)/2} \xi_\alpha, \quad (x,\xi) \in \Omega \times \mathbb{R}^{n,2}.$$

Define the function $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ by

$$B(x,s,\xi) = s|s|^\gamma \Big(\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p\Big), \quad (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2},$$

where $\gamma > 0$. Then the functions A_α, $\alpha \in \Lambda$, satisfy inequalities (2.1)-(2.4), and the function B satisfies inequalities (2.5), (2.6). □

Observe that the coefficients of the biharmonic operator $\Delta^2 u$ do not satisfy condition (2.3).

3. Proof of Theorem 2.1

Step 1. Let for every $i \in \mathbb{N}$, $T_i : \mathbb{R} \to \mathbb{R}$ be the function such that

$$T_i(s) = \begin{cases} s & \text{if } |s| \leqslant i, \\ i \operatorname{sign} s & \text{if } |s| > i. \end{cases}$$

Now, for every $i \in \mathbb{N}$ we define the function $B_i : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ by

$$B_i(x,s,\xi) = T_i(B(x,s,\xi)), \quad (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2}.$$

From (1.3), (2.1)-(2.4) and (2.7) and the results of [14] on solvability of equations with pseudomonotone operators it follows that if $i \in \mathbb{N}$, then there exists a function $u_i \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ such that for every $v \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$,

$$\int_\Omega \left(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_i) D^\alpha v + B_i(x, u_i, \nabla_2 u_i) v \right) dx = \int_\Omega f v dx. \tag{3.1}$$

For every $i \in \mathbb{N}$ we set

$$\Phi_i = \sum_{|\alpha|=1} |D^\alpha u_i|^q + \sum_{|\alpha|=2} |D^\alpha u_i|^p .$$

Observe that for every $i \in \mathbb{N}$,

$$\int_\Omega \Phi_i \, dx \leqslant c_4, \tag{3.2}$$

where c_4 is a positive constant depending only on q, c, c_3, $\|g_3\|_{L^1(\Omega)}$, $\|g_5\|_{L^1(\Omega)}$ and $\|f\|_{L^{q^*/(q^*-1)}(\Omega)}$. In fact, fixing an arbitrary $i \in \mathbb{N}$ and putting into (3.1) the function u_i instead of v, we get

$$\int_\Omega \left(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_i) D^\alpha u_i + B_i(x, u_i, \nabla_2 u_i) u_i \right) dx = \int_\Omega f u_i \, dx.$$

This along with (2.3) and (2.6) implies that

$$c_3 \int_\Omega \Phi_i \, dx \leqslant \int_\Omega f u_i \, dx + \int_\Omega g_3 \, dx + \int_\Omega g_5 \, dx.$$

From this inequality, estimating the first addend in the right-hand side by means of Hölder's and Young's inequalities and (1.3), we deduce (3.2).

By virtue of (3.2) and (1.3) and the compactness of the embedding of $\overset{\circ}{W}{}^{1,q}(\Omega)$ into $L^\lambda(\Omega)$ with $\lambda < q^*$, there exist an increasing sequence $\{i_j\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ such that

$$u_{i_j} \to u \quad \text{weakly in} \quad \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega), \tag{3.3}$$

$$u_{i_j} \to u \quad \text{a.e. in } \Omega. \tag{3.4}$$

Step 2. Following [12], we will prove the uniform boundedness of the sequence $\{u_i\}$ and then will establish estimate (2.11). For this we will need the following auxiliary result proved in [12].

LEMMA 3.1. *Let φ be a nonincreasing nonnegative function on $[0,+\infty)$. Let $C>0$, $0\leqslant \tau_1<\tau_2$, $\gamma>1$ and $k_0\geqslant 0$. Let for every k and l such that $k_0<k<l<2k$ the following inequality holds:*

$$\varphi(l)\leqslant \frac{Ck^{\tau_1}}{(l-k)^{\tau_2}}[\varphi(k)]^{\gamma}.$$

Let $d>k_0$ and

$$d^{\tau_2-\tau_1}\geqslant 2^{\tau_1+(2\gamma-1)\tau_2/(\gamma-1)}C[\varphi(k_0)]^{\gamma-1}.$$

Then $\varphi(k_0+d)=0$.

Note that Lemma 3.1 is an analogue of the corresponding part of Stampacchia's lemma [20].

Let $r>n/q$, let the functions g_2, g_3, g_5 and f belong to $L^r(\Omega)$, and let M be a majorant for $L^r(\Omega)$-norms of the functions g_2, g_3, g_5 and f.

By c_i, $i=5,6,\dots$, we shall denote positive constants depending only on n, p, q, c, c_2, c_3, r, M and $\operatorname{meas}\Omega$.

Let us introduce some auxiliary numbers and functions. In view of the assumption on r, we have $(r-1)/r-1/q^*>0$. We set

$$r_1=\left(\frac{r-1}{r}-\frac{1}{q^*}\right)^{-1}, \quad \gamma=q^*\min\left\{\frac{1}{r_1(q-1)},\frac{r-1}{rq}\right\}. \tag{3.5}$$

By the first equality of (3.5), we have

$$\frac{1}{r_1}+\frac{1}{r}+\frac{1}{q^*}=1. \tag{3.6}$$

Furthermore, from the inequality $r>n/q$ and the definition of r_1 and γ it follows that

$$\gamma>1. \tag{3.7}$$

We fix an arbitrary $i\in\mathbb{N}$, and let φ be the function on $[0,+\infty)$ such that for every $s\in[0,+\infty)$,

$$\varphi(s)=\operatorname{meas}\{|u_i|\geqslant s\}.$$

By virtue of (1.3) and (3.2), for every $k>0$ we have $k^{q^*}\varphi(k)\leqslant c^{q^*}c_4^{q^*/q}$. Therefore,

$$\forall k\geqslant c(c_4+1), \qquad \varphi(k)<1. \tag{3.8}$$

Next, we set

$$t=\frac{2qpr}{q-2p}+2, \tag{3.9}$$

and let ψ be the function on $(0,+\infty)$ such that for every $s \in (0,+\infty)$,

$$\psi(s) = s - s^t + \frac{t-1}{t+1} s^{t+1}.$$

We set

$$k_0 = \max\{c(c_4+1),\ 6n(t-2)(c_2+n)/c_3\} \tag{3.10}$$

and fix an arbitrary number $k \geqslant k_0$.

Let $h_k : \mathbb{R} \to \mathbb{R}$ be the function such that

$$h_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ \left[\psi\left(\frac{|s|-k}{k}\right) + 1\right] k \operatorname{sign} s & \text{if } k < |s| < 2k, \\ \frac{2kt}{t+1} \operatorname{sign} s & \text{if } |s| \geqslant 2k. \end{cases}$$

We have $h_k \in C^2(\mathbb{R})$ and

$$|h_k| < 2k \quad \text{in } \mathbb{R}, \tag{3.11}$$

$$0 \leqslant h_k' \leqslant 1 \quad \text{in } \mathbb{R}, \tag{3.12}$$

$$|h_k''| \leqslant \frac{t^2}{k} \quad \text{in } \mathbb{R}. \tag{3.13}$$

Moreover, the following assertions hold:

$(*_1)$ if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $k \leqslant |s| \leqslant (1+\varepsilon)k$, then

$$|h_k''(s)| \leqslant \frac{t^2}{k} \varepsilon^{t-2};$$

$(*_2)$ if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $(1+\varepsilon)k \leqslant |s| \leqslant 2k$, then

$$|h_k''(s)| \leqslant \frac{t}{k\varepsilon}(1 - h_k'(s));$$

$(*_3)$ if $k < l \leqslant 2k$, $s \in \mathbb{R}$ and $|s| \geqslant l$, then

$$|s - h_k(s)| \geqslant \frac{2}{t+1}(l-k)\left(\frac{l-k}{k}\right)^{t-1}.$$

These assertions were proved in [12]. Assertion $(*_3)$ implies that the following assertion holds:

$(*_4)$ if $k < l \leqslant 2k$, then

$$\varphi(l) \leqslant \frac{t^{q^*} k^{(t-1)q^*}}{(l-k)^{tq^*}} \int_\Omega |u_i - h_k(u_i)|^{q^*} dx. \tag{3.14}$$

Let us estimate in a suitable way the integral in the right-hand side of inequality (3.14). This will allow us to apply Lemma 3.1 and obtain the uniform boundedness of the functions u_i.

Using properties (3.11)-(3.13), by analogy with Lemma 2.2 of [6], we establish that $h_k(u_i) \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ and the following assertions hold:

$(*_5)$ for every n-dimensional multi-index $\alpha, |\alpha| = 1$,

$$D^\alpha h_k(u_i) = h_k'(u_i) D^\alpha u_i \quad \text{a.e. in } \Omega;$$

$(*_6)$ for every n-dimensional multi-index $\alpha, |\alpha| = 2$,

$$|D^\alpha h_k(u_i) - h_k'(u_i) D^\alpha u_i| \leqslant |h_k''(u_i)| \sum_{|\beta|=1} |D^\beta u_i|^2 \quad \text{a.e. in } \Omega.$$

We set

$$I_k = \int_\Omega |f||u_i - h_k(u_i)|dx, \;\; J_k = \int_\Omega g_5 |u_i - h_k(u_i)|dx,$$
$$I_k' = \int_\Omega \bigg(\sum_{|\alpha|=2} |A_\alpha(x, \nabla_2 u_i)| \bigg) \bigg(\sum_{|\beta|=1} |D^\beta u_i|^2 \bigg) |h_k''(u_i)| dx.$$

Putting the function $u_i - h_k(u_i)$ into (3.1) instead of v, we obtain

$$\int_\Omega \bigg(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_i) D^\alpha (u_i - h_k(u_i)) \bigg) dx + \int_\Omega B_i(x, u_i, \nabla_2 u_i)(u_i - h_k(u_i))dx = \int_\Omega f(u_i - h_k(u_i))dx. \quad (3.15)$$

Using (2.6) and the fact that in the set $\{|u_i| \geqslant k\}$

$$0 \leqslant \frac{u_i - h_k(u_i)}{u_i} \leqslant \frac{|u_i - h_k(u_i)|}{k_0},$$

we establish that

$$\int_\Omega B_i(x, u_i, \nabla_2 u_i)(u_i - h_k(u_i))dx \geqslant -\frac{1}{k_0} J_k. \quad (3.16)$$

From (3.15) and (3.16) and assertions $(*_5)$ and $(*_6)$ we deduce that

$$\int_\Omega \bigg(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_i) D^\alpha u_i \bigg) (1 - h_k'(u_i)) dx \leqslant I_k + I_k' + \frac{1}{k_0} J_k.$$

Hence, using (2.3) and (3.12) and the fact that $h_k' = 1$ on $(-k, k)$, we get

$$c_3 \int_\Omega \Phi_i (1 - h_k'(u_i)) dx \leqslant \int_{\{|u_i| \geqslant k\}} g_3 \, dx + I_k + I_k' + \frac{1}{k_0} J_k. \quad (3.17)$$

Let us obtain suitable estimates for the addends in the right-hand side of inequality (3.17). Clearly,

$$\int_{\{|u_i| \geqslant k\}} g_3 \, dx \leqslant M[\varphi(k)]^{(r-1)/r}. \quad (3.18)$$

Using inequality (1.3), assertion $(*_5)$ and (3.6) and (3.12), we get

$$I_k \leqslant \frac{c_3}{8}\int_\Omega \Phi_i(1-h_k'(u_i))dx + c_5[\varphi(k)]^{q/(q-1)r_1}, \tag{3.19}$$

$$\frac{1}{k_0}J_k \leqslant \frac{c_3}{8}\int_\Omega \Phi_i(1-h_k'(u_i))dx + c_6[\varphi(k)]^{q/(q-1)r_1}. \tag{3.20}$$

In turn, as in [12], using (2.2), (3.2), (3.8)-(3.10) and (3.13) along with assertions $(*_1)$ and $(*_2)$, we find that

$$I_k' \leqslant \frac{c_3}{2}\int_\Omega \Phi_i(1-h_k'(u_i))dx + c_7[\varphi(k)]^{(r-1)/r}. \tag{3.21}$$

From (3.17)-(3.21) it follows that

$$c_3\int_\Omega \Phi_i(1-h_k'(u_i))dx \leqslant 4(M+c_7)[\varphi(k)]^{(r-1)/r} + 4(c_5+c_6)[\varphi(k)]^{q/(q-1)r_1}.$$

The result obtained along with inequality (1.3), assertion $(*_5)$ and the second equality of (3.5) allows us to conclude that

$$\int_\Omega |u_i - h_k(u_i)|^{q^*}dx \leqslant c_8[\varphi(k)]^\gamma.$$

From this and assertion $(*_4)$ we deduce that the following assertion holds:
$(*_7)$ if $k_0 \leqslant k < l \leqslant 2k$, then

$$\varphi(l) \leqslant \frac{c_9 k^{(t-1)q^*}}{(l-k)^{tq^*}}[\varphi(k)]^\gamma.$$

Using this assertion, inequality (3.7) and Lemma 3.1, we establish that for every $i\in\mathbb{N}$, $u_i \in L^\infty(\Omega)$ and

$$\|u_i\|_{L^\infty(\Omega)} \leqslant C_1, \tag{3.22}$$

where C_1 is a positive constant depending only on n, p, q, c, c_2, c_3, r, M and $\operatorname{meas}\Omega$. Now, from (3.22) and (3.4) we deduce (2.11).

Step 3. We set $\tilde{b} = \max\limits_{s\in[0,C_1]} b(s)$ and

$$\lambda = 2 + 4C_1\tilde{b}/c_3. \tag{3.23}$$

Let us show that

$$\lim_{j\to\infty}\int_\Omega \Phi_{i_j}|u_{i_j} - u|^\lambda dx = 0. \tag{3.24}$$

To this purpose, we fix $j\in\mathbb{N}$ and set

$$v_j = |u_{i_j} - u|^\lambda (u_{i_j} - u).$$

Using (2.11) and (3.22) and taking into account the inequality $\lambda > 1$, we establish that $v_j \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ and the following assertions hold:

$(*_8)$ for every n-dimensional multi-index $\alpha, |\alpha| = 1$,

$$D^\alpha v_j = (\lambda+1)|u_{i_j} - u|^\lambda D^\alpha(u_{i_j} - u) \quad \text{a.e. in } \Omega;$$

$(*_9)$ for every n-dimensional multi-index $\alpha, |\alpha| = 2$,

$$\begin{aligned} &|D^\alpha v_j - (\lambda+1)|u_{i_j} - u|^\lambda D^\alpha(u_{i_j} - u)| \\ &\qquad \leqslant \lambda(\lambda+1)|u_{i_j} - u|^{\lambda-1} \sum_{|\beta|=1} |D^\beta(u_{i_j} - u)|^2 \quad \text{a.e. in } \Omega. \end{aligned}$$

We set

$$\begin{aligned} \rho_j &= \int_\Omega |f||u_{i_j} - u|^{\lambda+1}\, dx, \\ \rho'_j &= \int_\Omega \left(\sum_{\alpha\in\Lambda} |A_\alpha(x,\nabla_2 u_{i_j})||D^\alpha u|\right)|u_{i_j} - u|^\lambda\, dx, \\ \rho''_j &= \int_\Omega |B(x,u_{i_j},\nabla_2 u_{i_j})||u_{i_j} - u|^\lambda\, dx, \\ \rho'''_j &= \int_\Omega \left(\sum_{|\alpha|=2} |A_\alpha(x,\nabla_2 u_{i_j})|\right)\left(\sum_{|\beta|=1} |D^\beta(u_{i_j} - u)|^2\right)|u_{i_j} - u|^{\lambda-1}\, dx. \end{aligned}$$

Since $v_j \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$, by virtue of (3.1), we have

$$\int_\Omega \left(\sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u_{i_j}) D^\alpha v_j + B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j}) v_j\right) dx = \int_\Omega f v_j\, dx. \tag{3.25}$$

This equality, assertions $(*_8)$ and $(*_9)$ and (2.11) and (3.22) imply that

$$\begin{aligned} &(\lambda+1)\int_\Omega \left(\sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u_{i_j}) D^\alpha u_{i_j}\right)|u_{i_j} - u|^\lambda\, dx \\ &\qquad \leqslant \rho_j + (\lambda+1)\rho'_j + 2C_1\rho''_j + \lambda(\lambda+1)\rho'''_j. \end{aligned}$$

Hence, using (2.3), we get

$$\begin{aligned} c_3(\lambda+1)\int_\Omega \Phi_{i_j}|u_{i_j} - u|^\lambda\, dx &\leqslant (\lambda+1)\int_\Omega g_3|u_{i_j} - u|^\lambda\, dx \\ &\quad + \rho_j + (\lambda+1)\rho'_j + 2C_1\rho''_j + \lambda(\lambda+1)\rho'''_j. \end{aligned} \tag{3.26}$$

Setting

$$\Phi = \sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p,$$

we observe that, by (3.2) and (3.3),

$$\int_\Omega \Phi\, dx \leqslant c_4. \tag{3.27}$$

Then, using Yong's inequality and (2.1) and (2.2), we find that

$$(\lambda+1)\rho_j' \leqslant \frac{(\lambda+1)c_3}{2}\int_\Omega \Phi_{i_j}|u_{i_j}-u|^\lambda\,dx + c_{10}(\lambda+1)\int_\Omega (\Phi+g_1+g_2)|u_{i_j}-u|^\lambda\,dx, \quad (3.28)$$

and using (2.5) and (3.22), we establish that

$$2C_1\rho_j'' \leqslant 2C_1\tilde{b}\int_\Omega \Phi_{i_j}|u_{i_j}-u|^\lambda\,dx + 2C_1\int_\Omega g_4|u_{i_j}-u|^\lambda\,dx. \quad (3.29)$$

From (3.23), (3.26), (3.28) and (3.29) it follows that

$$c_3\int_\Omega \Phi_{i_j}|u_{i_j}-u|^\lambda\,dx \leqslant (\lambda+1)c_{11}\int_\Omega \left(\Phi+\sum_{m=1}^{4} g_m\right)|u_{i_j}-u|^\lambda\,dx + \lambda(\lambda+1)\rho_j'''. \quad (3.30)$$

Finally, using Hölder's inequality and (2.2), (3.2) and (3.27), we obtain

$$\rho_j''' \leqslant \left[\int_\Omega \left(\sum_{|\alpha|=2}|A_\alpha(x,\nabla_2 u_{i_j})|\right)^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left[\int_\Omega\left(\sum_{|\beta|=1}|D^\beta(u_{i_j}-u)|\right)^{\frac{q}{2}}dx\right]^{\frac{2}{q}}$$

$$\times\left[\int_\Omega |u_{i_j}-u|^{\frac{(\lambda-1)qp}{q-2p}}\,dx\right]^{\frac{q-2p}{qp}} \leqslant c_{12}\left[\int_\Omega |u_{i_j}-u|^{\frac{(\lambda-1)qp}{q-2p}}\,dx\right]^{\frac{q-2p}{qp}}. \quad (3.31)$$

From (3.30), (3.31), (3.4) and (3.22) we infer (3.24).

Now, taking into account that for every $\varepsilon>0$ and for every $j\in\mathbb{N}$,

$$\int_\Omega \Phi_{i_j}|u_{i_j}-u|dx \leqslant \frac{\varepsilon}{2c_4}\int_\Omega \Phi_{i_j}\,dx + \left(\frac{2c_4}{\varepsilon}\right)^{\lambda-1}\int_\Omega \Phi_{i_j}|u_{i_j}-u|^\lambda\,dx$$

and using (3.2) and (3.24), we get

$$\lim_{j\to\infty}\int_\Omega \Phi_{i_j}|u_{i_j}-u|dx = 0. \quad (3.32)$$

Observe that the idea of using the function v_j to obtain (3.24) is suggested by [3] where a function of the same kind was utilized to get a property analogous to (3.24) in the case of degenerate second-order elliptic equations.

Step 4. For every $i\in\mathbb{N}$ we set

$$\tilde{\Phi}_i = \sum_{\alpha\in\Lambda}[A_\alpha(x,\nabla_2 u_i) - A_\alpha(x,\nabla_2 u)](D^\alpha u_i - D^\alpha u).$$

Let us demonstrate that

$$\lim_{j\to\infty}\int_\Omega \tilde{\Phi}_{i_j}\,dx = 0. \quad (3.33)$$

We fix $j \in \mathbb{N}$. Since $u_{i_j} - u \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$, by virtue of (3.1), we have

$$\int_\Omega \tilde{\Phi}_{i_j}\,dx = \int_\Omega f(u_{i_j} - u)dx - \int_\Omega B_{i_j}(x, u_{i_j}, \nabla_2 u_{i_j})(u_{i_j} - u)dx \\ - \int_\Omega \bigg(\sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u)(D^\alpha u_{i_j} - D^\alpha u)\bigg)dx. \quad (3.34)$$

The integrals in the right-hand side of (3.34) tend to zero as $j \to \infty$. In fact, using (2.5) and (3.22), we obtain

$$\left| \int_\Omega B_{i_j}(x, u_{i_j}, \nabla_2 u_{i_j})(u_{i_j} - u)dx \right| \leqslant \int_\Omega |B(x, u_{i_j}, \nabla_2 u_{i_j})||u_{i_j} - u|dx \\ \leqslant \tilde{b} \int_\Omega \Phi_{i_j}|u_{i_j} - u|dx + \int_\Omega g_4|u_{i_j} - u|dx.$$

This along with (3.32), (3.4) and (3.22) implies that

$$\lim_{j\to\infty} \int_\Omega B_{i_j}(x, u_{i_j}, \nabla_2 u_{i_j})(u_{i_j} - u)dx = 0. \quad (3.35)$$

By virtue of (2.7) and (3.3), we have

$$\lim_{j\to\infty} \int_\Omega f(u_{i_j} - u)dx = 0. \quad (3.36)$$

Using (2.1), (2.2) and (3.3), we get

$$\lim_{j\to\infty} \int_\Omega \bigg(\sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u)(D^\alpha u_{i_j} - D^\alpha u)\bigg)dx = 0. \quad (3.37)$$

From (3.34)-(3.37) we deduce (3.33).

Step 5. Let us show that

$$\forall \alpha \in \Lambda, \;\; D^\alpha u_{i_j} \to D^\alpha u \;\; \text{in measure}. \quad (3.38)$$

To this purpose, we introduce some auxiliary functions and sets.

Let for every $x \in \Omega$, $A_x : \mathbb{R}^{n,2} \times \mathbb{R}^{n,2} \to \mathbb{R}$ be the function such that for every pair $(\xi, \xi') \in \mathbb{R}^{n,2} \times \mathbb{R}^{n,2}$,

$$A_x(\xi, \xi') = \sum_{\alpha\in\Lambda} [A_\alpha(x,\xi) - A_\alpha(x,\xi')](\xi_\alpha - \xi'_\alpha).$$

Since A_α, $\alpha \in \Lambda$, are Carathéodory functions and for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^{n,2} \times \mathbb{R}^{n,2}$, $\xi \neq \xi'$, inequality (2.4) holds, there exists a set $E \subset \Omega$ with measure zero such that:

(i) for every $x \in \Omega \setminus E$ the function A_x is continuous in $\mathbb{R}^{n,2} \times \mathbb{R}^{n,2}$;

(ii) for every $x \in \Omega \setminus E$ and for every $\xi, \xi' \in \mathbb{R}^{n,2}$, $\xi \neq \xi'$, we have $A_x(\xi, \xi') > 0$.

For every $\theta > 0$ and for every $m > \theta$ we set

$$G_{\theta,m} = \left\{ (\xi, \xi') \in \mathbb{R}^{n,2} \times \mathbb{R}^{n,2} : \sum_{\alpha\in\Lambda} |\xi_\alpha - \xi'_\alpha| \geqslant \theta, \sum_{\alpha\in\Lambda} |\xi_\alpha| \leqslant m, \sum_{\alpha\in\Lambda} |\xi'_\alpha| \leqslant m \right\}.$$

Evidently, for every $\theta > 0$ and for every $m > \theta$ the set $G_{\theta,m}$ is nonempty, closed and bounded.

Let for every $\theta > 0$ and for every $m > \theta$, $\mu_{\theta,m} : \Omega \to \mathbb{R}$ be the function such that

$$\mu_{\theta,m}(x) = \begin{cases} \min\limits_{G_{\theta,m}} A_x & \text{if } x \in \Omega \setminus E, \\ 0 & \text{if } x \in E. \end{cases}$$

Using properties (i) and (ii) along with (2.1) and (2.2) and taking into account the fact that A_α, $\alpha \in \Lambda$, are Carathéodory functions, we establish that if $\theta > 0$ and $m > \theta$, then

$$\mu_{\theta,m} \geqslant 0 \text{ in } \Omega, \quad \mu_{\theta,m} > 0 \text{ a.e. in } \Omega, \tag{3.39}$$

$$\mu_{\theta,m} \in L^1(\Omega). \tag{3.40}$$

Next, we will need the following simple result.

LEMMA 3.2. *Let $\mu \in L^1(\Omega)$, $\mu > 0$ a.e. in Ω, and let $\{E_j\}$ be a sequence of measurable sets lying in Ω such that*

$$\lim_{j\to\infty} \int_{E_j} \mu \, dx = 0.$$

Then

$$\lim_{j\to\infty} \text{meas } E_j = 0.$$

Concerning the proof of this result see for instance [7].

Now, we pass to the immediate proof of assertion (3.38). We fix $\theta > 0$ and $\varepsilon > 0$. Using (3.2), we obtain that for every $m > 0$ and for every $j \in \mathbb{N}$,

$$m \,\text{meas}\{\Phi_{i_j} \geqslant m\} \leqslant \int_{\{\Phi_{i_j} \geqslant m\}} \Phi_{i_j} \, dx \leqslant c_4.$$

Therefore, there exists $m > \max(1, \theta)$ such that

$$\sup_{j\in\mathbb{N}} \text{meas}\left\{ \sum_{\alpha\in\Lambda} |D^\alpha u_{i_j}| \geqslant m \right\} \leqslant \varepsilon, \quad \text{meas}\left\{ \sum_{\alpha\in\Lambda} |D^\alpha u| \geqslant m \right\} \leqslant \varepsilon. \tag{3.41}$$

For every $j \in \mathbb{N}$ we set

$$E_j = \left\{ \sum_{\alpha\in\Lambda} |D^\alpha u_{i_j}| \leqslant m, \sum_{\alpha\in\Lambda} |D^\alpha u| \leqslant m, \sum_{\alpha\in\Lambda} |D^\alpha u_{i_j} - D^\alpha u| \geqslant \theta \right\}.$$

Let $j \in \mathbb{N}$ and $x \in E_j \setminus E$. We have

$$\sum_{\alpha\in\Lambda} |D^\alpha u_{i_j}| \leqslant m, \quad \sum_{\alpha\in\Lambda} |D^\alpha u| \leqslant m, \quad \sum_{\alpha\in\Lambda} |D^\alpha u_{i_j} - D^\alpha u| \geqslant \theta.$$

Hence $(\nabla_2 u_{i_j}(x), \nabla_2 u(x)) \in G_{\theta,m}$. Then, by virtue of the definition of $\mu_{\theta,m}$ and A_x, we have $\mu_{\theta,m}(x) \leqslant \tilde{\Phi}_{i_j}(x)$.

Now, we conclude that for every $j \in \mathbb{N}$,

$$\int_{E_j} \mu_{\theta,m}\, dx \leqslant \int_{E_j} \tilde{\Phi}_{i_j}\, dx \leqslant \int_\Omega \tilde{\Phi}_{i_j}\, dx.$$

This and (3.33) imply that

$$\lim_{j\to\infty} \int_{E_j} \mu_{\theta,m}\, dx = 0.$$

Hence, taking into account (3.39) and (3.40) and applying Lemma 3.2, we deduce that

$$\lim_{j\to\infty} \operatorname{meas} E_j = 0. \tag{3.42}$$

Obviously, for every $j \in \mathbb{N}$,

$$\operatorname{meas}\Big\{ \sum_{\alpha\in\Lambda} |D^\alpha u_{i_j} - D^\alpha u| \geqslant \theta \Big\} \leqslant \operatorname{meas}\Big\{ \sum_{\alpha\in\Lambda} |D^\alpha u_{i_j}| > m \Big\} + \operatorname{meas}\Big\{ \sum_{\alpha\in\Lambda} |D^\alpha u| > m \Big\} + \operatorname{meas} E_j.$$

From this and (3.41) and (3.42) we infer (3.38).

We remark that in the proof of assertion (3.38) we used some ideas of [6, 7].

By virtue of (3.38) and F. Riesz's theorem, for every $\alpha \in \Lambda$ there exists a subsequence of the sequence $\{D^\alpha u_{i_j}\}$ that converges to $D^\alpha u$ a.e. in Ω. Taking it into account, we may assume, without loss of generality, that

$$\forall \alpha \in \Lambda, \;\; D^\alpha u_{i_j} \to D^\alpha u \;\; \text{a.e. in } \Omega. \tag{3.43}$$

Step 6. Let us prove that the following assertion holds:

$(*_{10})$ for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $G \subset \Omega$, meas $G < \delta$, we have

$$\limsup_{j\to\infty} \int_G \Phi_{i_j}\, dx \leqslant \varepsilon. \tag{3.44}$$

We set

$$c_{13} = (c_1 c_4 + \|g_1\|_{L^1(\Omega)})^{\frac{q-1}{q}}, \;\; c_{14} = (c_2 c_4 + \|g_2\|_{L^1(\Omega)})^{\frac{p-1}{p}}. \tag{3.45}$$

Let $\varepsilon > 0$. By virtue of the property of absolute continuity of Lebesgue integral, there exists $\delta > 0$ such that for every measurable set $G \subset \Omega$, meas $G < \delta$, we have

$$\int_G g_3\, dx \leqslant \frac{c_3 \varepsilon}{2}, \tag{3.46}$$

$$\int_G \left(\sum_{|\alpha|=1} |D^\alpha u|^q \right) dx \leqslant \left(\frac{c_3 \varepsilon}{4c_{13}} \right)^q, \quad \int_G \left(\sum_{|\alpha|=2} |D^\alpha u|^p \right) dx \leqslant \left(\frac{c_3 \varepsilon}{4c_{14}} \right)^p. \tag{3.47}$$

We fix an arbitrary measurable set $G \subset \Omega$ such that $\operatorname{meas} G < \delta$. For every $j \in \mathbb{N}$ we set

$$\sigma_j = \int_G \left(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u)(D^\alpha u_{i_j} - D^\alpha u) \right) dx,$$

$$\sigma_j' = \int_G \left(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha u \right) dx.$$

Using (2.1), (2.2) and (3.3), we establish that

$$\lim_{j \to \infty} \sigma_j = 0. \tag{3.48}$$

Now, we fix $j \in \mathbb{N}$. By means of (2.3) we get

$$c_3 \int_G \Phi_{i_j} \, dx \leqslant \int_\Omega \tilde{\Phi}_{i_j} \, dx + \int_G g_3 \, dx + \sigma_j + \sigma_j'. \tag{3.49}$$

Using Hölder's inequality along with (2.1), (2.2), (3.2) and (3.45), we obtain

$$\begin{aligned} \sigma_j' &\leqslant \left[\int_G (c_1 \Phi_{i_j} + g_1) \, dx \right]^{\frac{q-1}{q}} \left[\int_G \left(\sum_{|\alpha|=1} |D^\alpha u|^q \right) dx \right]^{\frac{1}{q}} \\ &\quad + \left[\int_G (c_2 \Phi_{i_j} + g_2) \, dx \right]^{\frac{p-1}{p}} \left[\int_G \left(\sum_{|\alpha|=2} |D^\alpha u|^p \right) dx \right]^{\frac{1}{p}} \\ &\leqslant c_{13} \left[\int_G \left(\sum_{|\alpha|=1} |D^\alpha u|^q \right) dx \right]^{\frac{1}{q}} + c_{14} \left[\int_G \left(\sum_{|\alpha|=2} |D^\alpha u|^p \right) dx \right]^{\frac{1}{p}}. \end{aligned}$$

This and (3.47) imply that

$$\sigma_j' \leqslant c_3 \varepsilon / 2. \tag{3.50}$$

Using (3.46), (3.49) and (3.50), we get

$$c_3 \int_G \Phi_{i_j} \, dx \leqslant \int_\Omega \tilde{\Phi}_{i_j} \, dx + \sigma_j + c_3 \varepsilon.$$

This along with (3.33) and (3.48) implies (3.44). Thus assertion $(*_{10})$ holds.

Step 7. Let us show that the following assertions hold:

$(*_{11})$ for every function $v \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$,

$$\lim_{j \to \infty} \int_\Omega \left(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v \right) dx = \int_\Omega \left(\sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \right) dx;$$

$(*_{12})$ for every function $v \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega B_{i_j}(x, u_{i_j}, \nabla_2 u_{i_j}) v\, dx = \int_\Omega B(x, u, \nabla_2 u) v\, dx.$$

In fact, let $v \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$. In view of (3.43), we have

$$\sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v \to \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \quad \text{a.e. in } \Omega. \tag{3.51}$$

Let $\varepsilon > 0$. Reasoning by analogy with the proof of inequality (3.50), we establish that there exists $\varepsilon_1 > 0$ such that for every measurable set $G \subset \Omega$, meas $G \leqslant \varepsilon_1$, and for every $j \in \mathbb{N}$ the following inequalities hold:

$$\int_G \Big| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v \Big| dx \leqslant \frac{\varepsilon}{4}, \quad \int_G \Big| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \Big| dx \leqslant \frac{\varepsilon}{4}. \tag{3.52}$$

By virtue of (3.51) and Egoroff's theorem, there exists a measurable set $\Omega_1 \subset \Omega$ such that

$$\operatorname{meas}(\Omega \setminus \Omega_1) \leqslant \varepsilon_1, \tag{3.53}$$

$$\sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v \to \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \ \text{ uniformly in } \Omega_1.$$

Then there exists $j_1 \in \mathbb{N}$ such that for every $j \in \mathbb{N}$, $j \geqslant j_1$, we have

$$\int_{\Omega_1} \Big| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v - \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \Big| dx \leqslant \frac{\varepsilon}{2}. \tag{3.54}$$

Let us fix $j \in \mathbb{N}$, $j \geqslant j_1$. Using (3.52)-(3.54), we obtain

$$\Big\| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v - \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \Big\|_{L^1(\Omega)}$$
$$\leqslant \int_{\Omega\setminus\Omega_1} \Big| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v \Big| dx + \int_{\Omega\setminus\Omega_1} \Big| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \Big| dx + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

Therefore,

$$\Big\| \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v - \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v \Big\|_{L^1(\Omega)} \to 0.$$

Thus assertion $(*_{11})$ holds.

Further, let $v \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. Owing to (3.43) and (3.4), we have

$$B_{i_j}(x, u_{i_j}, \nabla_2 u_{i_j}) v \to B(x, u, \nabla_2 u) v \quad \text{a.e. in } \Omega. \tag{3.55}$$

Let $\varepsilon > 0$. Using (2.5) and (3.22), assertion $(*_{10})$ and the boundedness of the function v, we establish that there exists $\varepsilon_2 > 0$ such that for every measurable set $G \subset \Omega$, $\operatorname{meas} G \leqslant \varepsilon_2$, the following inequalities hold:

$$\limsup_{j\to\infty} \int_G |B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v|dx \leqslant \frac{\varepsilon}{3}, \tag{3.56}$$

$$\int_G |B(x,u,\nabla_2 u)v|dx \leqslant \frac{\varepsilon}{3}. \tag{3.57}$$

In view of (3.55), there exists a measurable set $\Omega_2 \subset \Omega$ such that

$$\operatorname{meas}(\Omega \setminus \Omega_2) \leqslant \varepsilon_2 \tag{3.58}$$

and $B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v \to B(x,u,\nabla_2 u)v$ uniformly in Ω_2. Then there exists $j_2 \in \mathbb{N}$ such that for every $j \in \mathbb{N}$, $j \geqslant j_2$, we have

$$\int_{\Omega_2} |B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v - B(x,u,\nabla_2 u)v|dx \leqslant \frac{\varepsilon}{3}. \tag{3.59}$$

We fix $j \in \mathbb{N}$, $j \geqslant j_2$. Using (3.57)-(3.59), we obtain

$$\begin{aligned}
&\int_\Omega |B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v - B(x,u,\nabla_2 u)v|dx \\
&\qquad \leqslant \frac{\varepsilon}{3} + \int_{\Omega\setminus\Omega_2} |B(x,u,\nabla_2 u)v|dx + \int_{\Omega\setminus\Omega_2} |B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v|dx \\
&\qquad \leqslant \frac{2\varepsilon}{3} + \int_{\Omega\setminus\Omega_2} |B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v|dx.
\end{aligned}$$

This along with (3.56) and (3.58) implies that

$$\limsup_{j\to\infty} \int_\Omega |B_{i_j}(x,u_{i_j},\nabla_2 u_{i_j})v - B(x,u,\nabla_2 u)v|dx \leqslant \varepsilon.$$

Hence, taking into account the arbitrariness of ε, we deduce that assertion $(*_{12})$ holds.

From (3.1) and assertions $(*_{11})$ and $(*_{12})$ it follows that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \left(\sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u) D^\alpha v + B(x,u,\nabla_2 u)v \right) dx = \int_\Omega f v\, dx.$$

The established properties of the function u allow us to conclude that u is a generalized solution of problem (2.8), (2.9). This completes the proof of Theorem 2.1.

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