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EXISTENCE OF SOLUTIONS FOR THE COUPLED SYSTEMS OF SECOND AND FOURTH ORDER ELLIPTIC EQUATIONS

LEI JI AND CHUN-LEI TANG

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Abstract. In this paper, under the nonquadraticity condition, we obtain two existence theorems of nontrivial solutions for a coupled system of second and fourth order elliptic equations.

1. Introduction and main results

Consider the following coupled system of second and fourth order elliptic equations given by

$$\begin{cases} \Delta^2 y + k(y-z)^+ = f_1(x,y,z) & \text{in } \Omega, \\ -\Delta z - k(y-z)^+ = f_2(x,y,z) & \text{in } \Omega, \\ \Delta y = y = 0 & \text{on } \partial\Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain in R^N $(N \geqslant 3)$ with smooth boundary $\partial\Omega$.

System (1) comes from the mathematical model of the suspension bridge presented in Lazer and McKenna [14]. Many authors, using degree theory and the variational method, investigated the equations corresponding to the suspension bridge. These results can be seen in [4], [8], [9], [13], [15], [16] and [17].

In [3], by the least action principle and the mountain pass theorem, the author studied the existence of nontrivial solutions for the following system under the perturbation $g = (g_1, g_2)$ being sublinear and superlinear with respect to $u = (y,z)$, respectively.

$$\begin{cases} y_{xxxx} + k(y-z)^+ = g_1(x,y,z) & \text{in } (0,\pi), \\ -z_{xx} - k(y-z)^+ = g_2(x,y,z) & \text{in } (0,\pi), \\ y(0) = y(\pi) = y_{xx}(0) = y_{xx}(\pi) = 0, \\ z(0) = z(\pi) = 0. \end{cases} \tag{2}$$

The other relevant studies can be found in [10] and [11]. In [2], the authors investigated problem (1) under the potential $F(x,u)$ $\big(\nabla_u F(x,u) = (f_1(x,u), f_2(x,u))\big)$ being a homogenous function using the variational approach. In [1], the author continued the

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work of [2] and extended the result in [12] to system (1) in which the nonlinearities are more general than those in the result of [2] and obtained a nontrivial solution by the mountain pass theorem. The result is the following theorem.

THEOREM A (see [1]). *Let* $\nabla_u F(x,u) = f(x,u) = (f_1(x,u), f_2(x,u)))$. *Suppose that* $F(x,u)$ *satisfies the following assumptions:*

(F_1) $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ *is a Caratheodory function and there exist* p *such that*

$$2 < p < 2^* = 2N/(N-2)$$

and constant $C > 0$ *such that for all* $u = (y,z) \in \mathbb{R}^2$ *and almost every* $x \in \Omega$,

$$|f_1(x,u)| + |f_2(x,u)| \leqslant C(1 + |u|^{p-1});$$

(F_2) *there exist* $\theta \in (0, \frac{1}{2})$ *and* $M > 0$ *such that*

$$0 < F(x,u) \leqslant \theta u \cdot \nabla_u F(x,u) \quad \text{for almost every } x \in \Omega, \text{ if } |u| \geqslant M;$$

(F_3) $\lim\limits_{|u| \to 0} \frac{F(x,u)}{|u|^2} = 0$ *uniformly for a.e.* $x \in \Omega$.

Let λ_1 *be the first eigenvalue of* $(-\Delta, H_0^1(\Omega))$ *and* $\lambda = \min\{\lambda_1, \lambda_1^2\}$. *Assume that* $k > -\lambda/2$. *Then system* (1) *has at least one nontrivial solution.*

It should be noted that condition (F_2) is the well-known Ambrosetti-Rabinowitz condition. It usually plays a very important role in verifying that the corresponding functional has a Mountain-Pass geometry and showing a related $(PS)_c$ sequence is bounded. In this paper, motivated by [1] and [6], we consider system (1) under the nonquadraticity condition which was introduced by D. G. Costa and C. A. Magalhaes in [6]. Using the linking theorem, we obtain two existence results of nontrivial solutions for system (1).

Let $\Omega \subset R^N (N \geqslant 3)$ be a bounded smooth domain. By H and V we denote the spaces of $L^2(\Omega) \times L^2(\Omega)$ and $\left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$, respectively. For $u = (y,z)$, we denote by

$$\|u\|_H^2 = \|y\|_{L^2(\Omega)}^2 + \|z\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|u\|_V^2 = \int_\Omega |\Delta y|^2 \, dx + \int_\Omega |\nabla z|^2 \, dx,$$

the norms on H and V, respectively. By $\|u\|_r$ we denote the norm of $u = (y,z) \in L^r(\Omega) \times L^r(\Omega)$. The embedding from V into H is continuous, and moreover it is actually compact.

Let λ_1 be the first eigenvalue, which is positive, of the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

and φ_1 is the eigenfunction corresponding to λ_1, with $\varphi_1(x) > 0$ for all $x \in \Omega$. We can easily know that λ_1^2 is the first eigenvalue of the eigenvalue problem

$$\Delta^2 u = \lambda u \quad \text{in } \Omega, \quad \Delta u = u = 0 \quad \text{on } \partial\Omega.$$

Let

$$\underline{\lambda} = \min\{\lambda_1, \lambda_1^2\}, \quad \overline{\lambda} = \max\{\lambda_1, \lambda_1^2\}.$$

Then $\underline{\lambda} > 0$, $\overline{\lambda} > 0$. It follows from the Poincaré inequality that

$$\|u\|_V^2 \geqslant \underline{\lambda}\|u\|_H^2$$

for all $u \in V$.

Now we state the assumptions imposed on F in this paper, where $\nabla_u F(x,u) = f(x,u) = (f_1(x,u), f_2(x,u))$.

(H_1) $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Caratheodory function, that is, $f(x,u)$ is measurable in x for each $u = (y,z) \in \mathbb{R}^2$ and continuous in $u = (y,z) \in \mathbb{R}^2$ for almost every $x \in \Omega$, and there exist $2 < p < 2^* = 2N/(N-2)$, a_0, $b_0 > 0$ such that for all $u = (y,z) \in \mathbb{R}^2$ and almost every $x \in \Omega$,

$$|f_1(x,u)| + |f_2(x,u)| \leqslant a_0|u|^{p-1} + b_0.$$

(H_2) There exist $q > 2$ and $a_1 > 0$ such that

$$\limsup_{|u|\to\infty} \frac{F(x,u)}{|u|^q} \leqslant a_1 < \infty \text{ uniformly for a.e. } x \in \Omega.$$

(H_3) There exist $\frac{N}{2}(q-2) < \mu \leqslant q$ and $b_1 > 0$ such that

$$\liminf_{|u|\to\infty} \frac{f(x,u)u - 2F(x,u)}{|u|^\mu} \geqslant b_1 > 0 \text{ uniformly for a.e. } x \in \Omega.$$

The main results in this paper are the following theorems.

THEOREM 1. *Suppose that* $-\frac{1}{2}\underline{\lambda} < k \leqslant 0$ *and* (H_1)-(H_3) *hold. Assume that there exist* a_2, $b_2 > 0$ *such that*

$$\limsup_{|u|\to 0} \frac{2F(x,u)}{|u|^2} \leqslant a_2 < \underline{\lambda} + 2k \quad \text{and} \quad \liminf_{|u|\to\infty} \frac{2F(x,u)}{|u|^2} \geqslant b_2 > \overline{\lambda} \tag{3}$$

uniformly for a.e. $x \in \Omega$. *Then there exists at least one nontrivial solution of system* (1).

THEOREM 2. *Suppose that* $k \geqslant 0$ *and* (H_1)-(H_3) *hold. Assume that there exist* a_3, $b_3 > 0$ *such that*

$$\limsup_{|u|\to 0} \frac{2F(x,u)}{|u|^2} \leqslant a_3 < \underline{\lambda} \quad \text{and} \quad \liminf_{|u|\to\infty} \frac{2F(x,u)}{|u|^2} \geqslant b_3 > \overline{\lambda} + 2k \tag{4}$$

uniformly for a.e. $x \in \Omega$. *Then there exists at least one nontrivial solution of system* (1).

COROLLARY 1. *Suppose that* $k > -\frac{1}{2}\underline{\lambda}$ *and* (H_1)-(H_3) *hold. Assume that*

$$\limsup_{|u|\to 0} \frac{2F(x,u)}{|u|^2} = 0 \quad \text{and} \quad \liminf_{|u|\to\infty} \frac{2F(x,u)}{|u|^2} = +\infty$$

uniformly for a.e. $x \in \Omega$. *Then system* (1) *has at least one nontrivial solution.*

REMARK 1. There are functions $F(x,u)$ satisfying our corollary and not satisfying condition (F_2) in Theorem A. For example, let $F(x,u)=|u|^2\ln(1+|u|^2)$. It is easy to see that $F(x,u)$ satisfies our corollary, but doesn't satisfy (F_2) of Theorem A.

2. Proof of theorems

We shall find weak solutions of problem (1) in space V. We have known that the weak solutions $u=(y,z)$ of problem (1) are the critical points of the following functional defined in space V:

$$J(u)=\frac{1}{2}\int_\Omega\left(|\Delta y|^2+|\nabla z|^2+k\left((y-z)^+\right)^2\right)dx-\int_\Omega F(x,u)\,dx,$$

where $u=(y,z)\in V$ and $\nabla_u F(x,u)=f(x,u)=(f_1(x,u),f_2(x,u))$. It is easy to know that the functional J is well defined and $J\in C^1(V,R)$ under condition (H_1), and moreover,

$$\langle J'(u),v\rangle=\int_\Omega\Big(\Delta y\Delta\xi+\nabla z\nabla\eta+k(y-z)^+\xi-k(y-z)^+\eta \\ -f_1(x,u)\xi-f_2(x,u)\eta\Big)dx$$

for all $u=(y,z),\ v=(\xi,\ \eta)\in V$.

For convenience, we state the definition of condition $(C)_c$ introduced by Cerami in [7] (see also [5]), abstract linking concept and theorem (see [5]).

A functional $J\in C^1(E,R)$ is said to satisfy condition $(C)_c,\ c\in R$, if, whenever $\{u_n\}\subset E$ is such that $J(u_n)\to c$ and $(1+\|u_n\|)\|J'(u_n)\|\to 0$, then $\{u_n\}$ possesses a convergent subsequence.

DEFINITION 1. (see [5]) Let S be a closed subset of a real Hilbert space X. Let Q be a Hilbert manifold with relative boundary ∂Q. We shall say that S and ∂Q link if:

(1°) $S\cap\partial Q=\emptyset$;

(2°) for any map $\phi\in C^0(X,X)$ such that $\phi|_{\partial Q}=id$ there holds

$$\phi(Q)\cap S\neq\emptyset.$$

The following remark give a type of linking geometry structure which is a special case of Proposition 2.2 in [5].

REMARK 2. Let X be a Hilbert space, $x_0\in X$ with $\|x_0\|=1$. Let $t_0>\rho>0$, and $S=\{x\in X\mid \|x\|=\rho\}$, $Q=\{tx_0\mid 0\leqslant t\leqslant t_0\}$ with relative boundary $\partial Q=\{0,\ t_0x_0\}$. Then S and ∂Q link.

THEOREM B (see [5, Theorem 2.3]). *Let X be a real Hilbert space. Suppose that $J \in C^1(X,R)$ and satisfies condition $(C)_c$ for all $c > 0$. There exists a closed subset $S \subset X$ and a Hilbert manifold $Q \subset X$ with boundary ∂Q verifying the following conditions:*

(i) $\sup_{u\in\partial Q} J(u) \leqslant \alpha < \beta \leqslant \inf_{u\in S} J(u)$ *for some* $0 \leqslant \alpha < \beta$*;*

(ii) S *and* ∂Q *link;*

(iii) $\sup_{u\in Q} J(u) < +\infty$.

Then J possesses a critical value $c \geqslant \beta$.

LEMMA 1. *Assume that $k > -\frac{1}{2}\underline{\lambda}$ and (H_1)-(H_3) hold. Then J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$.*

Proof. From the Sobolev embedding theorem, we can easily know that the embedding of $V \to L^p(\Omega)\times L^p(\Omega)$ is compact, for any $p < 2^* = 2N/(N-2)$. And combining with condition (H_1), we see that the map $K : V \to V^*$ given by

$$K(u) = \big(k(y-z)^+, -k(y-z)^+\big) - \big(f_1(x,u), f_2(x,u)\big)$$

is compact.

Now let $c \in \mathbb{R}$ and $\{u_n\} \subset V$ being such that

$$J(u_n) \to c \quad \text{and} \quad (1+\|u_n\|)\|J'(u_n)\| \to 0. \tag{5}$$

It suffices to show that $\big\{\|u_n\|_V\big\}$ is bounded.

From (5) we obtain that there exists $M > 0$ such that

$$|J(u_n)| \leqslant M \quad \text{and} \quad (1+\|u_n\|)\|J'(u_n)\| \leqslant M. \tag{6}$$

Using (H_3), for some $M_1 > 0$, when $|u| \geqslant M_1$ we have

$$\begin{aligned} \frac{f(x,u)u - 2F(x,u)}{|u|^\mu} &\geqslant \frac{1}{2}b_1 > 0, \text{ i.e.} \\ f(x,u)u - 2F(x,u) &\geqslant \frac{1}{2}b_1|u|^\mu. \end{aligned} \tag{7}$$

When $|u| < M_1$, using the subcritical growth condition (H_1) one has

$$\begin{aligned} \Big|f(x,u)u - 2F(x,u)\Big| &\leqslant |f(x,u)|\cdot|u| + 2|F(x,u)| \\ &\leqslant \big(\big|f_1(x,u)\big| + \big|f_2(x,u)\big|\big)\cdot|u| \\ &\qquad + 2\left|\int_0^1 (\nabla_u F(x,su),u)\,ds\right| \\ &\leqslant (a_0|u|^{p-1} + b_0)\cdot|u| + 2\int_0^1 |f(x,su)|\cdot|u|\,ds \end{aligned}$$

$$\leqslant (a_0|u|^p + b_0|u|) + 2\int_0^1 (a_0|su|^{p-1} + b_0)\cdot|u|\,ds$$
$$\leqslant (a_0|u|^p + b_0|u|) + 2(a_0|u|^{p-1} + b_0)\cdot|u|$$
$$\leqslant C_0 \tag{8}$$

for some $C_0 > 0$. Then from (7) and (8) there exists $C_1 > 0$ such that

$$f(x,u)u - 2F(x,u) \geqslant \frac{1}{2}b_1|u|^\mu - C_1$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. So one has

$$\begin{aligned}
3M &\geqslant 2J(u_n) - \langle J'(u_n), u_n\rangle \\
&= \int_\Omega \Big(|\Delta y_n|^2 + |\nabla z_n|^2 + k\big((y_n - z_n)^+\big)^2\Big)\,dx - 2\int_\Omega F(x,u_n)\,dx \\
&\quad - \int_\Omega \Big(|\Delta y_n|^2 + |\nabla z_n|^2 + k(y_n - z_n)^+(y_n - z_n) - f(x,u_n)u_n\Big)\,dx \\
&= \int_\Omega \Big(f(x,u_n)u_n - 2F(x,u_n)\Big)\,dx \\
&\geqslant \int_\Omega \Big(\frac{1}{2}b_1|u_n|^\mu - C_1\Big)\,dx \\
&= \frac{1}{2}b_1\int_\Omega |u_n|^\mu\,dx - C_1\mathrm{meas}(\Omega)
\end{aligned}$$

by (6). Hence $\left\{\int_\Omega |u_n|^\mu\,dx\right\}$ is a bounded sequence, that is, there exists $C_2 > 0$ such that

$$\int_\Omega |u_n|^\mu\,dx \leqslant C_2 \tag{9}$$

for all n. Using (H_2), for some $M_2 > 0$, when $|u| \geqslant M_2$ we obtain

$$\frac{F(x,u)}{|u|^q} \leqslant 2a_1, \text{ i.e.}$$
$$F(x,u) \leqslant 2a_1|u|^q.$$

When $|u| < M_2$, as before, there exists $C_3 > 0$ such that $F(x,u) \leqslant C_3$. So we have

$$F(x,u) \leqslant 2a_1|u|^q + C_3 \tag{10}$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. On one hand, from (6) and (10) we obtain

$$\begin{aligned}
J(u_n) + \int_\Omega F(x,u_n)\,dx &\leqslant M + \int_\Omega \big(2a_1|u_n|^q + C_3\big)\,dx \\
&= M + 2a_1\|u_n\|_q^q + C_3\mathrm{meas}(\Omega) \\
&\overset{\triangle}{=} 2a_1\|u_n\|_q^q + C_4.
\end{aligned} \tag{11}$$

On the other hand, when $-\frac{1}{2}\underline{\lambda} < k \leqslant 0$ one has

$$\begin{aligned} J(u_n)+\int_\Omega F(x,u_n)\,dx &= \frac{1}{2}\int_\Omega \left(|\Delta y_n|^2+|\nabla z_n|^2+k\big((y_n-z_n)^+\big)^2\right)dx \\ &= \frac{1}{2}\|u_n\|_V^2+\frac{1}{2}k\int_\Omega \big((y_n-z_n)^+\big)^2\,dx \\ &\geqslant \frac{1}{2}\|u_n\|_V^2+\frac{k}{\underline{\lambda}}\|u_n\|_V^2 \\ &= \left(\frac{1}{2}+\frac{k}{\underline{\lambda}}\right)\|u_n\|_V^2 \\ &\geqslant l\|u_n\|_V^2 \end{aligned} \tag{12}$$

for some $0 < l < \frac{1}{2}+\frac{k}{\underline{\lambda}} \leqslant \frac{1}{2}$. If $k>0$ we have

$$\begin{aligned} J(u_n)+\int_\Omega F(x,u_n)\,dx &= \frac{1}{2}\int_\Omega \left(|\Delta y_n|^2+|\nabla z_n|^2+k\big((y_n-z_n)^+\big)^2\right)dx \\ &\geqslant \frac{1}{2}\|u_n\|_V^2 \\ &\geqslant l\|u_n\|_V^2. \end{aligned} \tag{13}$$

Thus from (11), (12) and (13), when $k > -\frac{1}{2}\underline{\lambda}$ we obtain

$$l\|u_n\|_V^2 \leqslant 2a_1\|u_n\|_q^q + C_4. \tag{14}$$

Therefore, use the Hölder inequality together with (9) and the Sobolev inequality $\|u\|_{2^*} \leqslant C\|u\|_V$ in the above estimate (14) to obtain

$$\begin{aligned} l\|u_n\|_V^2 &\leqslant 2a_1\|u_n\|_q^q + C_4 \\ &\leqslant 2a_1\left(\int_\Omega (|u_n|^{\frac{\mu(2^*-q)}{2^*-\mu}})^{\frac{2^*-\mu}{2^*-q}}\,dx\right)^{\frac{2^*-q}{2^*-\mu}} \cdot \left(\int_\Omega (|u_n|^{\frac{2^*(q-\mu)}{2^*-\mu}})^{\frac{2^*-\mu}{q-\mu}}\,dx\right)^{\frac{q-\mu}{2^*-\mu}} + C_4 \\ &= 2a_1\left(\int_\Omega |u_n|^\mu\,dx\right)^{\frac{2^*-q}{2^*-\mu}} \cdot \|u_n\|_{2^*}^{\frac{2^*(q-\mu)}{2^*-\mu}} + C_4 \\ &\leqslant C_5\|u_n\|_{2^*}^{\frac{2^*(q-\mu)}{2^*-\mu}} + C_4 \\ &\leqslant C_6\|u_n\|_V^{\frac{2^*(q-\mu)}{2^*-\mu}} + C_4 \end{aligned} \tag{15}$$

for some $C_5>0,\ C_6>0$. Finally, since we are taking $\frac{N}{2}(q-2) < \mu \leqslant q$, it follows that $\frac{2^*(q-\mu)}{2^*-\mu} < 2$, and hence, (15) implies that $\left\{\|u_n\|_V\right\}$ is bounded. Thus the functional J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$.

PROOF OF THEOREM 1. First, from Lemma 1 we know that J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$.

Second, assume that (3) holds, then there exist $\rho,\ \gamma>0$ such that $J(u)\geqslant\gamma>0$ if $\|u\|_V=\rho$. Moreover, there exists $u_0\in V$ such that $J(tu_0)\to-\infty$ as $t\to\infty$.

In view of (3) and condition (H_1), choosing

$$\varepsilon_0=\min\Big\{\frac{\underline{\lambda}+2k-a_2}{2},\frac{b_2-\overline{\lambda}}{2}\Big\},$$

then $a_2+\varepsilon_0<\underline{\lambda}+2k,\ b_2-\varepsilon_0>\overline{\lambda}$. For $\varepsilon_0>0$ there exist $A\geqslant 0$ and $B\geqslant 0$ such that

$$\frac{1}{2}(b_2-\varepsilon_0)|u|^2-B\leqslant F(x,u)\leqslant\frac{1}{2}(a_2+\varepsilon_0)|u|^2+A|u|^p \tag{16}$$

for all $u\in\mathbb{R}^2$ and a.e. $x\in\Omega$. Then we have

$$\begin{aligned}
J(u)&=\frac{1}{2}\int_\Omega\Big(|\Delta y|^2+|\nabla z|^2+k\big((y-z)^+\big)^2\Big)\,dx-\int_\Omega F(x,u)\,dx\\
&\geqslant\frac{1}{2}\|u\|_V^2+\frac{1}{2}\int_\Omega k\big((y-z)^+\big)^2\,dx-\int_\Omega\Big(\frac{1}{2}(a_2+\varepsilon_0)|u|^2+A|u|^p\Big)\,dx\\
&\geqslant\frac{1}{2}\|u\|_V^2+\Big(k-\frac{a_2+\varepsilon_0}{2}\Big)\|u\|_H^2-A\|u\|_p^p\\
&\geqslant\frac{1}{2}\|u\|_V^2+\Big(k-\frac{a_2+\varepsilon_0}{2}\Big)\frac{1}{\underline{\lambda}}\|u\|_V^2-A\|u\|_p^p\\
&\geqslant\frac{1}{2}\Big(1+\frac{2k-(a_2+\varepsilon_0)}{\underline{\lambda}}\Big)\|u\|_V^2-C_7\|u\|_V^p
\end{aligned}$$

for some $C_7>0$.

Therefore, since $p>2$, we obtain the estimate

$$J(u)\geqslant\frac{1}{4}\Big(1+\frac{2k-(a_2+\varepsilon_0)}{\underline{\lambda}}\Big)\rho^2\equiv\gamma>0,\ \forall\|u\|_V=\rho$$

with $\rho=\Big\{\frac{1}{4C_7}(1+\frac{2k-(a_2+\varepsilon_0)}{\underline{\lambda}})\Big\}^{\frac{1}{p-2}}>0$. By (16), one has

$$\begin{aligned}
J(u)&=\frac{1}{2}\int_\Omega\Big(|\Delta y|^2+|\nabla z|^2+k\big((y-z)^+\big)^2\Big)\,dx-\int_\Omega F(x,u)\,dx\\
&\leqslant\frac{1}{2}\|u\|_V^2-\int_\Omega\Big(\frac{1}{2}(b_2-\varepsilon_0)|u|^2-B\Big)\,dx\\
&=\frac{1}{2}\Big(\|u\|_V^2-(b_2-\varepsilon_0)\|u\|_H^2\Big)+B\,\mathrm{meas}(\Omega),
\end{aligned}$$

so we get

$$J(tu)\leqslant\frac{1}{2}t^2\Big(\|u\|_V^2-(b_2-\varepsilon_0)\|u\|_H^2\Big)+B\,\mathrm{meas}(\Omega). \tag{17}$$

Let $u_0=(\varphi_1,\varphi_1)$, φ_1 is the eigenfunction corresponding to λ_1, with $\varphi_1(x)>0$ for all $x\in\Omega$. Then we obtain

$$\|u_0\|_V^2=\int_\Omega|\Delta\varphi_1|^2\,dx+\int_\Omega|\nabla\varphi_1|^2\,dx$$

$$\begin{aligned}
&= \lambda_1^2 \|\varphi_1\|_{L^2(\Omega)}^2 + \lambda_1 \|\varphi_1\|_{L^2(\Omega)}^2 \\
&\leqslant \overline{\lambda}\left(\|\varphi_1\|_{L^2(\Omega)}^2 + \|\varphi_1\|_{L^2(\Omega)}^2 \right) \\
&= \overline{\lambda}\|u_0\|_H^2 \\
&< (b_2 - \varepsilon_0)\|u_0\|_H^2. \qquad (18)
\end{aligned}$$

From (17) and (18), one has

$$J(tu_0) \leqslant \frac{1}{2}t^2\left(\|u_0\|_V^2 - (b_2-\varepsilon_0)\|u_0\|_H^2 \right) + B\,\mathrm{meas}(\Omega) \to -\infty$$

when $t \to \infty$.

At last, taking $S = \{u \mid \|u\|_V = \rho\}$ and $Q = \{tu_0 \mid 0 \leqslant t \leqslant t_0\}$ with $t_0 > \rho$ being such that $J(t_0u_0) \leqslant 0$, then J has a critical value $c \geqslant \gamma > 0$ via Remark 2 by Theorem B. Hence system (1) has a nontrivial solution $u = (y,z) \in V$, which completes our proof. □

PROOF OF THEOREM 2. First, we also obtain that the functional J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$ from Lemma 1.

In view of (4) and condition (H_1), choosing $\varepsilon_1 = \min\left\{ \frac{\underline{\lambda}-a_3}{2}, \frac{b_3-(\overline{\lambda}+2k)}{2} \right\}$, then $a_3 + \varepsilon_1 < \underline{\lambda}$, $b_3 - \varepsilon_1 > \overline{\lambda} + 2k$. For $\varepsilon_1 > 0$, there exist $D \geqslant 0$ and $E \geqslant 0$ such that

$$\frac{1}{2}(b_3-\varepsilon_1)|u|^2 - E \leqslant F(x,u) \leqslant \frac{1}{2}(a_3+\varepsilon_1)|u|^2 + D|u|^p \qquad (19)$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. Then we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_\Omega \left(|\Delta y|^2 + |\nabla z|^2 + k\big((y-z)^+\big)^2 \right) dx - \int_\Omega F(x,u)\,dx \\
&\geqslant \frac{1}{2}\|u\|_V^2 - \int_\Omega \left(\frac{1}{2}(a_3+\varepsilon_1)|u|^2 + D|u|^p \right) dx \\
&\geqslant \frac{1}{2}\|u\|_V^2 - \frac{1}{2}(a_3+\varepsilon_1)\frac{1}{\underline{\lambda}}\|u\|_V^2 - C_8\|u\|_V^p \\
&= \frac{1}{2}\left(1 - \frac{a_3+\varepsilon_1}{\underline{\lambda}} \right)\|u\|_V^2 - C_8\|u\|_V^p
\end{aligned}$$

for some $C_8 > 0$.

Since $\frac{a_3+\varepsilon_1}{\underline{\lambda}} < 1$, $p > 2$, we obtain the estimate

$$J(u) \geqslant \frac{1}{4}\left(1 - \frac{a_3+\varepsilon_1}{\underline{\lambda}} \right)\delta^2 \equiv \zeta > 0, \ \ \forall \|u\|_V = \delta$$

with $\delta = \left(\frac{1}{4C_8}\left(1 - \frac{a_3+\varepsilon_1}{\underline{\lambda}}\right) \right)^{\frac{1}{p-2}} > 0$. By (19), one has

$$J(u) = \frac{1}{2}\int_\Omega \left(|\Delta y|^2 + |\nabla z|^2 + k\big((y-z)^+\big)^2 \right) dx - \int_\Omega F(x,u)\,dx$$

$$\leqslant \frac{1}{2}\|u\|_V^2 + k\int_\Omega (y^2+z^2)\,dx - \int_\Omega \Big(\frac{1}{2}(b_3-\varepsilon_1)|u|^2 - E\Big)\,dx$$
$$= \frac{1}{2}\Big(\|u\|_V^2 - \big((b_3-\varepsilon_1)-2k\big)\|u\|_H^2\Big) + E\,\mathrm{meas}\,\Omega,$$

so we have

$$J(tu) \leqslant \frac{1}{2}t^2\Big(\|u\|_V^2 - \big((b_3-\varepsilon_1)-2k\big)\|u\|_H^2\Big) + E\,\mathrm{meas}(\Omega). \tag{20}$$

Let $u_0 = (\varphi_1, \varphi_1)$, then one has

$$\begin{aligned}\|u_0\|_V^2 &= \int_\Omega |\Delta\varphi_1|^2\,dx + \int_\Omega |\nabla\varphi_1|^2\,dx\\ &= \lambda_1^2\|\varphi_1\|_{L^2(\Omega)}^2 + \lambda_1\|\varphi_1\|_{L^2(\Omega)}^2\\ &\leqslant \overline{\lambda}\big(\|\varphi_1\|_{L^2(\Omega)}^2 + \|\varphi_1\|_{L^2(\Omega)}^2\big)\\ &= \overline{\lambda}\|u_0\|_H^2\\ &< \big((b_3-\varepsilon_1)-2k\big)\|u_0\|_H^2. \end{aligned} \tag{21}$$

From (20) and (21), we obtain

$$J(tu_0) \leqslant \frac{1}{2}t^2\Big(\|u_0\|_V^2 - \big((b_3-\varepsilon_1)-2k\big)\|u_0\|_H^2\Big) + E\,\mathrm{meas}(\Omega) \to -\infty$$

when $t \to \infty$.

At last, taking $S = \{u \mid \|u\|_V = \delta\}$ and $Q = \{tu_0 \mid 0 \leqslant t \leqslant t_0\}$ with $t_0 > \delta$ being such that $J(t_0u_0) \leqslant 0$, then J has a critical value $d \geqslant \zeta > 0$ via Remark 2 by Theorem B. Hence system (1) has a nontrivial solution $u = (y,z) \in V$, which completes our proof. □

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Lei Ji and Chun-Lei Tang
School of Mathematics and Statistics
Southwest University
Chongqing 400715
People's Republic of China
e-mail: tangcl@swu.edu.cn