

EXISTENCE OF SOLUTIONS FOR THE COUPLED SYSTEMS OF SECOND AND FOURTH ORDER ELLIPTIC EQUATIONS

LEI JI AND CHUN-LEI TANG

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Abstract. In this paper, under the nonquadraticity condition, we obtain two existence theorems of nontrivial solutions for a coupled system of second and fourth order elliptic equations.

1. Introduction and main results

Consider the following coupled system of second and fourth order elliptic equations given by

$$\begin{cases} \Delta^2 y + k(y - z)^+ = f_1(x, y, z) & \text{in } \Omega, \\ -\Delta z - k(y - z)^+ = f_2(x, y, z) & \text{in } \Omega, \\ \Delta y = y = 0 & \text{on } \partial\Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is an open bounded domain in R^N ($N \geq 3$) with smooth boundary $\partial\Omega$.

System (1) comes from the mathematical model of the suspension bridge presented in Lazer and McKenna [14]. Many authors, using degree theory and the variational method, investigated the equations corresponding to the suspension bridge. These results can be seen in [4], [8], [9], [13], [15], [16] and [17].

In [3], by the least action principle and the mountain pass theorem, the author studied the existence of nontrivial solutions for the following system under the perturbation $g = (g_1, g_2)$ being sublinear and superlinear with respect to $u = (y, z)$, respectively.

$$\begin{cases} y_{xxxx} + k(y - z)^+ = g_1(x, y, z) & \text{in } (0, \pi), \\ -z_{xx} - k(y - z)^+ = g_2(x, y, z) & \text{in } (0, \pi), \\ y(0) = y(\pi) = y_{xx}(0) = y_{xx}(\pi) = 0, \\ z(0) = z(\pi) = 0. \end{cases} \quad (2)$$

The other relevant studies can be found in [10] and [11]. In [2], the authors investigated problem (1) under the potential $F(x, u)$ ($\nabla_u F(x, u) = (f_1(x, u), f_2(x, u))$) being a homogenous function using the variational approach. In [1], the author continued the

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work of [2] and extended the result in [12] to system (1) in which the nonlinearities are more general than those in the result of [2] and obtained a nontrivial solution by the mountain pass theorem. The result is the following theorem.

THEOREM A (see [1]). *Let $\nabla_u F(x, u) = f(x, u) = (f_1(x, u), f_2(x, u))$. Suppose that $F(x, u)$ satisfies the following assumptions:*

(F₁) $f : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a Caratheodory function and there exist p such that

$$2 < p < 2^* = 2N/(N - 2)$$

and constant $C > 0$ such that for all $u = (y, z) \in \mathbb{R}^2$ and almost every $x \in \Omega$,

$$|f_1(x, u)| + |f_2(x, u)| \leq C(1 + |u|^{p-1});$$

(F₂) there exist $\theta \in (0, \frac{1}{2})$ and $M > 0$ such that

$$0 < F(x, u) \leq \theta u \cdot \nabla_u F(x, u) \text{ for almost every } x \in \Omega, \text{ if } |u| \geq M;$$

(F₃) $\lim_{|u| \rightarrow 0} \frac{F(x, u)}{|u|^2} = 0$ uniformly for a.e. $x \in \Omega$.

Let λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and $\lambda = \min\{\lambda_1, \lambda_1^2\}$. Assume that $k > -\lambda/2$. Then system (1) has at least one nontrivial solution.

It should be noted that condition (F₂) is the well-known Ambrosetti-Rabinowitz condition. It usually plays a very important role in verifying that the corresponding functional has a Mountain-Pass geometry and showing a related $(PS)_c$ sequence is bounded. In this paper, motivated by [1] and [6], we consider system (1) under the nonquadraticity condition which was introduced by D. G. Costa and C. A. Magalhaes in [6]. Using the linking theorem, we obtain two existence results of nontrivial solutions for system (1).

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded smooth domain. By H and V we denote the spaces of $L^2(\Omega) \times L^2(\Omega)$ and $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, respectively. For $u = (y, z)$, we denote by

$$\|u\|_H^2 = \|y\|_{L^2(\Omega)}^2 + \|z\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|u\|_V^2 = \int_{\Omega} |\Delta y|^2 dx + \int_{\Omega} |\nabla z|^2 dx,$$

the norms on H and V , respectively. By $\|u\|_r$ we denote the norm of $u = (y, z) \in L^r(\Omega) \times L^r(\Omega)$. The embedding from V into H is continuous, and moreover it is actually compact.

Let λ_1 be the first eigenvalue, which is positive, of the eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and φ_1 is the eigenfunction corresponding to λ_1 , with $\varphi_1(x) > 0$ for all $x \in \Omega$. We can easily know that λ_1^2 is the first eigenvalue of the eigenvalue problem

$$\Delta^2 u = \lambda u \text{ in } \Omega, \quad \Delta u = u = 0 \text{ on } \partial\Omega.$$

Let

$$\underline{\lambda} = \min\{\lambda_1, \lambda_1^2\}, \quad \bar{\lambda} = \max\{\lambda_1, \lambda_1^2\}.$$

Then $\underline{\lambda} > 0, \bar{\lambda} > 0$. It follows from the Poincaré inequality that

$$\|u\|_V^2 \geq \underline{\lambda} \|u\|_H^2$$

for all $u \in V$.

Now we state the assumptions imposed on F in this paper, where $\nabla_u F(x, u) = f(x, u) = (f_1(x, u), f_2(x, u))$.

(H₁) $f : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a Caratheodory function, that is, $f(x, u)$ is measurable in x for each $u = (y, z) \in \mathbb{R}^2$ and continuous in $u = (y, z) \in \mathbb{R}^2$ for almost every $x \in \Omega$, and there exist $2 < p < 2^* = 2N/(N - 2)$, $a_0, b_0 > 0$ such that for all $u = (y, z) \in \mathbb{R}^2$ and almost every $x \in \Omega$,

$$|f_1(x, u)| + |f_2(x, u)| \leq a_0 |u|^{p-1} + b_0.$$

(H₂) There exist $q > 2$ and $a_1 > 0$ such that

$$\limsup_{|u| \rightarrow \infty} \frac{F(x, u)}{|u|^q} \leq a_1 < \infty \text{ uniformly for a.e. } x \in \Omega.$$

(H₃) There exist $\frac{N}{2}(q - 2) < \mu \leq q$ and $b_1 > 0$ such that

$$\liminf_{|u| \rightarrow \infty} \frac{f(x, u)u - 2F(x, u)}{|u|^\mu} \geq b_1 > 0 \text{ uniformly for a.e. } x \in \Omega.$$

The main results in this paper are the following theorems.

THEOREM 1. *Suppose that $-\frac{1}{2}\underline{\lambda} < k \leq 0$ and (H₁)-(H₃) hold. Assume that there exist $a_2, b_2 > 0$ such that*

$$\limsup_{|u| \rightarrow 0} \frac{2F(x, u)}{|u|^2} \leq a_2 < \underline{\lambda} + 2k \text{ and } \liminf_{|u| \rightarrow \infty} \frac{2F(x, u)}{|u|^2} \geq b_2 > \bar{\lambda} \tag{3}$$

uniformly for a.e. $x \in \Omega$. Then there exists at least one nontrivial solution of system (1).

THEOREM 2. *Suppose that $k \geq 0$ and (H₁)-(H₃) hold. Assume that there exist $a_3, b_3 > 0$ such that*

$$\limsup_{|u| \rightarrow 0} \frac{2F(x, u)}{|u|^2} \leq a_3 < \underline{\lambda} \text{ and } \liminf_{|u| \rightarrow \infty} \frac{2F(x, u)}{|u|^2} \geq b_3 > \bar{\lambda} + 2k \tag{4}$$

uniformly for a.e. $x \in \Omega$. Then there exists at least one nontrivial solution of system (1).

COROLLARY 1. *Suppose that $k > -\frac{1}{2}\underline{\lambda}$ and (H₁)-(H₃) hold. Assume that*

$$\limsup_{|u| \rightarrow 0} \frac{2F(x, u)}{|u|^2} = 0 \text{ and } \liminf_{|u| \rightarrow \infty} \frac{2F(x, u)}{|u|^2} = +\infty$$

uniformly for a.e. $x \in \Omega$. Then system (1) has at least one nontrivial solution.

REMARK 1. There are functions $F(x, u)$ satisfying our corollary and not satisfying condition (F_2) in Theorem A. For example, let $F(x, u) = |u|^2 \ln(1 + |u|^2)$. It is easy to see that $F(x, u)$ satisfies our corollary, but doesn't satisfy (F_2) of Theorem A.

2. Proof of theorems

We shall find weak solutions of problem (1) in space V . We have known that the weak solutions $u = (y, z)$ of problem (1) are the critical points of the following functional defined in space V :

$$J(u) = \frac{1}{2} \int_{\Omega} \left(|\Delta y|^2 + |\nabla z|^2 + k((y - z)^+)^2 \right) dx - \int_{\Omega} F(x, u) dx,$$

where $u = (y, z) \in V$ and $\nabla_u F(x, u) = f(x, u) = (f_1(x, u), f_2(x, u))$. It is easy to know that the functional J is well defined and $J \in C^1(V, R)$ under condition (H_1) , and moreover,

$$\begin{aligned} \langle J'(u), v \rangle = \int_{\Omega} \left(\Delta y \Delta \xi + \nabla z \nabla \eta + k(y - z)^+ \xi - k(y - z)^+ \eta \right. \\ \left. - f_1(x, u) \xi - f_2(x, u) \eta \right) dx \end{aligned}$$

for all $u = (y, z), v = (\xi, \eta) \in V$.

For convenience, we state the definition of condition $(C)_c$ introduced by Cerami in [7] (see also [5]), abstract linking concept and theorem (see [5]).

A functional $J \in C^1(E, R)$ is said to satisfy condition $(C)_c, c \in R$, if, whenever $\{u_n\} \subset E$ is such that $J(u_n) \rightarrow c$ and $(1 + \|u_n\|)\|J'(u_n)\| \rightarrow 0$, then $\{u_n\}$ possesses a convergent subsequence.

DEFINITION 1. (see [5]) Let S be a closed subset of a real Hilbert space X . Let Q be a Hilbert manifold with relative boundary ∂Q . We shall say that S and ∂Q link if:

- (1°) $S \cap \partial Q = \emptyset$;
- (2°) for any map $\phi \in C^0(X, X)$ such that $\phi|_{\partial Q} = id$ there holds

$$\phi(Q) \cap S \neq \emptyset.$$

The following remark give a type of linking geometry structure which is a special case of Proposition 2.2 in [5].

REMARK 2. Let X be a Hilbert space, $x_0 \in X$ with $\|x_0\| = 1$. Let $t_0 > \rho > 0$, and $S = \{x \in X \mid \|x\| = \rho\}, Q = \{tx_0 \mid 0 \leq t \leq t_0\}$ with relative boundary $\partial Q = \{0, t_0x_0\}$. Then S and ∂Q link.

THEOREM B (see [5, Theorem 2.3]). *Let X be a real Hilbert space. Suppose that $J \in C^1(X, \mathbb{R})$ and satisfies condition $(C)_c$ for all $c > 0$. There exists a closed subset $S \subset X$ and a Hilbert manifold $Q \subset X$ with boundary ∂Q verifying the following conditions:*

- (i) $\sup_{u \in \partial Q} J(u) \leq \alpha < \beta \leq \inf_{u \in S} J(u)$ for some $0 \leq \alpha < \beta$;
- (ii) S and ∂Q link;
- (iii) $\sup_{u \in Q} J(u) < +\infty$.

Then J possesses a critical value $c \geq \beta$.

LEMMA 1. *Assume that $k > -\frac{1}{2}\underline{\lambda}$ and (H_1) - (H_3) hold. Then J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$.*

Proof. From the Sobolev embedding theorem, we can easily know that the embedding of $V \rightarrow L^p(\Omega) \times L^p(\Omega)$ is compact, for any $p < 2^* = 2N/(N - 2)$. And combining with condition (H_1) , we see that the map $K : V \rightarrow V^*$ given by

$$K(u) = (k(y - z)^+, -k(y - z)^+) - (f_1(x, u), f_2(x, u))$$

is compact.

Now let $c \in \mathbb{R}$ and $\{u_n\} \subset V$ being such that

$$J(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\| \rightarrow 0. \tag{5}$$

It suffices to show that $\{\|u_n\|_V\}$ is bounded.

From (5) we obtain that there exists $M > 0$ such that

$$|J(u_n)| \leq M \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\| \leq M. \tag{6}$$

Using (H_3) , for some $M_1 > 0$, when $|u| \geq M_1$ we have

$$\begin{aligned} \frac{f(x, u)u - 2F(x, u)}{|u|^\mu} &\geq \frac{1}{2}b_1 > 0, \quad \text{i.e.} \\ f(x, u)u - 2F(x, u) &\geq \frac{1}{2}b_1|u|^\mu. \end{aligned} \tag{7}$$

When $|u| < M_1$, using the subcritical growth condition (H_1) one has

$$\begin{aligned} \left| f(x, u)u - 2F(x, u) \right| &\leq |f(x, u)| \cdot |u| + 2|F(x, u)| \\ &\leq (|f_1(x, u)| + |f_2(x, u)|) \cdot |u| \\ &\quad + 2 \left| \int_0^1 (\nabla_u F(x, su), u) ds \right| \\ &\leq (a_0|u|^{p-1} + b_0) \cdot |u| + 2 \int_0^1 |f(x, su)| \cdot |u| ds \end{aligned}$$

$$\begin{aligned}
 &\leq (a_0|u|^p + b_0|u|) + 2 \int_0^1 (a_0|su|^{p-1} + b_0) \cdot |u| ds \\
 &\leq (a_0|u|^p + b_0|u|) + 2(a_0|u|^{p-1} + b_0) \cdot |u| \\
 &\leq C_0
 \end{aligned}
 \tag{8}$$

for some $C_0 > 0$. Then from (7) and (8) there exists $C_1 > 0$ such that

$$f(x, u)u - 2F(x, u) \geq \frac{1}{2}b_1|u|^\mu - C_1$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. So one has

$$\begin{aligned}
 3M &\geq 2J(u_n) - \langle J'(u_n), u_n \rangle \\
 &= \int_\Omega \left(|\Delta y_n|^2 + |\nabla z_n|^2 + k((y_n - z_n)^+)^2 \right) dx - 2 \int_\Omega F(x, u_n) dx \\
 &\quad - \int_\Omega \left(|\Delta y_n|^2 + |\nabla z_n|^2 + k(y_n - z_n)^+(y_n - z_n) - f(x, u_n)u_n \right) dx \\
 &= \int_\Omega \left(f(x, u_n)u_n - 2F(x, u_n) \right) dx \\
 &\geq \int_\Omega \left(\frac{1}{2}b_1|u_n|^\mu - C_1 \right) dx \\
 &= \frac{1}{2}b_1 \int_\Omega |u_n|^\mu dx - C_1 \text{meas}(\Omega)
 \end{aligned}$$

by (6). Hence $\left\{ \int_\Omega |u_n|^\mu dx \right\}$ is a bounded sequence, that is, there exists $C_2 > 0$ such that

$$\int_\Omega |u_n|^\mu dx \leq C_2 \tag{9}$$

for all n . Using (H_2) , for some $M_2 > 0$, when $|u| \geq M_2$ we obtain

$$\begin{aligned}
 \frac{F(x, u)}{|u|^q} &\leq 2a_1, \text{ i.e.} \\
 F(x, u) &\leq 2a_1|u|^q.
 \end{aligned}$$

When $|u| < M_2$, as before, there exists $C_3 > 0$ such that $F(x, u) \leq C_3$. So we have

$$F(x, u) \leq 2a_1|u|^q + C_3 \tag{10}$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. On one hand, from (6) and (10) we obtain

$$\begin{aligned}
 J(u_n) + \int_\Omega F(x, u_n) dx &\leq M + \int_\Omega (2a_1|u_n|^q + C_3) dx \\
 &= M + 2a_1\|u_n\|_q^q + C_3 \text{meas}(\Omega) \\
 &\triangleq 2a_1\|u_n\|_q^q + C_4.
 \end{aligned}
 \tag{11}$$

On the other hand, when $-\frac{1}{2}\underline{\lambda} < k \leq 0$ one has

$$\begin{aligned}
 J(u_n) + \int_{\Omega} F(x, u_n) dx &= \frac{1}{2} \int_{\Omega} \left(|\Delta y_n|^2 + |\nabla z_n|^2 + k((y_n - z_n)^+)^2 \right) dx \\
 &= \frac{1}{2} \|u_n\|_V^2 + \frac{1}{2} k \int_{\Omega} ((y_n - z_n)^+)^2 dx \\
 &\geq \frac{1}{2} \|u_n\|_V^2 + \frac{k}{\underline{\lambda}} \|u_n\|_V^2 \\
 &= \left(\frac{1}{2} + \frac{k}{\underline{\lambda}} \right) \|u_n\|_V^2 \\
 &\geq l \|u_n\|_V^2
 \end{aligned} \tag{12}$$

for some $0 < l < \frac{1}{2} + \frac{k}{\underline{\lambda}} \leq \frac{1}{2}$. If $k > 0$ we have

$$\begin{aligned}
 J(u_n) + \int_{\Omega} F(x, u_n) dx &= \frac{1}{2} \int_{\Omega} \left(|\Delta y_n|^2 + |\nabla z_n|^2 + k((y_n - z_n)^+)^2 \right) dx \\
 &\geq \frac{1}{2} \|u_n\|_V^2 \\
 &\geq l \|u_n\|_V^2.
 \end{aligned} \tag{13}$$

Thus from (11), (12) and (13), when $k > -\frac{1}{2}\underline{\lambda}$ we obtain

$$l \|u_n\|_V^2 \leq 2a_1 \|u_n\|_q^q + C_4. \tag{14}$$

Therefore, use the Hölder inequality together with (9) and the Sobolev inequality $\|u\|_{2^*} \leq C \|u\|_V$ in the above estimate (14) to obtain

$$\begin{aligned}
 l \|u_n\|_V^2 &\leq 2a_1 \|u_n\|_q^q + C_4 \\
 &\leq 2a_1 \left(\int_{\Omega} (|u_n|^{\frac{\mu(2^*-q)}{2^*-\mu}})^{\frac{2^*-\mu}{2^*-q}} dx \right)^{\frac{2^*-q}{2^*-\mu}} \cdot \left(\int_{\Omega} (|u_n|^{\frac{2^*(q-\mu)}{2^*-\mu}})^{\frac{2^*-\mu}{q-\mu}} dx \right)^{\frac{q-\mu}{2^*-\mu}} + C_4 \\
 &= 2a_1 \left(\int_{\Omega} |u_n|^{\mu} dx \right)^{\frac{2^*-q}{2^*-\mu}} \cdot \|u_n\|_{2^*}^{\frac{2^*(q-\mu)}{2^*-\mu}} + C_4 \\
 &\leq C_5 \|u_n\|_{2^*}^{\frac{2^*(q-\mu)}{2^*-\mu}} + C_4 \\
 &\leq C_6 \|u_n\|_V^{\frac{2^*(q-\mu)}{2^*-\mu}} + C_4
 \end{aligned} \tag{15}$$

for some $C_5 > 0, C_6 > 0$. Finally, since we are taking $\frac{N}{2}(q-2) < \mu \leq q$, it follows that $\frac{2^*(q-\mu)}{2^*-\mu} < 2$, and hence, (15) implies that $\{\|u_n\|_V\}$ is bounded. Thus the functional J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$.

PROOF OF THEOREM 1. First, from Lemma 1 we know that J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$.

Second, assume that (3) holds, then there exist $\rho, \gamma > 0$ such that $J(u) \geq \gamma > 0$ if $\|u\|_V = \rho$. Moreover, there exists $u_0 \in V$ such that $J(tu_0) \rightarrow -\infty$ as $t \rightarrow \infty$.

In view of (3) and condition (H_1) , choosing

$$\varepsilon_0 = \min \left\{ \frac{\underline{\lambda} + 2k - a_2}{2}, \frac{b_2 - \bar{\lambda}}{2} \right\},$$

then $a_2 + \varepsilon_0 < \underline{\lambda} + 2k, b_2 - \varepsilon_0 > \bar{\lambda}$. For $\varepsilon_0 > 0$ there exist $A \geq 0$ and $B \geq 0$ such that

$$\frac{1}{2}(b_2 - \varepsilon_0)|u|^2 - B \leq F(x, u) \leq \frac{1}{2}(a_2 + \varepsilon_0)|u|^2 + A|u|^p \tag{16}$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. Then we have

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\Omega} (|\Delta y|^2 + |\nabla z|^2 + k((y-z)^+)^2) dx - \int_{\Omega} F(x, u) dx \\ &\geq \frac{1}{2} \|u\|_V^2 + \frac{1}{2} \int_{\Omega} k((y-z)^+)^2 dx - \int_{\Omega} \left(\frac{1}{2}(a_2 + \varepsilon_0)|u|^2 + A|u|^p \right) dx \\ &\geq \frac{1}{2} \|u\|_V^2 + \left(k - \frac{a_2 + \varepsilon_0}{2} \right) \|u\|_H^2 - A \|u\|_p^p \\ &\geq \frac{1}{2} \|u\|_V^2 + \left(k - \frac{a_2 + \varepsilon_0}{2} \right) \frac{1}{\underline{\lambda}} \|u\|_V^2 - A \|u\|_p^p \\ &\geq \frac{1}{2} \left(1 + \frac{2k - (a_2 + \varepsilon_0)}{\underline{\lambda}} \right) \|u\|_V^2 - C_7 \|u\|_p^p \end{aligned}$$

for some $C_7 > 0$.

Therefore, since $p > 2$, we obtain the estimate

$$J(u) \geq \frac{1}{4} \left(1 + \frac{2k - (a_2 + \varepsilon_0)}{\underline{\lambda}} \right) \rho^2 \equiv \gamma > 0, \forall \|u\|_V = \rho$$

with $\rho = \left\{ \frac{1}{4C_7} \left(1 + \frac{2k - (a_2 + \varepsilon_0)}{\underline{\lambda}} \right) \right\}^{\frac{1}{p-2}} > 0$. By (16), one has

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\Omega} (|\Delta y|^2 + |\nabla z|^2 + k((y-z)^+)^2) dx - \int_{\Omega} F(x, u) dx \\ &\leq \frac{1}{2} \|u\|_V^2 - \int_{\Omega} \left(\frac{1}{2}(b_2 - \varepsilon_0)|u|^2 - B \right) dx \\ &= \frac{1}{2} \left(\|u\|_V^2 - (b_2 - \varepsilon_0) \|u\|_H^2 \right) + B \text{meas}(\Omega), \end{aligned}$$

so we get

$$J(tu) \leq \frac{1}{2} t^2 \left(\|u\|_V^2 - (b_2 - \varepsilon_0) \|u\|_H^2 \right) + B \text{meas}(\Omega). \tag{17}$$

Let $u_0 = (\varphi_1, \varphi_1)$, φ_1 is the eigenfunction corresponding to λ_1 , with $\varphi_1(x) > 0$ for all $x \in \Omega$. Then we obtain

$$\|u_0\|_V^2 = \int_{\Omega} |\Delta \varphi_1|^2 dx + \int_{\Omega} |\nabla \varphi_1|^2 dx$$

$$\begin{aligned}
 &= \lambda_1^2 \|\varphi_1\|_{L^2(\Omega)}^2 + \lambda_1 \|\varphi_1\|_{L^2(\Omega)}^2 \\
 &\leq \bar{\lambda} \left(\|\varphi_1\|_{L^2(\Omega)}^2 + \|\varphi_1\|_{L^2(\Omega)}^2 \right) \\
 &= \bar{\lambda} \|u_0\|_H^2 \\
 &< (b_2 - \varepsilon_0) \|u_0\|_H^2.
 \end{aligned} \tag{18}$$

From (17) and (18), one has

$$J(tu_0) \leq \frac{1}{2} t^2 \left(\|u_0\|_V^2 - (b_2 - \varepsilon_0) \|u_0\|_H^2 \right) + B \text{meas}(\Omega) \rightarrow -\infty$$

when $t \rightarrow \infty$.

At last, taking $S = \{u \mid \|u\|_V = \rho\}$ and $Q = \{tu_0 \mid 0 \leq t \leq t_0\}$ with $t_0 > \rho$ being such that $J(t_0u_0) \leq 0$, then J has a critical value $c \geq \gamma > 0$ via Remark 2 by Theorem B. Hence system (1) has a nontrivial solution $u = (y, z) \in V$, which completes our proof. \square

PROOF OF THEOREM 2. First, we also obtain that the functional J satisfies condition $(C)_c$ for every $c \in \mathbb{R}$ from Lemma 1.

In view of (4) and condition (H_1) , choosing $\varepsilon_1 = \min \left\{ \frac{\lambda - a_3}{2}, \frac{b_3 - (\bar{\lambda} + 2k)}{2} \right\}$, then $a_3 + \varepsilon_1 < \underline{\lambda}$, $b_3 - \varepsilon_1 > \bar{\lambda} + 2k$. For $\varepsilon_1 > 0$, there exist $D \geq 0$ and $E \geq 0$ such that

$$\frac{1}{2}(b_3 - \varepsilon_1)|u|^2 - E \leq F(x, u) \leq \frac{1}{2}(a_3 + \varepsilon_1)|u|^2 + D|u|^p \tag{19}$$

for all $u \in \mathbb{R}^2$ and a.e. $x \in \Omega$. Then we have

$$\begin{aligned}
 J(u) &= \frac{1}{2} \int_{\Omega} \left(|\Delta y|^2 + |\nabla z|^2 + k((y - z)^+)^2 \right) dx - \int_{\Omega} F(x, u) dx \\
 &\geq \frac{1}{2} \|u\|_V^2 - \int_{\Omega} \left(\frac{1}{2}(a_3 + \varepsilon_1)|u|^2 + D|u|^p \right) dx \\
 &\geq \frac{1}{2} \|u\|_V^2 - \frac{1}{2}(a_3 + \varepsilon_1) \frac{1}{\underline{\lambda}} \|u\|_V^2 - C_8 \|u\|_V^p \\
 &= \frac{1}{2} \left(1 - \frac{a_3 + \varepsilon_1}{\underline{\lambda}} \right) \|u\|_V^2 - C_8 \|u\|_V^p
 \end{aligned}$$

for some $C_8 > 0$.

Since $\frac{a_3 + \varepsilon_1}{\underline{\lambda}} < 1$, $p > 2$, we obtain the estimate

$$J(u) \geq \frac{1}{4} \left(1 - \frac{a_3 + \varepsilon_1}{\underline{\lambda}} \right) \delta^2 \equiv \zeta > 0, \quad \forall \|u\|_V = \delta$$

with $\delta = \left(\frac{1}{4C_8} \left(1 - \frac{a_3 + \varepsilon_1}{\underline{\lambda}} \right) \right)^{\frac{1}{p-2}} > 0$. By (19), one has

$$J(u) = \frac{1}{2} \int_{\Omega} \left(|\Delta y|^2 + |\nabla z|^2 + k((y - z)^+)^2 \right) dx - \int_{\Omega} F(x, u) dx$$

$$\begin{aligned} &\leq \frac{1}{2} \|u\|_V^2 + k \int_{\Omega} (y^2 + z^2) dx - \int_{\Omega} \left(\frac{1}{2} (b_3 - \varepsilon_1) |u|^2 - E \right) dx \\ &= \frac{1}{2} \left(\|u\|_V^2 - ((b_3 - \varepsilon_1) - 2k) \|u\|_H^2 \right) + E \operatorname{meas} \Omega, \end{aligned}$$

so we have

$$J(tu) \leq \frac{1}{2} t^2 \left(\|u\|_V^2 - ((b_3 - \varepsilon_1) - 2k) \|u\|_H^2 \right) + E \operatorname{meas}(\Omega). \quad (20)$$

Let $u_0 = (\varphi_1, \varphi_1)$, then one has

$$\begin{aligned} \|u_0\|_V^2 &= \int_{\Omega} |\Delta \varphi_1|^2 dx + \int_{\Omega} |\nabla \varphi_1|^2 dx \\ &= \lambda_1^2 \|\varphi_1\|_{L^2(\Omega)}^2 + \lambda_1 \|\varphi_1\|_{L^2(\Omega)}^2 \\ &\leq \bar{\lambda} (\|\varphi_1\|_{L^2(\Omega)}^2 + \|\varphi_1\|_{L^2(\Omega)}^2) \\ &= \bar{\lambda} \|u_0\|_H^2 \\ &< ((b_3 - \varepsilon_1) - 2k) \|u_0\|_H^2. \end{aligned} \quad (21)$$

From (20) and (21), we obtain

$$J(tu_0) \leq \frac{1}{2} t^2 \left(\|u_0\|_V^2 - ((b_3 - \varepsilon_1) - 2k) \|u_0\|_H^2 \right) + E \operatorname{meas}(\Omega) \rightarrow -\infty$$

when $t \rightarrow \infty$.

At last, taking $S = \{u \mid \|u\|_V = \delta\}$ and $Q = \{tu_0 \mid 0 \leq t \leq t_0\}$ with $t_0 > \delta$ being such that $J(t_0 u_0) \leq 0$, then J has a critical value $d \geq \zeta > 0$ via Remark 2 by Theorem B. Hence system (1) has a nontrivial solution $u = (y, z) \in V$, which completes our proof. \square

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Lei Ji and Chun-Lei Tang
School of Mathematics and Statistics
Southwest University
Chongqing 400715
People's Republic of China
e-mail: tangcl@swu.edu.cn