

## LONG TIME ASYMPTOTICS OF SUB-THRESHOLD SOLUTIONS OF A SEMILINEAR CAUCHY PROBLEM

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*Abstract.* We show that the solution of the semilinear heat equation  $u_t = u_{xx} + u^p$  (with  $x \in \mathbb{R}$ ,  $p > 3$ , and nonnegative Cauchy data) behaves for large  $t$  like the solution of the corresponding linear problem plus a small correction of order  $t^{-1/2-c}$ , where  $c := 1/2$ , if  $p \geq 4$ , and  $c = (p-3)/2$ , if  $3 < p < 4$ . The result is known in special cases like small initial data. We prove it here for positive sub-threshold initial data satisfying some assumptions. Part of our results are contained in the recent work [10], but the motivation of this paper is to provide a new method leading to somewhat more general space-time estimates.

### 1. Main result

We consider the long time asymptotical behaviour of the solution of the classical semilinear diffusion equation and the associated Cauchy problem for the unknown function  $u = u(x, t)$ ,  $x \in \mathbb{R}$ ,  $t \geq 0$ :

$$u_t = u_{xx} + u^p \quad \text{for } x \in \mathbb{R}, t > 0, \tag{1.1}$$

$$u(x, 0) = h(x) \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

It is assumed that the initial data  $h \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$  is nonnegative. Moreover,  $p$  is assumed larger than the Fujita-exponent, i.e.  $p > 3$ . To simplify arguments, we only deal with the case the  $x$ -space dimension is one.

Given an  $h$ , let us denote by  $T_{\max}(h)$  the maximal existence time of the solution  $u$  to (1.1)-(1.2) with initial data  $h$  (see [23], Section 16 for terminology). Assuming that  $f \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$  is also nonnegative, we set

$$\alpha^* := \sup\{\alpha > 0 \mid T_{\max}(\alpha f) = \infty\}. \tag{1.3}$$

In this paper we consider initial data  $h$  which is of the form  $\alpha f$  for any  $\alpha$ ,  $0 < \alpha < \alpha^*$  for some fixed  $f$ . By this assumption, the (so-called sub-threshold) solution  $u$  exists for all  $t$ . More precisely, for example by [21] it is known that if  $f$  is even ( $f(x) = f(|x|)$ )

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for a.e.  $x$ ) and decays fast enough ( $\lim_{x \rightarrow \infty} f(x)x^\gamma = 0$  for some  $\gamma > 2/(p - 1)$ ), then there exists  $C \geq 1$  such that

$$e^{t\partial_x^2}(\alpha f)(x, t) \leq u(x, t) \leq C e^{t\partial_x^2}(\alpha f)(x, t) \quad \text{for } x \in \mathbb{R}, t > 1, \tag{1.4}$$

where  $e^{t\partial_x^2}(\alpha f)$  denotes the solution of Cauchy problem for the linear heat equation in  $\mathbb{R}$  with initial data  $\alpha f$ .

The asymptotical large time behaviour of  $u$  is studied in more detail also in, for example, [1], [2], [5], [6], [7], [8], [9], [11], [10], [12], [13], [14], [18], [19], [20], [21]. We cite the result of [14], Proposition 3, which essentially states that, for “all” positive initial data  $h = \alpha f$  with  $\alpha < \alpha^*$ , there exists a number  $A$  such that

$$\|u(\cdot, t) - A\varphi(\cdot, t)\|_\infty \leq \varepsilon(t)(t + 1)^{-\frac{1}{2}}, \tag{1.5}$$

where  $0 < \varepsilon(t) \rightarrow 0$ , as  $t \rightarrow \infty$ . Here  $\|\cdot\|_\infty$  denotes the sup-norm with respect to the variable  $x$ , and

$$\varphi(x, t) := \frac{1}{\sqrt{4\pi(t + 1)}} e^{-\frac{1}{4} \frac{x^2}{t+1}}, \tag{1.6}$$

so (1.5) presents the solution of the superlinear problem as a perturbation of the linear Cauchy problem.

On the other hand, in [25] (see also [1], [26]) it was proven for *small initial data* and  $p \geq 4$  that the solution  $u$  has the form

$$u = A\varphi(x, t) + v, \tag{1.7}$$

where  $A \in \mathbb{R}$  is a constant and  $v$  is small for large  $t$  in the sense that

$$\|v(\cdot, t)\|_\infty \leq \mathcal{O}(1/t). \tag{1.8}$$

In view of this result, the exponent  $-1/2$  on the right hand side of (1.5) is not yet optimal. Recently, in [11] the authors improved the result of [14] (for comparison, a particular case of their general results is the bound  $\|v(\cdot, t)\|_\infty \leq Ct^{-3/4}$  in case  $p = 4$ ; take  $f(u) = u^4$  and  $h(u) = u^3$  in Theorem 1.2. of [11].)

Parallel to the recent work [10], we want to present the accurate asymptotical behaviour (1.7), (1.8) of  $u$  for a large class of positive initial data  $h = \alpha f$  with  $\alpha < \alpha^*$ . The results of [14], [11] are thus improved by providing a better convergence exponent, and the result of [25] by releasing the smallness assumption of the initial data. Also we improve the approach of [1], [25], by considering all  $p > 3$  instead of  $p \geq 4$ . Part of our results is contained in [10]; we perform a comparison in Remark 1.4 below.

To state the main result we introduce for all  $t \in \mathbb{R}^+$ , the weighted sup-norm

$$\|u(\cdot, t)\|_t := \|u\|_t := \sup_{x \in \mathbb{R}} \left(1 + \frac{|x|}{\sqrt{t + 1}}\right)^\gamma |u(x, t)|, \tag{1.9}$$

where  $\gamma > 1$  ( $\gamma$  is to be fixed by the assumptions of Theorem 1.1). Notice that for all  $1 < q < \infty$  and  $t \geq 1$ , the inequality

$$\|u(\cdot, t)\|_{L^q(\mathbb{R})} \leq Ct^{1/(2q)} \|u(\cdot, t)\|_t \tag{1.10}$$

holds. The proof is given at the end of Section 5.

THEOREM 1.1. Assume that  $p > 3$  and that the measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$0 \leq f(x) \leq \frac{C}{(1 + |x|)^{\gamma+2}}, \quad x \in \mathbb{R}, \tag{1.11}$$

for some  $\gamma > 1$ . Assume that  $h = \alpha f$  and  $\alpha < \alpha^*$ , and that the solution  $u = u(x, t)$  to the Cauchy problem

$$u_t = u_{xx} + u^p \quad \text{on } \mathbb{R} \times \mathbb{R}^+ \tag{1.12}$$

$$u(x, 0) = h(x) \quad \text{for all } x \in \mathbb{R}, \tag{1.13}$$

satisfies the pointwise estimate

$$0 \leq u(x, t) \leq \frac{C}{\sqrt{t+1}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma} \tag{1.14}$$

for all  $x, t$ . (The number  $\gamma$  is fixed by (1.11), (1.14) for the rest of the paper.)

Then there exists a number  $A \in \mathbb{R}$  such that for  $t \geq 1$

$$u = A\varphi + v \quad \text{with } \|v(\cdot, t)\|_t \leq Ct^{-\frac{1}{2} - \frac{1}{2}(p^*-3)}, \tag{1.15}$$

and  $\varphi$  is the Gaussian (1.6) and  $p^* := \min(4, p)$ . In particular,  $v$  satisfies

$$\|v(\cdot, t)\|_\infty \leq Ct^{-\frac{1}{2} - \frac{1}{2}(p^*-3)}, \tag{1.16}$$

$$\|v(\cdot, t)\|_{L^q(\mathbb{R})} \leq Ct^{-\frac{1}{2} + \frac{1}{2q} - \frac{1}{2}(p^*-3)}, \tag{1.17}$$

for  $1 < q < \infty$  and, if  $p \geq 4$ ,  $\|v(\cdot, t)\|_\infty \leq Ct^{-1}$  for  $t \geq 1$ .

The estimates (1.16) and (1.17) readily follow from (1.15) and (1.10). The number  $A$  will be equal to  $\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} u(x, t) dx$ , see Definition 3.2. In view of the definition of the norm  $\|\cdot\|_t$ , the result (1.15) yields a rather detailed estimate, not just for large time behaviour, but also the space-time convergence of  $v$ .

The assumption of the theorem is satisfied at least in the following two cases.

COROLLARY 1.2. Let  $f, \alpha$  and  $h$  be as in the previous theorem, and let  $u$  be the solution of (1.12)-(1.13). If  $f$  is an even function, then the pointwise estimate (1.14) (and thus also (1.15)) always hold.

*Proof.* That (1.14) holds in case of such an  $f$ , is proven in Remark 4.3.(ii) of [21], formula (35), see also p. 127 of [18].  $\square$

COROLLARY 1.3. Let  $f, \alpha$  and  $h$  be as in Theorem 1.1, and assume that the solution  $u$  of (1.12)-(1.13) satisfies

$$t^{\frac{1}{p-1}} \|u(\cdot, t)\|_\infty \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{1.18}$$

Then (1.14) is satisfied and thus (1.15) holds.

*Proof.* One can again follow the arguments on the few lines preceding (35) of [21], or, alternatively, Section 2, the proof of Theorem 1.2. of [2]. The idea of proof is, given  $u$  satisfying (1.18), to consider the solution  $\tilde{u}$  of the linear Cauchy problem

$$\tilde{u}_t = \tilde{u}_{xx} + \|u(\cdot, t)\|_\infty^{p-1} \tilde{u} \quad \text{on } \mathbb{R} \times \mathbb{R}^+ \tag{1.19}$$

$$\tilde{u}(x, 0) = h(x) \quad \text{for all } x \in \mathbb{R}. \tag{1.20}$$

From the comparison principle it follows that  $u(x, t) \leq \tilde{u}(x, t)$  for all  $x$  and  $t$ . On the other hand, the solution to (1.19)-(1.20) is given by

$$\tilde{u}(x, t) = U(x, t) \exp\left(\int_0^t \|u(\cdot, s)\|_\infty^{p-1} ds\right), \tag{1.21}$$

where  $U = e^{t\partial_x^2} h$  is the solution of the heat equation and thus satisfies (1.14). The property (1.14) follows for  $u$  from (1.21) and the fact that the integral  $\int_0^\infty \|u(\cdot, s)\|_\infty^{p-1} ds$  converges, by Lemma 2.3 of [2] and arguments around it.  $\square$

REMARK 1.4. Another form of the result (1.15) was recently obtained in the reference [10], Corollary 1.6, using completely different methods. Let us make a review for comparison.

Our result yields space-time estimates for the solution, due to the definition of the norm  $\|\cdot\|_t$ , whereas the result of [10] is only formulated in terms of  $L^q(\mathbb{R}^N)$ -norms; for  $1 < q < \infty$  these are gotten in our work as a consequence of the definition of the norm  $\|\cdot\|_t$ . The obtained time-rate is the same in both works.

As for the initial data, the citation assumes  $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and moreover

$$\int_{\mathbb{R}^N} (1 + |x|^K) |f(x)| dx < \infty$$

for some  $K > 1$ , which for  $N = 1$  is weaker than (1.11) and do not presume positivity. The a priori assumption on the existence and behaviour of the solution  $u$  is slightly weaker in the present paper; compare Corollaries 1.2 and 1.3 with (1.16) of [10]. Moreover, our work could easily be extended to some unbounded initial data: indeed, initial data is only handled (in addition to (2.3)) in Lemma 2.2. Inspecting the proof one finds that it would be enough just to assume  $f$  (equivalently  $h = \alpha f$ ) be in  $L^1(\mathbb{R})$  and satisfy

$$\int_{|x| \geq |y|} |f(x)| dx \leq C(1 + |y|)^{-\gamma-1} \tag{1.22}$$

for all  $y \in \mathbb{R}$ . Of course, there are unbounded functions satisfying these requirements.

REMARK 1.5. [22]. If the function  $f$  decays exponentially as  $|x| \rightarrow \infty$ , it is possible to deduce the result (1.15) for the sub-threshold solution using the rescaled equation, see [13], and the abstract asymptotic analysis presented in Section 51 of [23]. We leave the details to the reader.

The proof of Theorem 1.1 is based on the fact that suitable, quite tricky integrations by parts of the nonlinear term of (2.1) can distinguish between the leading  $A\varphi$  and perturbative  $v$  terms. We derive an integral equation (3.6) for  $v$  (to this end we need all the a priori assumptions on  $u$ ), and use the above mentioned integrations by parts, together with standard weighted sup-norm estimates, to prove the smallness of  $v$ .

It might be good to keep in mind that for the linear Cauchy problem, the solution  $u$  can always be written as  $A\varphi + v$  simply by decomposing the initial data  $h$  as the sum  $A(4\pi)^{-1/2}e^{-x^2/4} + g$ , where  $A = \int h$  and thus  $\int g = 0$ . For such a  $g$  one has  $\|e^{t\partial_x^2}g\|_\infty = \mathcal{O}(1/t)$  for large  $t$  (see Lemma 2.2 below).

Notation. By  $C, c, C'$  (respectively,  $c_n$ ), and so on, we denote strictly positive constants which are independent on the variables  $x, t, n$  etc. (resp. depend on  $n$  only) and which may vary from place to place but not in the same inequality. We write  $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ . We have

$$e^{t\partial_x^2}h = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)}h(y)dy,$$

if  $h \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ . The Gaussian function  $\varphi$  is defined in (1.6). Finally,  $\mathcal{O}(1/t)$  refers to the usual Landau symbol.

### 2. The key Lemma 2.4.

The rest of the paper is occupied by the proof of Theorem 1.1. From now on we assume that  $f, \alpha$  and  $h$  meet the assumptions of the theorem,  $u$  is a global classical solution of (1.12)-(1.13), and the condition (1.14) holds. A classical solution is always a mild solution, hence,  $u$  satisfies the integral equation

$$u(x,t) = e^{t\partial_x^2}h(x) + \int_0^t e^{(t-s)\partial_x^2}u(\cdot,s)^p ds. \tag{2.1}$$

Let  $A \geq 0$  and define

$$v := u - A\varphi \text{ i.e. } u = A\varphi + v. \tag{2.2}$$

We shall fix  $A$  in the next section. Until that, the results hold for all  $A$ .

DEFINITION 2.1. We denote

$$A_0 := \int_{-\infty}^{\infty} h(x)dx \quad \text{and} \quad g := h - A_0\varphi(x,0). \tag{2.3}$$

Notice that then  $\int_{-\infty}^{\infty} g(x)dx = 0$ , since  $\int_{-\infty}^{\infty} \varphi(x,0)dx = 1$ .

Moreover, let

$$\mathcal{M} := \mathcal{M}_{u,A}(x,t) := u^p - A^p\varphi^p, \tag{2.4}$$

and

$$\mathcal{R} := \mathcal{R}_A(x, t) := A^p \int_0^t e^{(t-s)\partial_x^2} \varphi^p ds - BA^p \varphi, \tag{2.5}$$

where  $B$  is the numerical constant

$$B := \frac{2}{(4\pi)^{\frac{p}{2}} \sqrt{p} (3-p)}. \tag{2.6}$$

So,  $A_0$  are  $g$  are now fixed, whereas  $\mathcal{R}$  and  $\mathcal{M}$  depend on the choice of  $A$ .

At this point the reader might have a look at (3.6) in order to find motivation for Lemmas 2.2, 2.3 and 2.4: they contain the crucial estimates for the terms of  $v$ .

If  $w = w(x)$ ,  $x \in \mathbb{R}$ , is a function which decays fast enough as  $|x| \rightarrow \infty$  and satisfies  $\int w(x)dx = 0$ , then  $e^{t\partial_x^2} w$  is of order  $t^{-1}$  for large  $t$ . (The vanishing integral means that the Fourier transform of  $w$  vanishes at 0, and the estimate can be obtained by working on the Fourier side.) In the first lemma we specify this phenomenon for the particular norm we are using.

LEMMA 2.2. *For  $t \geq 1$  we have*

$$\|e^{t\partial_x^2} g\|_t \leq \frac{C}{t}. \tag{2.7}$$

*Proof.* Integrating by parts,

$$\begin{aligned} e^{(t+1)\partial_x^2} g(x) &= \frac{C}{\sqrt{t+1}} \int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-y)^2/(t+1)} (h(y) - A_0\varphi(y, 0)) dy \\ &= - \int_{-\infty}^{\infty} \frac{C'(x-y)}{(t+1)^{3/2}} e^{-\frac{1}{4}(x-y)^2/(t+1)} \int_{-\infty}^y (h(z) - A_0\varphi(z, 0)) dz dy. \end{aligned} \tag{2.8}$$

Here  $h(z) - A_0\varphi(z, 0)$  is a function whose integral over the real line vanishes and which moreover is bounded by  $C(1 + |z|)^{-\gamma-2}$ . Hence,

$$\left| \int_{-\infty}^y \left( h(z) - \frac{A_0}{\sqrt{4\pi}} e^{-\frac{1}{4}z^2} \right) dz \right| \leq \frac{C}{(1 + |y|)^{\gamma+1}}. \tag{2.9}$$

Moreover, for some constant  $c > 0$ ,

$$\frac{|x-y|}{(t+1)^{3/2}} e^{-\frac{1}{4}(x-y)^2/(t+1)} \leq \frac{C}{t+1} e^{-c(x-y)^2/(t+1)}$$

and thus we can bound (2.8) by

$$\int_{-\infty}^{\infty} \frac{C}{t+1} e^{-c(x-y)^2/(t+1)} \frac{1}{(1 + |y|)^{\gamma+1}} dy. \tag{2.10}$$

Dividing the integration domain to two parts, the integral over the set  $\{|x - y| \geq |x|/2\}$  is bounded by

$$\frac{C}{t+1} e^{-c'x^2/(t+1)} \int_{|x-y| \geq |x|/2} \frac{1}{(1+|y|)^{\gamma+1}} dy \leq \frac{C'}{t+1} e^{-c'x^2/(t+1)}. \tag{2.11}$$

For the remaining integral one uses the facts that the length of integration interval is  $|x|$  and that  $|y|$  can be approximated by  $|x|$ :

$$\begin{aligned} \frac{C}{t+1} \int_{|x-y| \leq |x|/2} e^{-c(x-y)^2/(t+1)} \frac{1}{(1+|y|)^{\gamma+1}} dy \\ \leq \frac{C'}{t+1} \frac{1}{(1+|x|)^{\gamma+1}} \int_{|x-y| \leq |x|/2} dy \leq \frac{C'}{t+1} \frac{1}{(1+|x|)^{\gamma}}. \end{aligned}$$

This and the estimate (2.11) are small enough to imply the statement.  $\square$

It is possible to estimate also  $\mathcal{R}$  using rather explicit calculations. We remark that this seems to be the leading term of  $v$ , see (3.6), and compare with Lemmas 2.2 and 2.4.

LEMMA 2.3. *We have for  $t \geq 1$*

$$\|\mathcal{R}_A(\cdot, t)\|_t \leq Ct^{-\frac{1}{2} - \frac{1}{2}(p^*-3)}. \tag{2.12}$$

The constant  $C$  actually depends on  $A$ . We postpone the proof until Section 4.

The last of the three lemmas is the key for our main result. It will be used more than once: first, for the choice of the constant  $A$  (the lemma holds for arbitrary  $A$ ) then, to produce the desired estimate for  $v$  by two iteration steps.

LEMMA 2.4. *Let  $v$  be as in (2.2). Assume that, for some  $n = 0, 1$ , the following estimates are proven for  $t > 0$ :*

$$\|v\|_t \leq C(t+1)^{-\frac{1}{2} - \frac{n}{4}(p^*-3)} \tag{2.13}$$

and

$$|\mathcal{M}(x, t)| \leq C(t+1)^{-\frac{n}{4}(p^*-3) - \frac{p}{2}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-p\gamma}. \tag{2.14}$$

Then we even have for  $t \geq 1$

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\partial_x^2} \mathcal{M}(\cdot, s) ds - \varphi(\cdot, t) \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y, s) dy ds \right\|_t \\ \leq C(t+1)^{-\frac{1}{2} - \frac{n+1}{4}(p^*-3)}. \end{aligned} \tag{2.15}$$

Here the constants  $C$  may depend on  $A$  and  $n$ . The proof is postponed until Section 5.

### 3. Choice of the constant $A$ and proof of Theorem 1.1.

The first consequence of Lemma 2.4 is the following.

LEMMA 3.1. *The limit*

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} u(x, t) dx \tag{3.1}$$

exists.

*Proof.* Let  $A \geq 0$  be arbitrary and  $v$  as in (2.2). We have  $\|\varphi(\cdot, t)\|_t \leq C(t+1)^{-1/2}$  and, by (1.14),  $\|u(\cdot, t)\|_t \leq C(t+1)^{-1/2}$ , hence, the same holds also for  $v$ . This means, (2.13) holds for  $n = 0$ . It is also clear from these estimates and (2.4) that  $\mathcal{M}$  satisfies (2.14) for  $n = 0$ . Hence, Lemma 2.4 is applicable with  $n = 0$ .

Straightforward replacement of  $u = A\varphi + v$  into (2.1) implies

$$\begin{aligned} A\varphi + v = u &= \left( BA^p + A_0 + \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y, s) dy ds \right) \varphi + e^{t\partial_x^2} g \\ &+ \left( \int_0^t e^{(t-s)\partial_x^2} \mathcal{M}(\cdot, s) ds - \varphi \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y, s) dy ds \right) + \mathcal{R}. \end{aligned} \tag{3.2}$$

We now notice that the integral  $\int_{-\infty}^\infty dx$  of the second and third rows of (3.2) tend to 0, as  $t \rightarrow \infty$ . Indeed, by Lemma 2.2, (2.15), (2.12) and the definition of the norm  $\|\cdot\|_t$  (1.9), all of these terms have the pointwise bound

$$\frac{C}{(t+1)^{\frac{1}{2}+\varepsilon}} \left( 1 + \frac{|x|}{\sqrt{t+1}} \right)^{-\gamma} \tag{3.3}$$

for some  $\varepsilon > 0$ . The  $x$ -integral of (3.3) vanishes for  $t \rightarrow \infty$ , and the claim follows.

On the other hand  $\int_{-\infty}^\infty \varphi dx$  is the constant 1. The lemma follows from (3.2).  $\square$

DEFINITION 3.2. From now on, we fix  $A$  by defining

$$A := \lim_{t \rightarrow \infty} \int_{-\infty}^\infty u(x, t) dx. \tag{3.4}$$

With this choice it is possible to formulate the separate equations satisfied by  $A$  and  $v$ . The latter of them, (3.6) below, together with Lemma 2.4, will be crucial to derive the desired smallness estimates for  $v$ .



COROLLARY 3.3. *The following equations are satisfied by  $A$  and  $v$ :*

$$A = A_0 + BA^p + \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y, s) dy ds, \tag{3.5}$$

$$v(x, t) = e^{t\partial_x^2} g(x) + \mathcal{R}(x, t) + \int_0^t e^{(t-s)\partial_x^2} \mathcal{M}(\cdot, s) ds - \varphi(x, t) \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y, s) dy ds. \tag{3.6}$$

*Proof.* It follows from the remarks in the proof of Lemma 3.1, that the limit  $t \rightarrow \infty$  of the integral  $\int_{-\infty}^\infty dx$  of the right hand side of (3.2) is

$$A_0 + BA^p + \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y, s) dy ds. \tag{3.7}$$

Hence, (3.5) follows by applying (3.4) and  $\int_{-\infty}^\infty \varphi dx = 1$  on the left hand side of (3.2). Then (3.6) follows from (3.2).  $\square$

PROOF OF THEOREM 1.1. We need to prove the estimate (1.15) for  $v$ . Since  $v$  satisfies the equation (3.6) and also the estimates (2.7), (2.12) hold, there remains to use Lemma 2.4 to bound the  $\mathcal{M}$ -term in (3.6). The lemma has to be used two times to get a good enough bound.

*First step.* We take  $n = 0$ . It was already remarked in the proof of Lemma 3.1 that (2.13) and (2.14) hold. As a conclusion, we obtain (2.15) for  $n = 0$ .

*Induction step.* Assume that (2.15) is proven for  $n$  (which is only either 0 or 1; bigger  $n$  are not needed and the argument would not even work for them). We claim that this implies (2.13) and (2.14) for  $n + 1$ .

Indeed (2.13) is gotten immediately from Lemmas 2.2 and 2.3, and (3.6). As for (2.14) for  $n + 1$ , we use the mean value theorem in the form

$$|\xi^p - \zeta^p| \leq p|\xi - \zeta|z^{p-1},$$

where  $\xi$  and  $\zeta$  are positive numbers and  $z \in [\xi, \zeta]$ . We take

$$\xi := u(x, t) \text{ and } \zeta := A\varphi(x, t)$$

and observe (by what we have just proven)

$$|u(x, t) - A\varphi(x, t)| = |v(x, t)| \leq C(t + 1)^{-\frac{1}{2} - \frac{n+1}{4}(p^*-3)} (1 + |x|/\sqrt{t+1})^{-\gamma}, \tag{3.8}$$

and use the bound

$$C(t + 1)^{-1/2} (1 + |x|/\sqrt{t+1})^{-\gamma}$$

satisfied by both  $u(x, t)$  and  $\varphi(x, t)$ :

$$|\mathcal{M}(x, t)| = |u(x, t)^p - A^p\varphi(x, t)^p|$$

$$\begin{aligned} &\leq Cp|v(x,t)|(t+1)^{-(p-1)/2}(1+|x|/\sqrt{t+1})^{-(p-1)\gamma} \\ &\leq C(t+1)^{-\frac{p}{2}-\frac{n+1}{4}(p^*-3)}(1+|x|/\sqrt{t+1})^{-p\gamma}. \end{aligned} \tag{3.9}$$

Finally, it suffices to observe that (1.15) is gotten from (2.7), (2.12), (2.15) with  $n = 1$ , and (3.6).  $\square$

**4. Proof of Lemma 2.3.**

We estimate  $\mathcal{R}$ , (2.5), using the method of [25], (44)-(49). We recall that for all positive parameters  $a$  and  $b$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-a(x-y)^2-by^2} dy &= \int_{-\infty}^{\infty} e^{-ax^2+\frac{a^2}{a+b}x^2-(\frac{a}{\sqrt{a+b}}x-\sqrt{a+by})^2} dy \\ &= \frac{\sqrt{\pi}}{\sqrt{a+b}} e^{-abx^2/(a+b)}, \end{aligned} \tag{4.1}$$

hence, the expression  $e^{(t-s)\partial_x^2} \varphi(\cdot, s)^p$  equals  $(4\pi)^{-\frac{1}{2}(p+1)}$  times

$$\begin{aligned} &\frac{1}{\sqrt{t-s}} \int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-y)^2/(t-s)} \left( \frac{1}{\sqrt{s+1}} e^{-\frac{1}{4}y^2/(s+1)} \right)^p dy \\ &= \frac{1}{\sqrt{t-s}} \frac{1}{(s+1)^{p/2}} \frac{C}{\sqrt{(t-s)^{-1}+p(s+1)^{-1}}} e^{-\frac{1}{4}x^2/(t-\rho(s))} \\ &= (s+1)^{-p/2+1/2} \frac{C}{\sqrt{p(t+1)-(p-1)(s+1)}} e^{-\frac{1}{4}x^2/(t-\rho(s))}, \end{aligned} \tag{4.2}$$

where  $\rho(s) = s - s/p - 1/p$ . We can write (4.2) as

$$\begin{aligned} &(s+1)^{-p/2+1/2} \frac{1}{\sqrt{p}} \frac{1}{\sqrt{t+1}} \frac{1}{\sqrt{1-\frac{p-1}{p}\frac{s+1}{t+1}}} e^{-\frac{1}{4}x^2/(t-\rho(s))} \\ &= \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{p}} \frac{(s+1)^{-p/2+1/2+n}}{(t+1)^{n+1/2}} e^{-\frac{1}{4}x^2/(t-\rho(s))}, \end{aligned} \tag{4.3}$$

where the coefficients  $c_n$  come from the binomial series, hence  $|c_n| = \mathcal{O}(n^{-3/2})$ . Integrating termwise by parts (for the exceptional case  $-p/2 + 1/2 + n = -1$ , see the comment below) we obtain

$$\begin{aligned} &\left[ \sum_{n=0}^{\infty} c'_n \frac{(s+1)^{n+3/2-p/2}}{(t+1)^{n+1/2}} e^{-\frac{1}{4}x^2/(t-\rho(s))} \right]_{s=0}^t \\ &+ \int_0^t \sum_{n=0}^{\infty} c''_n \frac{(s+1)^{n+3/2-p/2}}{(t+1)^{n+1/2}} \frac{x^2}{(t-\rho(s))^2} e^{-\frac{1}{4}x^2/(t-\rho(s))} ds, \end{aligned} \tag{4.4}$$

where

$$c'_n = \frac{1}{n + 3/2 - p/2} \frac{c_n}{\sqrt{p}}, \tag{4.5}$$

and  $c''_n$  equals  $c'_n$  multiplied by some number depending on  $p$  only.

Except for the term with  $n = 0$  and  $\int_{s=0}$ , the sum of the other terms of the first line is bounded by

$$C(t + 1)^{-\frac{1}{2} - \frac{1}{2}(p^* - 3)} e^{-cx^2/(t+1)}.$$

Notice that for some  $c$ ,  $0 < c < 1$ , we have  $\rho(s) \leq ct$  for all  $0 \leq s \leq t$ . The second line of (4.4) also satisfies this bound, since we can use the estimate

$$\left| \frac{x^2}{(t - \rho(s))^2} e^{-\frac{1}{4}x^2/(t - \rho(s))} \right| \leq \frac{C}{t - \rho(s)} e^{-cx^2/(t - \rho(s))}.$$

All of these terms thus satisfy (2.12).

It may happen that  $-p/2 + 1/2 + n = -1$  for some  $n$ , and then the above reasoning cannot be used. But this is only possible if  $p = 5, 7, 9, \dots$ , which is so large a number that the corresponding term in (4.3) is easily seen to satisfy (2.12).

We have shown that

$$\int_0^t e^{(t-s)\partial_x^2} \varphi^p ds = B \frac{1}{\sqrt{4\pi(t+1)}} e^{-\frac{1}{4}x^2/(t-\rho(0))} + \mathcal{O}((t+1)^{-\frac{1}{2} - \frac{1}{2}(p^* - 3)} e^{-cx^2/(t+1)}), \tag{4.6}$$

where  $B$  is as in (2.6):

$$B = \frac{c'_0}{(4\pi)^{\frac{p}{2}}} = \frac{2}{(4\pi)^{\frac{p}{2}} \sqrt{p} (3 - p)}. \tag{4.7}$$

So we are left with  $(t + 1)^{-1/2} e^{-\frac{1}{4}x^2/(t-\rho(0))}$ . But using standard Taylor series developments this can also be written as

$$(t + 1)^{-1/2} e^{-\frac{1}{4}x^2/(t+1)} = \sqrt{4\pi} \varphi(x, t)$$

plus an expression bounded by  $C(t + 1)^{-1} e^{-cx^2/(t+1)}$ , we have

$$\begin{aligned} & \left| e^{-\frac{1}{4}x^2/(t+1)} - e^{-\frac{1}{4}x^2/(t-\rho(0))} \right| \tag{4.8} \\ &= e^{-\frac{1}{4}x^2/(t+1)} \left| 1 - e^{-\frac{1}{4}(1+\rho(0))x^2(t+1)^{-1}(t-\rho(0))^{-1}} \right| \\ &\leq C e^{-\frac{1}{4}x^2/(t+1)} \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{C(1+\rho(0))x^2}{(t+1)(t-\rho(0))} \right)^m \\ &\leq C' e^{-\frac{1}{4}x^2/(t+1)} \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{C'x^2}{(t+1)^2} \right)^m \leq \frac{C''}{t+1} e^{-cx^2/(t+1)}. \tag{4.9} \end{aligned}$$

This, together with (4.6), (2.5), proves (2.12).  $\square$

**5. Proofs of Lemma 2.4 and inequality (1.10).**

Before the proofs we record the following inequality (see e.g. [25], Lemma 4): for every  $a > 0$ ,  $b > 2$  and  $1 > \varepsilon > 0$  there exists a  $C > 0$  such that

$$\int_{-\infty}^{\infty} e^{-a(x-y)^2/(t-s)} (1 + |y|/\sqrt{s+1})^{-b} dy \leq C\sqrt{s+1}(1 + |x|/\sqrt{t+1})^{-b+1+\varepsilon}. \tag{5.1}$$

We proceed with the proof of Lemma 2.4. For (2.15) we use the method of [25], Lemma 3. However, to see the difference with [25], consider the case  $p \geq 4$ . Here we have to cope with a worse a priori bound for  $v$ : only  $\|v\|_t \leq C(t+1)^{-1/2}$  (for  $n = 0$ ) instead of  $C(t+1)^{-1}$  of the reference. This is so since we cannot assume small initial data and thus we cannot apply a fixed point argument here. It is a bit surprising that the arguments of [25] are still useful. Of course, another difference is that we also consider the cases  $3 < p < 4$  here.

A) The major part of the proof consists of producing and estimating some terms of the expression

$$\int_0^t e^{(t-s)\partial_x^2} \mathcal{M}(\cdot, s) ds. \tag{5.2}$$

This is done in the present subsection. We denote by  $-\Gamma$  the assumed decay order of  $\mathcal{M}$ , i.e.

$$\Gamma := \frac{n}{4}(p^* - 3) + \frac{p}{2} \tag{5.3}$$

and notice that  $\Gamma > 3/2$ . We aim to show that most of the terms of  $\int_0^t e^{(t-s)\partial_x^2} \mathcal{M}(\cdot, s) ds$  satisfy the norm estimate  $\|\cdot\|_t \leq C(t+1)^{-\Gamma+1}$ . Notice that

$$-\frac{1}{2} - \frac{n+1}{4}(p^* - 3) > -\Gamma + 1,$$

except for one term which will be estimated by  $(t+1)^{-\frac{1}{2} - \frac{n+1}{4}(p^* - 3)}$  (see (5.12) below), and another one which approximates the second term on the left hand side of (2.15) (see (5.19) below). This will imply the lemma.

Defining

$$\int_y^{\infty} \mathcal{M}(z, s) dz =: \mathcal{F}(y, s) \quad \text{for } y \geq 0 \tag{5.4}$$

and

$$\int_{-\infty}^y \mathcal{M}(z, s) dz =: \mathcal{G}(y, s) \quad \text{for } y \leq 0; \tag{5.5}$$

we obtain from (2.14) the estimates for  $\mathcal{F} := \mathcal{F}(x, t) := \mathcal{F}_{u,A}(x, t)$  and  $\mathcal{G} := \mathcal{G}(x, t) := \mathcal{G}_{u,A}(x, t)$ ,

$$\begin{aligned} |\mathcal{F}(y, s)| &\leq C(s+1)^{-\Gamma+1/2} \left(1 + \frac{|y|}{\sqrt{s+1}}\right)^{-p\gamma+1} \text{ for } y \geq 0 \text{ and} \\ |\mathcal{G}(y, s)| &\leq C(s+1)^{-\Gamma+1/2} \left(1 + \frac{|y|}{\sqrt{s+1}}\right)^{-p\gamma+1} \text{ for } y \leq 0. \end{aligned} \tag{5.6}$$

We apply integration by parts. In order to obtain a properly behaving integral function of  $\mathcal{M}(y, \cdot)$  we have to split the  $y$ -integration domain to two parts:

$$\begin{aligned} \sqrt{4\pi}e^{(t-s)\partial_x^2} \mathcal{M}(\cdot, s) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}(x-y)^2/(t-s)} \mathcal{M}(y, s) dy \\ &= - \left[ \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{F}(y, s) \right]_{y=0}^{\infty} + \int_0^{\infty} \frac{1}{2} \frac{x-y}{(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{F}(y, s) dy \\ &\quad + \left[ \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{G}(y, s) \right]_{y=-\infty}^0 - \int_{-\infty}^0 \frac{1}{2} \frac{x-y}{(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{G}(y, s) dy. \end{aligned} \tag{5.7}$$

Here the first term equals  $\frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \mathcal{F}(0, s)$ . Moreover, we obtain from the inequality  $\sqrt{z}e^{-z} \leq Ce^{-z/2}$  ( $z \geq 0$ ),

$$\begin{aligned} \left| \int_0^{\infty} \frac{x-y}{(t-s)^{3/2}} e^{-\frac{1}{4}(x-y)^2/(t-s)} \mathcal{F}(y, s) dy \right| \\ \leq \int_0^{\infty} \frac{C}{t-s} e^{-\frac{1}{8}(x-y)^2/(t-s)} |\mathcal{F}(y, s)| dy. \end{aligned} \tag{5.8}$$

To proceed with the evaluation of (5.2) we divide the  $s$ -integration interval into  $[0, t/2]$  and  $[t/2, t]$ .

To treat the first interval, using (5.6), and the estimate (5.1) we get for (5.8) the bound

$$\frac{C}{(t-s)(s+1)^{\Gamma-1}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-p\gamma+2+\varepsilon}. \tag{5.9}$$

Integrating this  $\int_0^{t/2} ds$  and taking a small enough  $\varepsilon$ , we get

$$\int_0^{t/2} \frac{C}{(t-s)(s+1)^{\Gamma-1}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-p\gamma+2+\varepsilon} ds$$

$$\leq \frac{C'}{t+1} \int_0^{t/2} \frac{1}{(s+1)^{\Gamma-1}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma} ds. \tag{5.10}$$

There are three cases (recall that  $\Gamma$  is always larger than  $3/2$ ):

If  $\Gamma < 2$ , then (5.10) is bounded by

$$\frac{C}{(t+1)^{\Gamma-1}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma}. \tag{5.11}$$

If  $\Gamma = 2$ , then the  $s$ -integral diverges logarithmically, and (5.10) is thus bounded by

$$\frac{C \log(t+2)}{t+1} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma} \leq \frac{C'}{(t+1)^{\frac{1}{2} + \frac{1}{4}(p^*-3)}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma} \tag{5.12}$$

since  $3 < p^* \leq 4$ . (Remark. The logarithmic divergence appearing here is the only reason why a two-step lemma and proof are needed. Could we obtain for  $n = 0$  an estimate with  $(t+1)^{-\frac{1}{2} - \frac{1}{4}(p^*-3)}$  instead of  $(t+1)^{-\frac{1}{2} - \frac{1}{4}(p^*-3)}$ , the case  $n = 1$  would not be needed.)

If  $\Gamma > 2$ , then the  $s$ -integral is bounded and we directly get the bound

$$\frac{C}{t+1} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma}. \tag{5.13}$$

Integrating (5.8) as  $\int_{t/2}^t ds$  is different: we have  $t - s \leq s + 1$ , hence, we obtain by (5.6) the bound

$$\begin{aligned} & \int_{t/2}^t \int_0^\infty \frac{C}{(t-s)} e^{-\frac{1}{8}(x-y)^2/(t-s)} \frac{1}{(s+1)^{\Gamma-1/2}} \left(1 + \frac{|y|}{\sqrt{t+1}}\right)^{-p\gamma+1} dy ds \\ & \leq C \int_{t/2}^t \frac{1}{(s+1)^{\Gamma-1/2}} \frac{1}{t-s} \\ & \quad \cdot \left( \int_{|x-y| \geq |x|/2} e^{-\frac{1}{16}(x-y)^2/(t-s) - \frac{1}{16}(x-y)^2/(t-s)} \left(1 + \frac{|y|}{\sqrt{t+1}}\right)^{-p\gamma+1} dy \right. \\ & \quad \left. + \int_{|x-y| \leq |x|/2} e^{-\frac{1}{8}(x-y)^2/(t-s)} \left(1 + \frac{|y|}{\sqrt{t+1}}\right)^{-p\gamma+1} dy \right) ds \\ & \leq C' \int_{t/2}^t \frac{1}{(s+1)^{\Gamma-1/2}} \frac{1}{t-s} \cdot \left( e^{-\frac{1}{64}x^2/t} \int_{|x-y| \geq |x|/2} e^{-\frac{1}{16}(x-y)^2/(t-s)} dy \right. \\ & \quad \left. + \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-p\gamma+1} \int_{|x-y| \leq |x|/2} e^{-\frac{1}{8}(x-y)^2/(t-s)} dy \right) ds \end{aligned}$$

$$\begin{aligned}
 &\leq C'' \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-p\gamma+1} \int_{t/2}^t \frac{1}{(s+1)^{\Gamma-1/2}} \frac{1}{t-s} \int_{-\infty}^{\infty} e^{-\frac{1}{16}(x-y)^2/(t-s)} dy ds \\
 &\leq C''' \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-p\gamma+1} \int_{t/2}^t \frac{1}{(s+1)^{\Gamma-1/2}} \frac{1}{\sqrt{t-s}} ds \\
 &\leq \frac{C''''}{(t+1)^{\Gamma-1}} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-\gamma}.
 \end{aligned} \tag{5.14}$$

Similar representations and bounds apply to the terms with  $\mathcal{G}$ . So we are left with

$$\int_0^t \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \mathcal{F}(0,s) ds, \tag{5.15}$$

and the similar integral for  $\mathcal{G}$ , which contain some explicit terms of order  $(t+1)^{-1/2}$  only, as we shall see.

Integrating again by parts yields, for some constants  $B_1, B_2, B_3 \in \mathbb{R}$ ,

$$\begin{aligned}
 &\int_0^{t/2} \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \mathcal{F}(0,s) ds \\
 &= - \left[ \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \int_s^{\infty} \mathcal{F}(0,\sigma) d\sigma \right]_{s=0}^{t/2} \\
 &\quad + \int_0^{t/2} \left( \frac{B_1}{(t-s)^{3/2}} + \frac{B_2 x^2}{(t-s)^{5/2}} \right) e^{-\frac{1}{4}x^2/(t-s)} \int_s^{\infty} \mathcal{F}(0,\sigma) d\sigma ds \\
 &=: \frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/t} \int_0^{\infty} \mathcal{F}(0,\sigma) d\sigma + \frac{B_3}{\sqrt{t}} e^{-\frac{1}{2}x^2/t} \int_{t/2}^{\infty} \mathcal{F}(0,\sigma) d\sigma + \mathcal{Y}(x,t).
 \end{aligned} \tag{5.16}$$

From (5.6) we obtain

$$\left| \int_s^{\infty} \mathcal{F}(0,\sigma) d\sigma \right| \leq C(s+1)^{-\Gamma+3/2};$$

hence, recalling that  $3/2 < \Gamma$ , we get the bound

$$\begin{aligned}
 |\mathcal{Y}(x,t)| &\leq C \int_0^{t/2} \left( \frac{1}{(t-s)^{3/2}} + \frac{1x^2}{(t-s)^{5/2}} \right) e^{-\frac{1}{4}x^2/(t-s)} (s+1)^{-\Gamma+3/2} ds \\
 &\leq C' \int_0^{t/2} \frac{1}{(t-s)^{3/2}} \frac{1}{(s+1)^{\Gamma-3/2}} e^{-\frac{1}{8}x^2/(t-s)} ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq C'' \frac{1}{(t+1)^{3/2}} \int_0^{t/2} \frac{1}{(s+1)^{\Gamma-3/2}} e^{-\frac{1}{8}x^2/t} ds \\
 &\leq C''' \frac{1}{(t+1)^{3/2}} \left( \frac{1}{(t+1)^{\Gamma-5/2}} + 1 \right) e^{-\frac{1}{8}x^2/t} \\
 &\leq C'''' \left( \frac{1}{(t+1)^{3/2}} + \frac{1}{(t+1)^{\Gamma-1}} \right) \left( 1 + \frac{|x|}{\sqrt{t+1}} \right)^{-\gamma}. \tag{5.17}
 \end{aligned}$$

(In case  $\Gamma = 5/2$  there appears an additional factor  $\log(t+1)$ , but even in this case the obtained bound is much better than  $(t+1)^{-\frac{1}{2}-\frac{1}{2}(p^*-3)}$ , which would suffice.)

The second term of the last line of (5.16) easily leads to a smaller bound because of (5.6) and the fact that  $\Gamma > 3/2$ .

Moreover, returning to (5.15),

$$\left| \int_{t/2}^t \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \mathcal{F}(0,s) ds \right| \tag{5.18}$$

is easily seen to be at most  $C(t+1)^{-\Gamma+1} e^{-cx^2/(t+1)}$ , since, in (5.6),  $(s+1)^{-\Gamma+1/2}$  can be replaced by  $C(t+1)^{-\Gamma+1/2}$ .

Thus, we have shown that the integral

$$\int_0^t ds \text{ of (5.7), i.e. } \sqrt{4\pi} \int_0^t e^{(t-s)\partial_x^2} \mathcal{M}(\cdot,s) ds,$$

is

$$\frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/t} \int_0^\infty (\mathcal{F}(0,s) + \mathcal{G}(0,s)) ds + \mathcal{W}(x,t), \tag{5.19}$$

where

$$\|\mathcal{W}(\cdot,t)\|_t \leq (t+1)^{-\frac{1}{2}-\frac{n+1}{4}(p^*-3)}.$$

Recall that

$$-\Gamma + 1 < -\frac{1}{2} - \frac{n+1}{4}(p^*-3).$$

B) To complete the proof of the lemma it is enough, in view of (5.19), to show that for  $t \geq 1$

$$\begin{aligned}
 &\left\| \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}x^2/t} \int_0^\infty (\mathcal{F}(0,s) + \mathcal{G}(0,s)) ds \right. \\
 &\quad \left. - \varphi(x,t) \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y,s) dy ds \right\|_t \leq \frac{C}{t+1}. \tag{5.20}
 \end{aligned}$$



But we have

$$\int_0^\infty (\mathcal{F}(0,s) + \mathcal{G}(0,s)) ds = \int_0^\infty \int_{-\infty}^\infty \mathcal{M}(y,s) dy ds, \tag{5.21}$$

by (5.4) and (5.5), hence (5.20) follows from

$$\begin{aligned} & \left| \frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/t} - \sqrt{4\pi} \varphi(x,t) \right| \\ & \leq \left| \frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/t} - \frac{1}{\sqrt{t+1}} e^{-\frac{1}{4}x^2/(t+1)} \right| \\ & = \left| \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+1}} \right| e^{-\frac{1}{4}x^2/t} + \frac{1}{\sqrt{t}} \left| e^{-\frac{1}{4}x^2/t} - e^{-\frac{1}{4}x^2/(t+1)} \right| \\ & \leq C \frac{1}{t+1} e^{-\frac{1}{4}x^2/(t+1)} + \frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/(t+1)} \left( 1 - e^{-\frac{1}{4}x^2 t^{-1} (t+1)^{-1}} \right) \\ & \leq C' \frac{1}{t+1} e^{-\frac{1}{4}x^2/(t+1)} + \frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/(t+1)} \sum_{m=1}^\infty \frac{1}{m!} \left( \frac{x^2}{4t(t+1)} \right)^m \\ & \leq \frac{C''}{t+1} e^{-\frac{1}{4}x^2/(t+1)} + \frac{C''}{t} e^{-cx^2/(t+1)}, \end{aligned} \tag{5.22}$$

to extract a factor  $t^{-1/2}$  from the sum on the second but last row and to still keep the remaining sum not too large, one needs to assume that  $t$  is not too small, say  $t \geq 2$ . Of course, this is possible.  $\square$

PROOF OF (1.10). Let  $u = u(x,t)$  be a function such that  $\|u(\cdot,t)\|_t \leq 1$  for some  $\gamma > 1$ . Then

$$|u(x,t)| \leq (1 + |x|(t+1)^{-1/2})^{-\gamma} \leq (1 + |x|(t+1)^{-1/2})^{-1},$$

and

$$\begin{aligned} \int_{-\infty}^\infty |u(x,t)|^q dx & \leq \int_{-\infty}^\infty \left( 1 + \frac{|x|}{\sqrt{t+1}} \right)^{-q} dx \\ & \leq \int_{|x| \leq \sqrt{t+1}} 1 dx + \int_{|x| \geq \sqrt{t+1}} \left( \frac{|x|}{\sqrt{t+1}} \right)^{-q} dx \\ & \leq 2\sqrt{t+1} + (t+1)^{q/2} \int_{|x| \geq \sqrt{t+1}} |x|^{-q} dx = C\sqrt{t+1}, \end{aligned}$$

hence,

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq Ct^{1/(2q)}$$

and the claim follows.

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