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LARGE TIME BEHAVIOR OF SOLUTIONS FOR THE GENERALIZED KADOMTSEV–PETVIASHVILI EQUATION

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Abstract. We consider the Cauchy problem for the generalized Kadomtsev-Petviashvili (KP) equation

$$\begin{cases} u_t + u_{xxx} + \sigma \partial_x^{-1} u_{yy} = -(u^\rho)_x, & (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \mathbb{R}^2, \end{cases}$$

where $\sigma = 1$ or $\sigma = -1$, $\partial_x^{-1} = \int_{-\infty}^{x} dx'$. Hayashi-Naumkin-Saut [2] have shown asymptotics of solutions for KP equation when $\rho \geqslant 3$ and the initial data is sufficiently small and regular. Our aim is to fill the gap of the proof of $\mathbf{L}^\infty$ time decay of small solutions obtained in [2] and improve their result on the regularity of the data.

1. Introduction

We study the Cauchy problem for the generalized Kadomtsev-Petviashvili equation

$$\begin{cases} u_t + u_{xxx} + \sigma \partial_x^{-1} u_{yy} = -(u^\rho)_x, & (x,y) \in \mathbb{R}^2, t \in \mathbb{R}, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\sigma = \pm 1$ and $\partial_x^{-1} = \int_{-\infty}^{x} dx'$. When $\rho = 2$ and $\sigma = -1$, (1.1) is known as the KPI equation, while $\rho = 2$ and $\sigma = 1$ is known as KPII equation. The KP equation models the propagation along the x-axis of nonlinear dispersive long waves on the surface of a fluid, when the variation along the y-axis proceeds slowly [4]. The case $\rho = 3, \sigma = -1$ has been found in the modeling of sound waves in antiferromagnetics [10].

Notation and Function Spaces. We denote the Lebesgue space as

$$\mathbf{L}^p = \left\{ \phi \in \mathbf{S}'; \|\phi\|_p = \left(\int\int |\phi(x,y)|^p \, dxdy \right)^{\frac{1}{p}} < \infty \right\},$$

if $1 \leqslant p < \infty$ and

$$\mathbf{L}^\infty = \left\{ \phi \in \mathbf{S}'; \|\phi\|_\infty = \sup\left\{ |\phi(x,y)|; (x,y) \in \mathbb{R}^2 \right\} < \infty \right\}$$

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if $p=\infty$. For simplicity we put $\|\phi\|=\|\phi\|_2$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s}=\left\{\varphi\in\mathbf{S}';\|\phi\|_{m,s,p}=\left\|\left(1+x^2+y^2\right)^{\frac{s}{2}}\left(1-\partial_x^2-\partial_y^2\right)^{\frac{m}{2}}\phi\right\|_p<\infty\right\},$$

$m,s\in\mathbb{R},1\leqslant p\leqslant\infty$. We also use the following notations $\mathbf{H}^{m,s}=\mathbf{H}_2^{m,s},\|\cdot\|_{m,s}=\|\cdot\|_{m,s,2}$. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C. By $\mathscr{F}\phi$ we denote the Fourier transform of the function ϕ. The inverse Fourier transform of ϕ is given by $\mathscr{F}^{-1}\phi$. We denote by $\mathscr{U}(t)$ the free KP evolution group defined on the space of functions $\phi\in\mathbf{L}^2$ such that $\partial_x^{-1}\phi\in\mathbf{L}^2$, where $\mathscr{F}\partial_x^{-1}\phi=\frac{\mathscr{F}\phi(\xi,\eta)}{i\xi}$, by the formula

$$\begin{aligned}\mathscr{U}(t)\phi(t)&=\mathscr{F}^{-1}\left(e^{it\left(\xi^3-\sigma\eta^2/\xi\right)}\mathscr{F}(t,\xi,\eta)\right)\\&=\int\int G\left(t,x-x',y-y'\right)\phi\left(t,x',y'\right)dx'dy',\end{aligned}$$

where $G(t,x,y)=\frac{1}{2\pi}\int\int e^{ix\xi+iy\eta+it\xi^3-it\sigma\eta^2/\xi}d\xi d\eta$. Integral equation associated with (1.1) is defined by

$$u(t)=\mathscr{U}(t)u_0-\int_0^t\mathscr{U}(t-s)\left(u^\rho\right)_x ds. \tag{1.2}$$

We introduce the following operators $\mathscr{J}_x=\mathscr{U}(t)x\mathscr{U}(-t)$ and $\mathscr{J}_y=\mathscr{U}(t)y\mathscr{U}(-t)$ which are used in paper [2]. These operators are useful to get the $\mathbf{L}^\infty$ time decay estimates solutions to (1.1). We use the operator $\mathscr{I}=x+2\partial_x^{-1}\partial_y y+3t\partial_x^{-1}\partial_t$ to get the estimate of $\mathscr{J}_x u_x$ since the operator $\mathscr{J}_x$ does not work well for the nonlinear term of (1.1). The operators $\mathscr{J}_x,\mathscr{J}_y,\mathscr{I}$ have following representations

$$\begin{aligned}\mathscr{J}_x&=\mathscr{U}(t)x\mathscr{U}(-t)=x-t(3\partial_x^2-\sigma\partial_x^{-2}\partial_y^2),\\\mathscr{J}_y&=\mathscr{U}(t)y\mathscr{U}(-t)=y-2\sigma t\partial_x^{-1}\partial_y,\end{aligned}$$

and

$$\mathscr{I}=x+2\partial_x^{-1}\partial_y y+3t\partial_x^{-1}\partial_t=\mathscr{J}_x+2\partial_x^{-1}\partial_y\mathscr{J}_y+3t\partial_x^{-1}\mathscr{L},$$

where $\mathscr{L}=\partial_t+\partial_x^3+\sigma\partial_x^{-1}\partial_y^2$ is the linear part of (1.1).

Hayashi-Naumkin-Saut [2] showed asymptotics of solutions for the generalized KP equation when $\rho\geqslant 3$ and the initial data is sufficiently small and regular. However their proof of $\mathbf{L}^\infty$ time decay of $\partial_x u$ needs a slight modification on the estimate of $\left\|\mathscr{J}_y u(t)\right\|_{3,0}$ (see (3.9) in [2]), where u is a solution of (1.1). More precisely, they used the inequality

$$\left\|u^{\rho-2}u_{xxx}\mathscr{J}_y u_x\right\|\leqslant C\|u\|_{2,0,q}^{\rho-1}\left(\|u\|_{3,0}+\left\|\mathscr{J}_y u_x\right\|_{2,0}\right)$$

with $2+\frac{2}{\rho-2}<q\leqslant 2\rho$. It seems that the estimate is not correct. Instead of the above inequality, we use the inequality

$$\left\|u^{\rho-2}u_{xxx}\mathscr{J}_y u_x\right\|\leqslant C\|u\|_\infty^{\rho-2}\left\|\mathscr{J}_y u_x\right\|_{1,0,4}\|u\|_{3,0}$$

which requires us the estimate of $\left\|\mathscr{J}_y u_x\right\|_{1,0,4}$. Therefore our main point in this paper is to prove a-priori estimate of $\left\|\mathscr{J}_y u_x\right\|_{1,0,4}$. We also improve their result on the regularity of the data which means that it is sufficient to get a-priori estimate of $\left\|\mathscr{J}_y u_x\right\|_4$ to get the desired result.

We defined the function spaces

$$\mathbf{X}_T = \left\{\phi \in C\left([-T,T];\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_T} < \infty\right\},$$

with

$$\|\phi\|_{\mathbf{X}_T} = \sup_{-T\leqslant t\leqslant T} \left\|\partial_x^{-1}\phi(t)\right\|_{4,0} + (1+|t|)^{1-\frac{2}{p}} \|\phi(t)\|_{1,0,p} + (1+|t|)^{\frac{1}{2}} \|\phi(t)\|_{2,0,4}$$

and

$$\mathbf{Y} = \{\phi \in \mathbf{X}_\infty; \|\phi\|_{\mathbf{Y}} < \infty\}$$

with

$$\|\phi\|_{\mathbf{Y}} = \|\phi\|_{\mathbf{X}_\infty} + \sup_{t\in\mathbb{R}} \left(\|\mathscr{J}_x\partial_x\phi(t)\| + \left\|\mathscr{J}_y\phi(t)\right\|_{2,0} + \left\|\mathscr{J}_y^2\partial_x\phi(t)\right\| \right).$$

In the next theorem we state a global existence of solutions for (1.1). The method of the proof is based on $\mathbf{L}^p - \mathbf{L}^q$ time decay estimate developed in [5], [8], [9].

THEOREM 1.1. *Let $\rho \geqslant 3$ be integer and the initial data $u_0 \in \mathbf{H}^{1,0}_{p'}, \partial_x^{-1}u_0 \in \mathbf{H}^{4,0}$ where $4 < p < \infty$ and $\frac{1}{p}+\frac{1}{p'} = 1$. Assume that $\left\|\partial_x^{-1}u_0\right\|_{4,0} + \|u_0\|_{1,0,p'} = \varepsilon$ and ε is sufficiently small. Then there exists a unique global solution $u \in \mathbf{X}_\infty$ of the Cauchy problem (1.1) satisfying the estimates*

$$\|u(t)\|_{1,0,p} \leqslant \sqrt{\varepsilon}\,(1+|t|)^{-1+\frac{2}{p}}, \left\|\partial_x^{-1}u(t)\right\|_{4,0} \leqslant \sqrt{\varepsilon}, \tag{1.3}$$

for all $t \in \mathbb{R}$.

REMARK 1.1. We may change the assumption $\rho \geqslant 3$ and integer to the assumption $\rho = 3$ or $\rho \geqslant 4$. In fact, our proof of Theorem 1.1 is valid when $\rho = 3$ or $\rho \geqslant 4$.

Next theorem is our main result. This result is an improvement of Theorem 1.2 in paper [2] since the assumption of our Theorem 1.2 in this paper is weaker than the assumption of paper [2].

THEOREM 1.2. *Let u be the solution of the Cauchy problem (1.1) obtained by Theorem 1.1. Assume also that $x\partial_x^i u_0 \in \mathbf{L}^2$, $yu_0 \in \mathbf{H}^{2,0}$ and $y^2\partial_x^i u_0 \in \mathbf{L}^2$ where $i = 0, 1$. Then $u \in \mathbf{Y}$ which implies the estimates*

$$\begin{aligned} \|u(t)\|_\infty &\leqslant C(1+|t|)^{-1}(\log(2+|t|))^\kappa, \\ \|\partial_x u(t)\|_\infty &\leqslant C(1+|t|)^{-1} \end{aligned} \tag{1.4}$$

for all $t \in \mathbb{R}$ where $\kappa = 1$ if $\rho = 3$ and $\kappa = 0$ if $\rho > 3$.

REMARK 1.2. The time decay estimate (1.3) and the second estimate of (1.4) have same decay rate of linear estimate of Lemma 2.2. While the first estimate of (1.4) differs from the linear estimate of Lemma 2.2 if $\rho = 3$.

We organize our paper as follows. In section 2 we state three lemmas which will be needed to prove Theorem 1.1 and Theorem 1.2. In section 3 we will prove our theorems. In what follows, for simplicity we only consider positive time t.

2. Preliminaries

LEMMA 2.1. *Let q,r be any numbers satisfying $1 \leqslant q,r \leqslant \infty$, and let j,m be any numbers satisfying $0 \leqslant j < m$. Then the following estimate is true:*

$$\left\|(-\partial_x^2-\partial_y^2)^{\frac{j}{2}}\varphi\right\|_p \leqslant C\left\|(-\partial_x^2-\partial_y^2)^{\frac{m}{2}}\varphi\right\|_r^{\alpha}\|\varphi\|_q^{1-\alpha},$$

where C is a constant depending only on m,j,r,α, $\frac{1}{p} = \frac{j}{2} + \alpha(\frac{1}{r} - \frac{m}{2}) + \frac{1-\alpha}{q}$ and the parameter α is arbitrary in the interval $\frac{j}{m} \leqslant \alpha \leqslant 1$ with the following exception: if $m - j - \frac{2}{r}$ is a nonnegative integer, then $\frac{j}{m} \leqslant \alpha < 1$.

For the proof of the lemma, see Friedman [1].

In the next lemma we give linear estimates for the linear part of the KP equation which was shown in [6].

LEMMA 2.2. *Let $\phi \in \mathbf{L}^p \cap \mathbf{L}^2, 1 \leqslant p \leqslant 2$ and $\partial_x^{-1}\phi \in \mathbf{L}^2$. Then*

$$\|\mathscr{U}(t)\phi\|_q \leqslant C|t|^{-1+\frac{2}{q}}\|\phi\|_p, \tag{2.1}$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

Proof of lemma 2.2. In [6], it was shown that

$$|G(t,x,y)| \leqslant C|t|^{-1},$$

for all $t \in \mathbb{R}$ and uniformly respect to $(x,y) \in \mathbb{R}^2$, therefore we obtain

$$\|\mathscr{U}(t)\phi\|_\infty \leqslant \left\|\int\int G\left(t,x-x',y-y'\right)\phi\left(x',y'\right)dx'dy'\right\|_\infty \leqslant C|t|^{-1}\|\phi\|_1.$$

$\mathscr{U}(t)$ is unitary operator in $\mathbf{L}^2$, namely $\|\mathscr{U}(t)\phi\| = \|\phi\|$ under the assumption on the data, then applying Riesz-Thorin theorem, we obtain the desired result (2.1).

In the next lemma we state the commutator relations which are our essential tool to prove Theorem 1.2. This lemma can be found in [2].

LEMMA 2.3. *The following commutator relations are valid*

$$[\mathscr{J}_x, \partial_y] = [\mathscr{J}_y, \partial_x] = 0,$$

$$[\mathscr{J}_x, \partial_x] = [\mathscr{J}_y, \partial_y] = -1,$$
$$[\mathscr{I}, \partial_x] = -1, \quad [\mathscr{I}, \partial_y] = -2\partial_x^{-1}\partial_y \tag{2.2}$$

and

$$[\mathscr{L}, \mathscr{J}_x] = [\mathscr{L}, \mathscr{J}_y] = 0, \quad [\mathscr{L}, \mathscr{I}] = 3\partial_x^{-1}\mathscr{L}. \tag{2.3}$$

3. Proof of Theorems

To prove Theorem 1.1 we use the following local existence theorem.

THEOREM 3.1. *Let $\rho \geqslant 3$ be an integer and the initial data $u_0 \in \mathbf{H}^{1,0}_{p'}, \partial_x^{-1}u_0 \in \mathbf{H}^{4,0}$, and $\left\|\partial_x^{-1}u_0\right\|_{4,0} + \|u_0\|_{1,0,p'} = \varepsilon$ where ε is sufficiently small, $4 < p < \infty$. Then there exist a finite interval $[0,T]$ with $T > 1$ and a unique solution u of (1.1) such that*

$$\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}.$$

For the proof of Theorem 3.1, see, e.g., [7], [3] and references cited therein.

Proof of Theorem 1.1. We prove the result by a contradiction argument. We assume the there exists a time T such that $\|u\|_{\mathbf{X}_T} \leqslant \sqrt{\varepsilon}$. By (1.2) we have for $0 \leqslant t \leqslant T$

$$\|u(t)\|_{1,0,p} \leqslant C(1+t)^{-1+\frac{2}{p}}\left(\|u_0\|_{1,0,p'} + \|u_0\|_{3,0}\right) + C\int_0^t (t-s)^{-1+\frac{2}{p}}\left\|u^{\rho-1}u_x\right\|_{1,0,p'} ds. \tag{3.1}$$

By Hölder's inequality with $\rho \leqslant \frac{3}{4}p, \rho+1 \leqslant p$

$$\begin{aligned}\left\|u^{\rho-1}u_{xx}\right\|_{p'} + \left\|u^{\rho-2}u_x^2\right\|_{p'} &\leqslant \|u\|^{\rho-1}_{\frac{(\rho-1)4p}{3p-4}}\|u_{xx}\|_4 + \|u\|^{\rho-2}_{\frac{(\rho-2)p}{p-3}}\|u_x\|_p^2 \\ &\leqslant C\varepsilon^{\frac{\rho}{2}}(1+s)^{-\left(\rho-2+\frac{2}{p}\right)}.\end{aligned}$$

If $\rho \geqslant \frac{3}{4}p, \rho+1 \geqslant p$, then

$$\begin{aligned}\left\|u^{\rho-1}u_{xx}\right\|_{p'} + \left\|u^{\rho-2}u_x^2\right\|_{p'} &\leqslant C\left(\|u\|^{\rho-1}_{1,0,p}\|u_{xx}\|_4 + \|u\|^{\rho-2}_{1,0,p}\|u_x\|_p^2\right) \\ &\leqslant C\varepsilon^{\frac{\rho}{2}}\left((1+s)^{-\left(1-\frac{2}{p}\right)(\rho-1)-\frac{1}{2}} + (1+s)^{-\left(1-\frac{2}{p}\right)\rho}\right) \\ &\leqslant C\varepsilon^{\frac{\rho}{2}}(1+s)^{-\left(1-\frac{2}{p}\right)\rho+\left(-\frac{1}{2}+\frac{2}{p}\right)}.\end{aligned}$$

We apply these estimates to the last term of the right hand side of (3.1) to find that

$$\|u\|_{1,0,p} \leqslant C\varepsilon(1+t)^{-1+\frac{2}{p}} + C\varepsilon^{\frac{\rho}{2}}\int_0^t (t-s)^{-1+\frac{2}{p}}(1+s)^{-\left(\rho-2+\frac{2}{p}\right)} ds$$

$$\begin{aligned}
&\leqslant C\varepsilon(1+t)^{-1+\frac{2}{p}} + C\varepsilon^{\frac{\rho}{2}}(1+t)^{-(\rho-2)} \\
&\leqslant C\varepsilon(1+t)^{-1+\frac{2}{p}}. \qquad (3.2)
\end{aligned}$$

Next we consider the a-priori estimate in the norm $\|\cdot\|_{2,0,4}$. We have by Lemma 2.2

$$\|u\|_{2,0,4} \leqslant C(1+t)^{-\frac{1}{2}}\left(\|u_0\|_{2,0,\frac{4}{3}} + \|u_0\|_{3,0}\right) + C\int_0^t (t-s)^{-\frac{1}{2}}\left\|u^{\rho-1}u_x\right\|_{2,0,\frac{4}{3}} ds.$$

By Hölder's inequality

$$\left\|u^{\rho-3}u_x^3\right\|_{\frac{4}{3}} \leqslant \|u\|_{1,0,4}^{\rho} \leqslant C\varepsilon^{\frac{\rho}{2}}(1+s)^{-\frac{\rho}{2}}$$

and

$$\begin{aligned}
\left\|u^{\rho-2}u_xu_{xx}\right\|_{\frac{4}{3}} + \left\|u^{\rho-1}u_{xxx}\right\|_{\frac{4}{3}} &\leqslant C\|u\|_{2,0,4}^{\rho} + C\|u\|_{4(\rho-1)}^{\rho-1}\|u\|_{3,0} \\
&\leqslant C\varepsilon^{\frac{\rho}{2}}(1+s)^{-\frac{\rho}{2}}.
\end{aligned}$$

Hence

$$\|u\|_{2,0,4} \leqslant C\varepsilon(1+t)^{-\frac{1}{2}} + C\varepsilon^{\frac{\rho}{2}}\int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{2}} ds \leqslant C\varepsilon(1+t)^{-\frac{1}{2}} \qquad (3.3)$$

for $0 \leqslant t \leqslant T$. By the definition of $\mathbf{X}_T$ and Lemma 2.1 we have

$$\|u\|_{1,0,\infty} \leqslant \|u\|_{1,0,p}^{\frac{p}{p+4}}\|u\|_{2,0,4}^{\frac{4}{p+4}} \leqslant C\sqrt{\varepsilon}(1+t)^{-\frac{p}{p+4}}. \qquad (3.4)$$

From the energy estimate we get for $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\frac{d}{dt}\|u_{xxx}\| &\leqslant C\Big(\|u\|_{1,0,\infty}^{\rho-1}\|u_x\| + \|u\|_{1,0,\infty}^{\rho-1}\|u_{xx}\| \\
&\qquad + \|u\|_{1,0,\infty}^{\rho-2}\|u_{xx}\|_4^2 + C\|u\|_{1,0,\infty}^{\rho-1}\|u_{xxx}\|\Big) \\
&\leqslant C\varepsilon^{\frac{\rho}{2}}\left((1+t)^{-(\rho-1)\frac{p}{p+4}} + (1+t)^{-(\rho-2)\frac{p}{p+4}-1}\right) \\
&\leqslant C\varepsilon^{\frac{\rho}{2}}(1+t)^{-(\rho-1)\frac{p}{p+4}}, \qquad (3.5)
\end{aligned}$$

where we have used the estimates (3.4) and (3.3). In the same way as in the proof of (3.5) we get

$$\frac{d}{dt}\left\|\partial_x^{-1}u\right\|_{4,0} \leqslant C\varepsilon^{\frac{\rho}{2}}(1+t)^{-(\rho-1)\frac{p}{p+4}}.$$

Therefore, we obtain for $0 \leqslant t \leqslant T$, $p > 4$,

$$\left\|\partial_x^{-1}u\right\|_{4,0} \leqslant C\varepsilon. \qquad (3.6)$$

From (3.2), (3.3) and (3.6) we have $\|u\|_{\mathbf{X}_T} \leqslant C\varepsilon$. We take ε satisfying $C\varepsilon < \sqrt{\varepsilon}$. Then we have the desired contradiction.

Proof of Theorem 1.2. By Theorem 1.1 we have by (3.4),

$$\|u(t)\|_{1,0,\infty} \leqslant C(1+t)^{-\frac{p}{p+4}} \tag{3.7}$$

for $0 \leqslant t \leqslant \infty$. We prove the following estimates:

$$\left\| \mathscr{J}_y u(t) \right\|_{2,0} \leqslant C, \left\| \mathscr{J}_y^2 u_x(t) \right\| \leqslant C, \| \mathscr{J}_x u_x(t) \| \leqslant C. \tag{3.8}$$

Multiplying (1.1) by $\mathscr{J}_y$, we get

$$\mathscr{L} \mathscr{J}_y u = -\rho u^{\rho-1} \mathscr{J}_y u_x. \tag{3.9}$$

Differentiating (3.9) twice in space, multiplying the resulting equation by $\mathscr{J}_y u_{xx}$ and integrating by parts in space to get

$$\begin{aligned} \frac{d}{dt} \left\| \mathscr{J}_y u_{xx} \right\| &\leqslant C \left(\left\| u^{\rho-3} u_x^2 \mathscr{J}_y u_x \right\| + \left\| u^{\rho-2} u_{xx} \mathscr{J}_y u_x \right\| + \left\| u^{\rho-2} u_x \mathscr{J}_y u_{xx} \right\| \right) \\ &\leqslant C(1+t)^{-(\rho-1)\frac{p}{p+4}} \left\| \mathscr{J}_y u_x \right\| + C(1+t)^{-\frac{1}{2}-(\rho-2)\frac{p}{p+4}} \left\| \mathscr{J}_y u_x \right\|_{1,0} \\ &\quad + C(1+t)^{-(\rho-1)\frac{p}{p+4}} \left\| \mathscr{J}_y u_{xx} \right\| \end{aligned}$$

by (3.7) and Lemma 2.2. Similarly, we have

$$\begin{aligned} \frac{d}{dt} \left(\left\| \mathscr{J}_y u_{xy} \right\| + \left\| \mathscr{J}_y u_{yy} \right\| \right) &\leqslant C(1+t)^{-(\rho-1)\frac{p}{p+4}} \|u\|_{3,0} \\ &\quad + C(1+t)^{-\frac{1}{2}-(\rho-2)\frac{p}{p+4}} \left\| \mathscr{J}_y u \right\|_{2,0}. \end{aligned}$$

We apply Gronwall's inequality to the above inequalities to get

$$\left\| \mathscr{J}_y u(t) \right\|_{2,0} \leqslant C \|y u_0\|_{2,0} \leqslant C. \tag{3.10}$$

So we obtain the first estimate of (3.8). Now, we consider the operator $\mathscr{I}$ to get the third estimate of (3.8) since the operator $\mathscr{J}_x$ does not act as the first order differential operator for the nonlinear term of (1.1). Multiplying both sides of (1.1) by $\mathscr{I}$, we get by Lemma 2.3,

$$\mathscr{L} \mathscr{I} u = -\rho u^{\rho-1} (\mathscr{I} u)_x + (3\rho-5) u^{\rho}. \tag{3.11}$$

Differentiating (3.11) in x and integrating by parts, we have

$$\begin{aligned} \frac{d}{dt} \|(\mathscr{I} u)_x\| &\leqslant C \left\| u^{\rho-2} u_x (\mathscr{I} u)_x \right\| + C \left\| u^{\rho-1} u_x \right\| \\ &\leqslant C(1+t)^{-(\rho-1)\frac{p}{p+4}} + C(1+t)^{-(\rho-1)\frac{p}{p+4}} \|(\mathscr{I} u)_x\| \end{aligned} \tag{3.12}$$

by (3.7). By Gronwall's inequality, we get

$$\| (\mathscr{I} u)_x (t) \| \leqslant C \left(\|u_0\| + \|x \partial_x u_0\| + \left\| y \partial_y u_0 \right\| \right) \leqslant C.$$

Therefore we obtain

$$\|\mathscr{I} u_x(t)\| \leqslant \|u\| + \|(\mathscr{I} u)_x\| \leqslant C \tag{3.13}$$

by Lemma 2.3. From (3.13) and the identity $\mathscr{I} = \mathscr{J}_x + 2\partial_x^{-1}\partial_y \mathscr{J}_y + 3t\partial_x^{-1}\mathscr{L}$ it follows that

$$\|\mathscr{J}_x u_x(t)\| \leqslant \|\mathscr{I} u_x\| + 2\left\|\partial_y \mathscr{J}_y u\right\| + 3t\|\mathscr{L} u\| \leqslant C. \tag{3.14}$$

So we obtain the third estimate of (3.8). Let us prove the second estimate of (3.8). To prove the estimate, we show the a-priori estimate of $\left\|\mathscr{J}_y u_x\right\|_4$. From the integral equation associated with (3.9), we have

$$\begin{aligned}
\left\|\mathscr{J}_y u_x\right\|_4 \leqslant{}& \|\mathscr{U}(t)(y\partial_x u_0)\|_4 \\
&+ \left\|\int_0^t \mathscr{U}(t-s)\left[\rho(\rho-1)u^{\rho-2}u_x \mathscr{J}_y u_x + \rho u^{\rho-1}\mathscr{J}_y u_{xx}\right] ds\right\|_4 \\
\leqslant{}& C(1+t)^{-\frac{1}{2}}\|y\partial_x u_0\|_{\frac{4}{3}} \\
&+ C\int_0^t (t-s)^{-\frac{1}{2}}\left(\left\|u^{\rho-2}u_x \mathscr{J}_y u_x\right\|_{\frac{4}{3}} + \left\|u^{\rho-1}\mathscr{J}_y u_{xx}\right\|_{\frac{4}{3}}\right) ds \\
\leqslant{}& C(1+t)^{-\frac{1}{2}} + C\int_0^t (t-s)^{-\frac{1}{2}}\|u\|_\infty^{\rho-2}\|u\|_{2,0,4}\left(\left\|\mathscr{J}_y u_x\right\| + \left\|\mathscr{J}_y u_{xx}\right\|\right) ds \\
\leqslant{}& C(1+t)^{-\frac{1}{2}} + C\int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-(\rho-2)\frac{p}{p+4}-\frac{1}{2}}\left\|\mathscr{J}_y u(t)\right\|_{2,0} ds \\
\leqslant{}& C(1+t)^{-\frac{1}{2}}
\end{aligned}$$

for $t > 1$. Since $\|\phi\|_4 \leqslant C\|\phi\|_{\frac{1}{2},0} \leqslant C\|\phi\|_{1,0}$ by Lemma 2.1, we have

$$\left\|\mathscr{J}_y u_x(t)\right\|_4 \leqslant \left\|\mathscr{J}_y u(t)\right\|_{2,0} \leqslant C$$

for $0 \leqslant t \leqslant 1$. Thus we get the estimate

$$\left\|\mathscr{J}_y u_x(t)\right\|_4 \leqslant C(1+t)^{-\frac{1}{2}} \tag{3.15}$$

for $0 \leqslant t \leqslant \infty$. We apply $\mathscr{J}_y^2\partial_x$ to (1.1) to get

$$\mathscr{L}\mathscr{J}_y^2 u_x = -\rho(\rho-1)u^{\rho-2}(\mathscr{J}_y u_x)^2 - \rho u^{\rho-1}\mathscr{J}_y^2 u_{xx}. \tag{3.16}$$

By the energy method, we have

$$\begin{aligned}
\frac{d}{dt}\left\|\mathscr{J}_y^2\partial_x u\right\| \leqslant{}& C\left\|u^{\rho-2}(\mathscr{J}_y u_x)^2\right\| + C\left\|u^{\rho-2}u_x\mathscr{J}_y^2 u_x\right\| \\
\leqslant{}& C\|u\|_\infty^{\rho-2}\left\|\mathscr{J}_y u_x\right\|_4^2 + C\|u\|_{1,0,\infty}^{\rho-1}\left\|\mathscr{J}_y^2 u_x\right\| \\
\leqslant{}& C(1+t)^{-(\rho-2)\frac{p}{p+4}-1} + C(1+t)^{-(\rho-1)\frac{p}{p+4}}\left\|\mathscr{J}_y^2 u_x\right\|
\end{aligned}$$

by (3.7) and (3.15). Applying Gronwall's inequality to the above, we get

$$\left\|\mathscr{J}_y^2 u_x(t)\right\| \leqslant C\left\|y^2\partial_x u_0\right\| \leqslant C. \tag{3.17}$$

Therefore we obtain the second estimate of (3.8). From (3.10), (3.14) and (3.17) it follows that $u \in \mathbf{Y}$. We now show the $\mathbf{L}^{\infty}$ time decay estimate of solutions of (1.1). We have by Lemma 2.2,

$$\begin{aligned}
\|u_x(t)\|_{\infty} &= \|\mathscr{U}(t)\mathscr{U}(-t)u_x\|_{\infty} \\
&\leqslant C(1+t)^{-1}\left(\|\mathscr{U}(-t)u_x\|_1 + \|u\|_{3,0}\right) \\
&\leqslant C(1+t)^{-1}\left(\left\|\left(1+|x|+y^2\right)\mathscr{U}(-t)u_x\right\| + \|u\|_{3,0}\right) \\
&\leqslant C(1+t)^{-1}\left(\|u_x\| + \|\mathscr{J}_x u_x\| + \left\|\mathscr{J}_y^2 u_x\right\| + \|u\|_{3,0}\right) \\
&\leqslant C(1+t)^{-1}
\end{aligned}$$

for $0 \leqslant t \leqslant \infty$. From (1.2), we have for $t > 1$,

$$\begin{aligned}
\|u(t)\|_{\infty} &\leqslant C(1+t)^{-1}\|u_0\|_1 \\
&\quad + C\int_0^{t-1}(t-s)^{-1}\left\|u^{\rho-1}u_x\right\|_1 ds + C\int_{t-1}^{t}\left\|u^{\rho-1}u_x\right\|_{2,0} ds \\
&\leqslant C(1+t)^{-1} + C\int_0^{t-1}(t-s)^{-1}(1+s)^{-1-(\rho-3)\frac{p}{p+4}} ds \\
&\quad + C\int_{t-1}^{t}(1+s)^{-(\rho-1)\frac{p}{p+4}} ds \\
&\leqslant C(1+t)^{-1}(\log(2+t))^{\kappa},
\end{aligned}$$

where $\kappa = 1$ when $\rho = 3$ and $\kappa = 0$ when $\rho > 3$. The case $0 \leqslant t \leqslant 1$ is following from $\|u\|_{\infty} \leqslant C\|u\|_{2,0} \leqslant C$, then we obtain the first estimate of (1.4).

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