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EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR A NONLINEAR THIRD–ORDER IMPULSIVE DYNAMIC SYSTEM ON TIME SCALES

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Abstract. In this paper, a nonlinear third-order impulsive dynamic system with boundary value conditions is studied on time scales. Some sufficient conditions for the existence of solutions are obtained by using Schauder's fixed point theorem.

1. Introduction

In recent years, much work has been done on the existence and uniqueness of solutions to boundary value problems for differential equations (see [12, 14, 21, 25]). Some theory and methods of nonlinear functional analysis, for example, the upper and lower solutions method and monotone iterative technique, fixed point theorem, the continuation method of topological degree, variational method and critical point theory, have been applied to those problems. At the same time, boundary value problems for impulsive differential equations and impulsive difference equations have received much attention [3-7, 9, 10, 15-19, 23] due to the potential applications in many fields such as physics, biology, engineering. Most boundary value problems for impulsive differential equations studied and used can be classified as either continuous or discrete. In order to unify the study of differential and difference equations, the theory of time scales was first introduced by Stefan Hilger [11] in 1990. Since then the theory of dynamic equations on time scales has become a new important branch (see, for example, [1, 2, 8, 20, 22, 24, 26]). In fact, both continuous and discrete systems are very important in implementing applications. Therefore, it is meaningful to study dynamic systems on time scales which can unify differential and difference systems.

In this paper, we will study the existence of solutions for the following boundary value problem for the nonlinear third-order impulsive dynamic system on time scales:

$$\begin{cases} -u^{\Delta^3}(t) = f(t,u(t),u^{\Delta}(t),u^{\Delta^2}(t)), \quad t \in [0,T]_{\mathbb{T}} \backslash \Omega, \\ \Delta u(t_k) = I_k,\ \Delta u^{\Delta}(t_k) = J_k,\ \Delta u^{\Delta^2}(t_k) = L_k, \quad k = 1,2,\ldots,m, \\ u(0) = \lambda u(\sigma(T)),\ u^{\Delta}(0) = \lambda u^{\Delta}(\sigma(T)),\ u^{\Delta^2}(0) = \lambda u^{\Delta^2}(\sigma(T)), \end{cases} \tag{1.1}$$

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where $\lambda \in \mathbb{R}$, $\lambda \neq 1$, $\Omega = \{t_1, t_2, \ldots, t_m\}$, and

$$\begin{aligned}
&\Delta u(t) = u(t^+) - u(t),\\
&\Delta u^{\Delta}(t) = u^{\Delta}(t^+) - u^{\Delta}(t),\\
&\Delta u^{\Delta^2}(t) = u^{\Delta^2}(t^+) - u^{\Delta^2}(t),\\
&I_k = I_k(u(t)),\ J_k = J_k(u(t), u^{\Delta}(t)),\ L_k = L_k(u(t), u^{\Delta}(t), u^{\Delta^2}(t)),\\
&k = 1, 2, \ldots, m.
\end{aligned}$$

Throughout this paper, we assume that:

(H_1) the function $f : [0,T]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the functions

$$I_k : \mathbb{R}^n \to \mathbb{R}^n,\ J_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \ \text{ and } \ L_k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ k = 1, 2, \ldots, m$$

are also continuous;

(H_2) $f(t_k^+, u, v, w) := \lim_{t \to t_k^+} f(t, u, v, w)$ for any $u, v, w \in \mathbb{R}^n$, $k = 1, 2, \cdots, m$;

(H_3) there exist nonnegative constants α, β_k, η_k and δ_k such that

$$\begin{aligned}
\alpha &= \limsup_{\|u\|+\|v\|+\|w\| \to \infty} \left(\max_{t \in [0,T]_{\mathbb{T}}} \frac{\|f(t,u,v,w)\|}{\|u\| + \|v\| + \|w\|} \right),\\
\beta_k &= \limsup_{\|u\| \to \infty} \left(\max_{t \in [0,T]_{\mathbb{T}}} \frac{\|I_k(u)\|}{\|u\|} \right),\\
\gamma_k &= \limsup_{\|u\|+\|v\| \to \infty} \left(\max_{t \in [0,T]_{\mathbb{T}}} \frac{\|J_k(u,v)\|}{\|u\| + \|v\|} \right),\\
\delta_k &= \limsup_{\|u\|+\|v\|+\|w\| \to \infty} \left(\max_{t \in [0,T]_{\mathbb{T}}} \frac{\|L_k(u,v,w)\|}{\|u\| + \|v\| + \|w\|} \right),
\end{aligned}$$

for any $u, v, w \in \mathbb{R}^n, k = 1, 2, \cdots, m$.

In order to define the solutions of system (1.1), we introduce and denote the Banach space $PC([0,T]_{\mathbb{T}}, \mathbb{R}^n)$ by

$$\begin{aligned}
PC([0,T]_{\mathbb{T}}, \mathbb{R}^n) = \big\{ u(t) \in C([0,T]_{\mathbb{T}} \backslash \Omega, \mathbb{R}^n) &: \text{the limits } u(t_k^-),\ u(t_k^+)\\
&\text{exist with } u(t_k^-) = u(t_k),\ k = 1, 2, \ldots, m \big\}
\end{aligned}$$

with the norm $\|u\|_0 = \sup_{t \in [0,T]_{\mathbb{T}}} \|u(t)\|$, where $\|\cdot\|$ is the usual Euclidean norm.

In a similar fashion to the above, we define

$$\begin{aligned}
PC^1([0,T]_{\mathbb{T}}, \mathbb{R}^n) = \big\{ u(t) \in PC([0,T]_{\mathbb{T}}, \mathbb{R}^n) &: u^{\Delta}(t) \in C([0,T]_{\mathbb{T}} \backslash \Omega, \mathbb{R}^n),\\
&\text{the limits } u^{\Delta}(t_k^-),\ u^{\Delta}(t_k^+) \text{ exist with}\\
&u^{\Delta}(t_k^-) = u^{\Delta}(t_k),\ k = 1, 2, \ldots, m \big\}
\end{aligned}$$

with the norm $\|u\|_1 = \max\{\|u\|_0, \|u^{\Delta}\|_0\}$, and

$$PC^2([0,T]_{\mathbb{T}},\mathbb{R}^n) = \big\{u(t) \in PC^1([0,T]_{\mathbb{T}},\mathbb{R}^n) : u^{\Delta^2}(t) \in C([0,T]_{\mathbb{T}}\backslash\Omega,\mathbb{R}^n), \text{ the limits } u^{\Delta^2}(t_k^-), u^{\Delta^2}(t_k^+) \text{ exist with } u^{\Delta^2}(t_k^-) = u^{\Delta^2}(t_k),\ k = 1,2,\ldots,m\big\}$$

with the norm $\|u\|_2 = \max\{\|u\|_1, \|u^{\Delta^2}\|_0\}$. $PC^1([0,T]_{\mathbb{T}},\mathbb{R}^n), PC^2([0,T]_{\mathbb{T}},\mathbb{R}^n)$ are Banach spaces with the norm $\|u\|_1, \|u\|_2$, respectively.

Our purpose of this paper is by employing Schauder's fixed point theorem to obtain some sufficient conditions for the existence of solutions of system (1.1) on time scales.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections. In Section 3, We establish our main results for the existence of solutions of system (1.1). In Section 4, an example is given to illustrate that our results are feasible and more general.

2. Preliminaries

In this section, we shall recall some definitions and lemmas which are used in what follows.

DEFINITION 2.1. ([1]) A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t.$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, defined $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. For the notations $[a,b]_{\mathbb{T}}, [a,b)_{\mathbb{T}}$ and so on, we will denote time scale intervals

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\},$$

where $a,b \in \mathbb{T}$ with $a < \rho(b)$.

DEFINITION 2.2. ([1]) The function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided that it is continuous at each right-dense point and has a left-sided limit at each point, write $f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T},\mathbb{R})$.

DEFINITION 2.3. ([1]) For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted by $f^{\Delta}(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - t]| \leqslant \varepsilon|\sigma(t) - s|, \forall s \in U.$$

DEFINITION 2.4. (see [1]) If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

LEMMA 2.1. (see [1]) *Assume that* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k, \alpha, \beta$ *be two constants, we have:*

(i) *if* $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t, *then* $(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t)$;

(ii) *if* f^{Δ} *exists, then* f *is continuous at* t.

LEMMA 2.2. ([1]) *If* $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, *and* $f, g \in C_{rd}$, *then:*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^a f(t)\Delta t = 0$.

A solution to system (1.1) is a function $u \in PC^2([0,T]_{\mathbb{T}}, \mathbb{R}^n)$ that satisfies (1.1) for each $t \in [0,T]_{\mathbb{T}}$.

Consider the following impulsive BVP on time scales:

$$\begin{cases} -u^{\Delta^3}(t) = h(t), \quad t \in [0,T]_{\mathbb{T}} \backslash \Omega, \\ \Delta u(t_k) = I_k, \ \Delta u^{\Delta}(t_k) = J_k, \ \Delta u^{\Delta^2}(t_k) = L_k, \quad k = 1, 2, \ldots, m, \\ u(0) = \lambda u(\sigma(T)), \ u^{\Delta}(0) = \lambda u^{\Delta}(\sigma(T)), \ u^{\Delta^2}(0) = \lambda u^{\Delta^2}(\sigma(T)), \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$ and $\lambda \neq 1$.

LEMMA 2.3. *Let*

$$p(t) = \int_0^t s\Delta s \quad \textit{and} \quad q(t,s) = p(t) - p(s) - ts + s^2.$$

If $h(t) \in PC([0,T]_{\mathbb{T}}, \mathbb{R}^n)$, *then* u *is the unique solution of the* $BVP(2.1)$ *if and only if*

$$\begin{aligned} u(t) = \int_0^{\sigma(T)} & G_1(t, \sigma(s))h(s)\Delta s + \sum_{k=1}^m G_2(t, t_k) I_k(u(t_k)) \\ & + \sum_{k=1}^m G_3(t, t_k) J_k(u(t_k), u^{\Delta}(t_k)) \\ & - \sum_{k=1}^m G_1(t, t_k) L_k(u(t_k), u^{\Delta}(t_k), u^{\Delta^2}(t_k)), \ \forall t \in [0, \sigma^3(T)]_{\mathbb{T}}, \end{aligned}$$

where for $0 \leqslant s \leqslant t \leqslant \sigma^3(T)$,

$$G_1(t,s) = \frac{\lambda}{(1-\lambda)^3}\Big[-\lambda(\sigma(T))^2 - \lambda(1-\lambda)\sigma(T)(\sigma(T)-s) - \lambda(1-\lambda)p^{\sigma}(T) - (1-\lambda)^2 q(\sigma(T),s) - \lambda(1-\lambda)\sigma(T)t - (1-\lambda)^2 t(\sigma(T)-s) - (1-\lambda)^2 p(t) - \lambda^{-1}(1-\lambda)^3 q(t,s)\Big]$$

and for $0 \leqslant t < s \leqslant \sigma(T)$,

$$G_1(t,s) = \frac{\lambda}{(1-\lambda)^3}\Big[-\lambda(\sigma(T))^2 - \lambda(1-\lambda)\sigma(T)(\sigma(T)-s) - \lambda(1-\lambda)p^{\sigma}(T) - (1-\lambda)^2 q(\sigma(T),s) - \lambda(1-\lambda)\sigma(T)t - (1-\lambda)^2 t(\sigma(T)-s) - (1-\lambda)^2 p(t)\Big];$$

$$G_2(t,s) = \begin{cases} \frac{1}{1-\lambda}, & 0 \leqslant s \leqslant t \leqslant \sigma^3(T), \\ \frac{\lambda}{1-\lambda}, & 0 \leqslant t < s \leqslant \sigma(T); \end{cases}$$

for $0 \leqslant s \leqslant t \leqslant \sigma^3(T)$,

$$G_3(t,s) = \frac{1}{(1-\lambda)^3}\Big[\lambda(1-\lambda)\sigma(T) - (1-\lambda)^2(\sigma(T)-t_k) + (1-\lambda)^2 t + \lambda^{-1}(1-\lambda)^3\big(t-t_k\big)\Big],$$

and for $0 \leqslant t < s \leqslant \sigma(T)$,

$$G_3(t,s) = \frac{1}{(1-\lambda)^3}\Big[\lambda(1-\lambda)\sigma(T) - (1-\lambda)^2(\sigma(T)-t_k) + (1-\lambda)^2 t\Big].$$

Proof. Assume that $u(t)$ is a solution of (2.1), then by integrating

$$u^{\Delta^3}(t) = -h(t),\ t \neq t_k, q\ (k = 1,2,\ldots,m)$$

step by step from 0 to t, we have

$$u^{\Delta^2}(t) = u^{\Delta^2}(0) - \int_0^t h(s)\Delta s + \sum_{0<t_k<t}[u^{\Delta^2}(t_k^+) - u^{\Delta^2}(t_k)],\ \forall t \in [0,\sigma(T)]_{\mathbb{T}}. \quad (2.2)$$

Similarly, we integrate $u^{\Delta^2}(t)$ and $u^{\Delta}(t)$, $t \neq t_k$, $(k = 1,2,\ldots,m)$ from 0 to t step by step to get

$$u^{\Delta}(t) = u^{\Delta}(0) + \int_0^t u^{\Delta^2}(s)\Delta s + \sum_{0<t_k<t}[u^{\Delta}(t_k^+) - u^{\Delta}(t_k)],\ \forall t \in [0,\sigma^2(T)]_{\mathbb{T}} \quad (2.3)$$

and

$$u(t) = u(0) + \int_0^t u^{\Delta}(s)\Delta s + \sum_{0<t_k<t}[u(t_k^+) - u(t_k)],\ \forall t \in [0,\sigma^3(T)]_{\mathbb{T}}. \quad (2.4)$$

Substituting (2.2) into (2.3), we obtain

$$u^{\Delta}(t)=u^{\Delta}(0)+u^{\Delta^2}(0)t-\int_0^t (t-\sigma(s))h(s)\Delta s+\sum_{0<t_k<t}[u^{\Delta}(t_k^+)-u^{\Delta}(t_k)]$$
$$+\sum_{0<t_k<t}[u^{\Delta^2}(t_k^+)-u^{\Delta^2}(t_k)](t-t_k),\ \forall t\in[0,\sigma^2(T)]_{\mathbb{T}}. \tag{2.5}$$

And substituting (2.5) into (2.4) to get

$$u(t)=u(0)+u^{\Delta}(0)t+u^{\Delta^2}(0)p(t)-\int_0^t q(t,\sigma(s))h(s)\Delta s$$
$$+\sum_{0<t_k<t}[u(t_k^+)-u(t_k)]+\sum_{0<t_k<t}[u^{\Delta}(t_k^+)-u^{\Delta}(t_k)](t-t_k)$$
$$+\sum_{0<t_k<t}[u^{\Delta^2}(t_k^+)-u^{\Delta^2}(t_k)]\int_{t_k}^t (s-t_k)\Delta s,\ \forall t\in[0,\sigma^3(T)]_{\mathbb{T}}. \tag{2.6}$$

By $u^{\Delta^2}(0)=\lambda u^{\Delta^2}(\sigma(T))$ and $\Delta u^{\Delta^2}(t_k)=L_k\big(u(t_k),u^{\Delta}(t_k),u^{\Delta^2}(t_k)\big)\,(k=1,2,\ldots,m)$, we have from (2.2) that

$$u^{\Delta^2}(0)=-\frac{\lambda}{1-\lambda}\Big(\int_0^{\sigma(T)} h(s)\Delta s-\sum_{k=1}^m L_k\Big). \tag{2.7}$$

By $u^{\Delta}(0)=\lambda u^{\Delta}(\sigma(T))$, $\Delta u^{\Delta}(t_k)=J_k(u(t_k),u^{\Delta}(t_k))(k=1,2,\ldots,m)$ and (2.7), from (2.5) it follows that

$$\begin{aligned}u^{\Delta}(0)&=\frac{\lambda}{1-\lambda}\Big[u^{\Delta^2}(0)\sigma(T)-\int_0^{\sigma(T)}(\sigma(T)-\sigma(s))h(s)\Delta s\\&\qquad+\sum_{k=1}^m L_k(\sigma(T)-t_k)+\sum_{k=1}^m J_k\Big]\\&=-\frac{\lambda^2\sigma(T)}{(1-\lambda)^2}\Big(\int_0^{\sigma(T)} h(s)\Delta s-\sum_{k=1}^m L_k\Big)\\&\qquad+\frac{\lambda}{1-\lambda}\Big[-\int_0^{\sigma(T)}(\sigma(T)-\sigma(s))h(s)\Delta s\\&\qquad+\sum_{k=1}^m L_k(\sigma(T)-t_k)+\sum_{k=1}^m J_k\Big].\end{aligned} \tag{2.8}$$

Note that $u(0)=\lambda u(\sigma(T))$, $\Delta u(t_k)=I_k(u(t_k))(k=1,2,\ldots,m)$, (2.7) and (2.8), so we have from (2.6) that

$$u(0)=\frac{\lambda}{1-\lambda}\Big[u^{\Delta}(0)\sigma(T)+u^{\Delta^2}(0)p(\sigma(T))-\int_0^{\sigma(T)} q(\sigma(T),\sigma(s))h(s)\Delta s+\sum_{k=1}^m I_k$$
$$+\sum_{k=1}^m J_k(\sigma(T)-t_k)+\sum_{k=1}^m L_k\int_{t_k}^{\sigma(T)}(s-t_k)\Delta s\Big]$$

$$
\begin{aligned}
= \frac{\lambda}{(1-\lambda)^3}\Big\{ \int_0^{\sigma(T)} & \Big[-\lambda(\sigma(T))^2 - \lambda(1-\lambda)\sigma(T)(\sigma(T)-\sigma(s)) \\
& - \lambda(1-\lambda)p^{\sigma}(T)\Big] h(s)\Delta s \\
& - \int_0^{\sigma(T)} (1-\lambda)^2 q(\sigma(T),\sigma(s))h(s)\Delta s \\
& + \sum_{k=1}^{m} \Big[(\lambda\sigma(T))^2 + \lambda(1-\lambda)(\sigma(T))^2(\sigma(T)-t_k) \\
& \quad - \lambda(1-\lambda)p^{\sigma}(T) + (1-\lambda)^2 \int_{t_k}^{\sigma(T)} (s-t_k)\Delta s\Big] L_k \\
& + \sum_{k=1}^{m} \Big[\lambda(1-\lambda)\sigma(T) - (1-\lambda)^2(\sigma(T)-t_k)\Big] J_k \\
& + \sum_{k=1}^{m} (1-\lambda)^2 I_k \Big\}.
\end{aligned} \tag{2.9}
$$

Then we substitute (2.7), (2.8) and (2.9) into (2.6) to obtain

$$
\begin{aligned}
u(t) = \frac{\lambda}{(1-\lambda)^3} \int_0^{\sigma(T)} & \Big[-\lambda(\sigma(T))^2 - \lambda(1-\lambda)\sigma(T)(\sigma(T)-\sigma(s)) \\
& - \lambda(1-\lambda)p^{\sigma}(T) - (1-\lambda)^2 q(\sigma(T),\sigma(s)) - \lambda(1-\lambda)\sigma(T)t \\
& - (1-\lambda)^2 t(\sigma(T)-\sigma(s)) - (1-\lambda)^2 p(t)\Big] h(s)\Delta s \\
& - \int_0^{t} q(t,\sigma(s))h(s)\Delta s \\
& + \sum_{k=1}^{m} \frac{\lambda}{(1-\lambda)^3}\Big[(\lambda\sigma(T))^2 + \lambda(1-\lambda)(\sigma(T))^2(\sigma(T)-t_k) - \lambda(1-\lambda)p^{\sigma}(T) \\
& \quad + (1-\lambda)^2 \int_{t_k}^{\sigma(T)} (s-t_k)\Delta s + \lambda(1-\lambda)\sigma(T)t \\
& \quad + (1-\lambda)^2 t(\sigma(T)-t_k) + (1-\lambda)^2 p(t)\Big] L_k \\
& + \sum_{0<t_k<t} L_k \int_{t_k}^{t} (s-t_k)\Delta s \\
& + \sum_{k=1}^{m} \frac{\lambda}{(1-\lambda)^3}\Big[\lambda(1-\lambda)\sigma(T) - (1-\lambda)^2(\sigma(T)-t_k) + (1-\lambda)^2 t\Big] J_k \\
& + \sum_{0<t_k<t} J_k\Big(t-t_k\Big) + \sum_{k=1}^{m} \frac{\lambda}{(1-\lambda)} I_k + \sum_{0<t_k<t} I_k \\
= \int_0^{\sigma(T)} G_1(t,\sigma(s))h(s)\Delta s & + \sum_{k=1}^{m} G_2(t,t_k)I_k + \sum_{k=1}^{m} G_3(t,t_k)J_k - \sum_{k=1}^{m} G_1(t,t_k)L_k.
\end{aligned}
$$

The proof of this lemma is complete. □

Recall that a mapping between Banach Spaces is compact if it is continuous and carries bounded sets into relatively compact sets.

We now introduce a mapping $\Phi : PC^2([0,\sigma^3(T)]_{\mathbb{T}},\mathbb{R}^n) \to PC^2([0,\sigma^3(T)]_{\mathbb{T}},\mathbb{R}^n)$ defined by

$$\begin{aligned}\Phi u(t) = \int_0^{\sigma(T)} & G_1(t,\sigma(s)) f(s,u(s),u^{\Delta}(s),u^{\Delta^2}(s))\Delta s \\ & + \sum_{k=1}^{m} G_2(t,t_k) I_k(u(t_k)) + \sum_{k=1}^{m} G_3(t,t_k) J_k(u(t_k),u^{\Delta}(t_k)) \\ & - \sum_{k=1}^{m} G_1(t,t_k) L_k(u(t_k),u^{\Delta}(t_k),u^{\Delta^2}(t_k)),\end{aligned} \tag{2.10}$$

where $t \in [0,\sigma^3(T)]_{\mathbb{T}}, G_1(t,s), G_2(t,s), G_3(t,s)$ are defined the same as those in Lemma 2.3.

LEMMA 2.4. *$\Phi : PC^2([0,\sigma^3(T)]_{\mathbb{T}},\mathbb{R}^n) \to PC^2([0,\sigma^3(T)]_{\mathbb{T}},\mathbb{R}^n)$ is a compact map.*

Proof. From (2.10) we know that

$$\begin{aligned}(\Phi u)^{\Delta}(t) = \int_0^{\sigma(T)} & G_1^{\Delta}(t,\sigma(s)) f\Delta s + \sum_{k=1}^{m} G_2^{\Delta}(t,t_k) I_k \\ & + \sum_{k=1}^{m} G_3^{\Delta}(t,t_k) J_k - \sum_{k=1}^{m} G_1^{\Delta}(t,t_k) L_k\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}(\Phi u)^{\Delta^2}(t) = \int_0^{\sigma(T)} & G_1^{\Delta^2}(t,\sigma(s)) f\Delta s + \sum_{k=1}^{m} G_2^{\Delta^2}(t,t_k) I_k \\ & + \sum_{k=1}^{m} G_3^{\Delta^2}(t,t_k) J_k - \sum_{k=1}^{m} G_1^{\Delta^2}(t,t_k) L_k.\end{aligned} \tag{2.12}$$

Then the continuality of f, I_k, J_k and L_k imply that Φ is a continuous map. On the other hand, for any bounded subset $A \subset PC^2([0,\sigma^3(T)]_{\mathbb{T}},\mathbb{R}^n)$, (2.11) implies $\{(\Phi u)^{\Delta}(t) : u(t) \in A\}$ is bounded subset of $PC^2([0,\sigma^3(T)]_{\mathbb{T}},\mathbb{R}^n)$. By (2.12), we also hava that $\{(\Phi u)^{\Delta^2}(t) : u(t) \in A\}$ is bounded. It follows from Lemma 2.4 in [24] that Φ is a compact map. The proof of this lemma is complete. □

From Lemma 2.3, we can easily derive the following lemma.

LEMMA 2.5. *It holds true: $u \in PC^2([0,T]_{\mathbb{T}},\mathbb{R}^n)$ is a solution of (1.1) if and only if $u \in PC^2([0,T]_{\mathbb{T}},\mathbb{R}^n)$ is a fixed point of Φ.*

THEOREM 2.1. (Schauder's Fixed Point Theorem). *Let E be a Banach space, $C \subset E$ be a nonempty bounded closed convex subset and $T : C \to C$ be a continuous compact mapping. Then T has a fixed point in C.*

3. Main results

In this section we state and prove our main result. For convenience, we introduce the following notations:

$$\begin{cases} \max\limits_{(t,s)\in[0,\sigma^3(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} |G_1(t,\sigma(s))| = G_0, \\ \max\limits_{(t,s)\in[0,\sigma^3(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} |G_2(t,s)| = Q_0, \\ \max\limits_{(t,s)\in[0,\sigma^3(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} |G_3(t,s)| \leqslant P_0, \\ \max\limits_{(t,s)\in[0,\sigma^3(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} |G_1^{\Delta}(t,\sigma(s))| = P_0, \\ \max\limits_{(t,s)\in[0,\sigma^3(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} |G_3^{\Delta}(t,s)| = Q_0, \\ \max\limits_{(t,s)\in[0,\sigma^3(T)]_{\mathbb{T}}\times[0,\sigma(T)]_{\mathbb{T}}} |G_1^{\Delta^2}(t,s)| = Q_0. \end{cases} \tag{3.1}$$

THEOREM 3.1. *Under the assumptions* (H_1)-(H_3)*, system* (1.1) *has at least one solution in* $PC^2([0,T]_{\mathbb{T}},\mathbb{R}^n)$*, if the following condition holds*

$$\eta = \max\{\eta_1,\eta_2,\eta_3\} < 1, \tag{3.2}$$

where

$$\begin{aligned} \eta_1 &= 3G_0\alpha\sigma(T) + Q_0\sum_{k=1}^m \beta_k + 2P_0\sum_{k=1}^m \gamma_k + 3G_0\sum_{k=1}^m \delta_k, \\ \eta_2 &= 3P_0\alpha\sigma(T) + 2Q_0\sum_{k=1}^m \gamma_k + 3P_0\sum_{k=1}^m \delta_k, \\ \eta_3 &= 3Q_0\alpha\sigma(T) + 3Q_0\sum_{k=1}^m \delta_k. \end{aligned} \tag{3.3}$$

Proof. By Lemma 2.4 it is sufficient to show that Φ has at least one fixed point in $PC^2([0,T]_{\mathbb{T}},\mathbb{R}^n)$. First, by (3.2) and (3.3), we can choose $\alpha' > \alpha, \beta_k' > \beta_k, \gamma_k' > \gamma_k, \delta_k' > \delta_k$ such that

$$\eta_1' = 3G_0\alpha'\sigma(T) + Q_0\sum_{k=1}^m \beta_k' + 2P_0\sum_{k=1}^m \gamma_k' + 3G_0\sum_{k=1}^m \delta_k' < 1, \tag{3.4}$$

$$\eta_2' = 3P_0\alpha'\sigma(T) + 2Q_0\sum_{k=1}^m \gamma_k' + 3P_0\sum_{k=1}^m \delta_k' < 1, \tag{3.5}$$

$$\eta_3' = 3Q_0\alpha'\sigma(T) + 3Q_0\sum_{k=1}^m \delta_k' < 1. \tag{3.6}$$

By (H_3) we can choose a positive number N such that

$$\|f(t,u,v,w)\| < \alpha'(\|u\|+\|v\|+\|w\|),\ \forall t\in[0,\sigma^3(T)]_{\mathbb{T}}, \|u\|+\|v\|+\|w\| \geqslant N.$$

Then

$$\|f(t,u,v,w)\| < \alpha'(\|u\|+\|v\|+\|w\|)+M,\ \forall t\in[0,\sigma^3(T)]_{\mathbb{T}}, \forall u,v,w\in\mathbb{R}^n, \quad (3.7)$$

where

$$M = \max_{t\in[0,\sigma^3(T)]_{\mathbb{T}}, \|u\|+\|v\|+\|w\|<N} \|f(t,u,v,w)\| < \infty.$$

Similarly, there exist positive constants E_k, F_k and $H_k, (k=1,2,\cdots,m)$ such that

$$\|I_k(u)\| < \beta_k'\|u\|+E_k,\ \forall u\in\mathbb{R}^n, \quad (3.8)$$

$$\|J_k(u,v)\| < \gamma_k'(\|u\|+\|v\|)+F_k,\ \forall u,v\in\mathbb{R}^n \quad (3.9)$$

and

$$\|L_k(u,v,w)\| < \delta_k'(\|u\|+\|v\|+\|w\|)+H_k,\ \forall u,v,w\in\mathbb{R}^n. \quad (3.10)$$

Then, by (2.10), (3.1), and (3.7)-(3.10), we have

$$\begin{aligned}
\|(\Phi u)(t)\|_0 &= \Big\| \int_0^{\sigma(T)} G_1(t,\sigma(s)) f\Delta s + \sum_{k=1}^m G_2(t,t_k)I_k \\
&\qquad + \sum_{k=1}^m G_3(t,t_k)J_k - \sum_{k=1}^m G_1(t,t_k)L_k \Big\|_0 \\
&\leqslant \sigma(T)G_0\Big[\alpha'\Big(\|u(t)\|_0+\|u^{\Delta}(t)\|_0+\|u^{\Delta^2}(t)\|_0\Big)+M\Big] \qquad (3.11)\\
&\qquad + Q_0\sum_{k=1}^m \Big(\beta_k'\|u(t)\|_0+E_k\Big) \\
&\qquad + P_0\sum_{k=1}^m \Big[\gamma_k'\Big(\|u(t)\|_0+\|u^{\Delta}(t)\|_0\Big)+F_k\Big] \\
&\qquad + G_0\sum_{k=1}^m \Big[\delta_k'\Big(\|u(t\|_0+\|u^{\Delta}(t)\|_0+\|u^{\Delta^2}(t)\|_0\Big)+H_k\Big] \\
&\leqslant \Big[3G_0\alpha'\sigma(T)+Q_0\sum_{k=1}^m\beta_k'+2P_0\sum_{k=1}^m\gamma_k'+3G_0\sum_{k=1}^m\delta_k'\Big]\|u(t)\|_2 \\
&\qquad + G_0\alpha'\sigma(T)M+Q_0\sum_{k=1}^m\beta_k'E_k \\
&\qquad + P_0\sum_{k=1}^m\gamma_k'F_k+G_0\sum_{k=1}^m\delta_k'H_k \\
&= \eta_1'\|u(t)\|_2+M^{(1)}, \qquad (3.12)
\end{aligned}$$

where $M^{(1)} = G_0\alpha'\sigma(T)M+Q_0\sum_{k=1}^m\beta_k'E_k+P_0\sum_{k=1}^m\gamma_k'F_k+G_0\sum_{k=1}^m\delta_k'H_k$ is a constant.

Similarly, differentiating both sides of (2.10), we can easily get

$$\begin{aligned}
\|(\Phi u)^{\Delta}(t)\|_0 &= \Big\| \int_0^{\sigma(T)} G_1^{\Delta}(t,\sigma(s)) f \Delta s + \sum_{k=1}^{m} G_2^{\Delta}(t,t_k) I_k \\
&\qquad + \sum_{k=1}^{m} G_3^{\Delta}(t,t_k) J_k - \sum_{k=1}^{m} G_1^{\Delta}(t,t_k) L_k \Big\|_0 \\
&\leqslant P_0 \sigma(T) \Big[\alpha' \Big(\|u(t)\|_0 + \|u^{\Delta}(t)\|_0 + \|u^{\Delta^2}(t)\|_0 \Big) + M \Big] \\
&\qquad + Q_0 \sum_{k=1}^{m} \Big[\gamma_k' \Big(\|u(t)\|_0 + \|u^{\Delta}(t)\|_0 \Big) + F_k \Big] \\
&\qquad + P_0 \sum_{k=1}^{m} \Big[\delta_k' \Big(\|u(t\|_0 + \|u^{\Delta}(t)\|_0 + \|u^{\Delta^2}(t)\|_0 \Big) + H_k \Big] \\
&\leqslant \Big[3P_0 \alpha' \sigma(T) + 2Q_0 \sum_{k=1}^{m} \gamma_k' + 3P_0 \sum_{k=1}^{m} \delta_k' \Big] \|u(t)\|_2 + P_0 \alpha' \sigma(T) M \\
&\qquad + Q_0 \sum_{k=1}^{m} \gamma_k' F_k + P_0 \sum_{k=1}^{m} \delta_k' H_k \\
&= \eta_2' \|u(t)\|_2 + M^{(2)}
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\|(\Phi u)^{\Delta^2}(t)\|_0 &= \Big\| \int_0^{\sigma(T)} G_1^{\Delta^2}(t,\sigma(s)) f \Delta s + \sum_{k=1}^{m} G_2^{\Delta^2}(t,t_k) I_k \\
&\qquad + \sum_{k=1}^{m} G_3^{\Delta^2}(t,t_k) J_k - \sum_{k=1}^{m} G_1^{\Delta^2}(t,t_k) L_k \Big\|_0 \\
&\leqslant Q_0 \sigma(T) \Big[\alpha' \Big(\|u(t)\|_0 + \|u^{\Delta}(t)\|_0 + \|u^{\Delta^2}(t)\|_0 \Big) + M \Big] \\
&\qquad + Q_0 \sum_{k=1}^{m} \Big[\delta_k' \Big(\|u(t\|_0 + \|u^{\Delta}(t)\|_0 + \|u^{\Delta^2}(t)\|_0 \Big) + H_k \Big] \\
&\leqslant \Big[3Q_0 \alpha' \sigma(T) + 3Q_0 \sum_{k=1}^{m} \delta_k' \Big] \|u(t)\|_2 + Q_0 \alpha' \sigma(T) M + Q_0 \sum_{k=1}^{m} \delta_k' H_k \\
&= \eta_3' \|u(t)\|_2 + M^{(3)},
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
M^{(2)} &= P_0 \alpha' \sigma(T) M + Q_0 \sum_{k=1}^{m} \gamma_k' F_k + P_0 \sum_{k=1}^{m} \delta_k' H_k, \\
M^{(3)} &= Q_0 \alpha' \sigma(T) M + Q_0 \sum_{k=1}^{m} \delta_k' H_k,
\end{aligned}$$

are two constants. Consequently, by (3.11)-(3.13) we have

$$\|(\Phi u)(t)\|_2 = \max\{\|(\Phi u)(t)\|_0, \|(\Phi u)^{\Delta}(t)\|_0, \|(\Phi u)^{\Delta^2}(t)\|_0\} \leqslant \eta' \|u(t)\|_2 + M',$$

where

$$\eta' = \max\{\eta_1', \eta_2', \eta_3'\} < 1, \quad M' = \max\{M^{(1)}, M^{(2)}, M^{(3)}\}.$$

Let $r = \frac{M'}{1-\eta'}$, $B_r = \{u \in PC^2([0,\sigma^3(T)], \mathbb{R}^n) | \|u\|_2 \leqslant r\}$. For $\forall u \in B_r$, we have

$$\|(\Phi u)(t)\|_2 \leqslant \eta' \|u(t)\|_2 + M' \leqslant \eta' r + M' = r.$$

So, we can easily get $\Phi(B_r) \subset B_r$. From Lemma 2.4, Φ is a complete continuous operator. By Theorem 2.1 (Schauder's Fixed Point Theorem), Φ has at least one fixed point in B_r. By Lemma 2.5, system (1.1) has at least one solution in $PC^2([0,T]_{\mathbb{T}}, \mathbb{R}^n)$. We complete the proof. □

REMARK 3.1. If the following conditions hold:

$$\frac{\|f(t,u,v,w)\|}{\|u\|+\|v\|+\|w\|} \to 0 \text{ and } \frac{\|L_k(u,v,w)\|}{\|u\|+\|v\|+\|w\|} \to 0 \text{ as } \|u\|+\|v\|+\|w\| \to \infty,$$

$$\frac{\|J_k(u,v)\|}{\|u\|+\|v\|} \to 0 \text{ as } \|u\|+\|v\| \to \infty,$$

$$\frac{\|I_k(u)\|}{\|u\|} \to 0 \text{ as } \|u\| \to \infty,$$

then (3.2) holds.

REMARK 3.2. If $\lambda = -1$, then system (1.1) is reduced into the following system:

$$\begin{cases} -u^{\Delta^3}(t) = f(t, u(t), u^{\Delta}(t), u^{\Delta^2}(t)), \quad t \in [0,T]_{\mathbb{T}} \backslash \Omega, \\ \Delta u(t_k) = I_k,\ \Delta u^{\Delta}(t_k) = J_k,\ \Delta u^{\Delta^2}(t_k) = L_k, \quad k = 1,2,\ldots,m, \\ u(0) = -u(\sigma(T)),\ u^{\Delta}(0) = -u^{\Delta}(\sigma(T)),\ u^{\Delta^2}(0) = -u^{\Delta^2}(\sigma(T)), \end{cases} \tag{3.15}$$

where

$$\begin{aligned} &\Delta u(t) = u(t^+) - u(t), \\ &\Delta u^{\Delta}(t) = u^{\Delta}(t^+) - u^{\Delta}(t), \\ &\Delta u^{\Delta^2}(t) = u^{\Delta^2}(t^+) - u^{\Delta^2}(t), \\ &I_k = I_k(u(t)),\ J_k = J_k(u(t), x^{\Delta}(t)),\ L_k = L_k(u(t), u^{\Delta}(t), u^{\Delta^2}(t)), \\ &\Omega = \{t_1, t_2, \ldots, t_m\},\ k = 1,2,\ldots,m. \end{aligned}$$

Then (3.14) is an anti-periodic boundary value problem. It is only a special case of system (1.1). Hence, our presented result is also valid to system (3.14).

4. Examples

In this section, we give an example to demonstrate the efficiencies of our result.

EXAMPLE 4.1. Consider the following boundary value problem:

$$\begin{cases} u'''(t) = f(t,u,v,w), \quad t \in [0,1]_{\mathbb{T}} \backslash \{\frac{1}{2}\}, \\ \Delta u(\frac{1}{2}) = \frac{1}{10^2} u(\frac{1}{2}),\ \Delta u'(\frac{1}{2}) = \frac{1}{10^2} u'(\frac{1}{2}),\ \Delta x''(\frac{1}{2}) = \frac{1}{10^2} x''(\frac{1}{2}), \\ x(0) = -2u(1),\ u'(0) = -2u'(1),\ u''(0) = -2u''(1), \end{cases} \tag{4.1}$$

where $\mathbb{T} = \mathbb{R}$ and

$$f(t,u,v,w) = \frac{1}{10^2} t \cos^2 u(t) + \frac{1}{10^2} \sin \sqrt{5} u'(t) + \frac{1}{10^2} \sin \sqrt{3} u''(t)$$

for all $(t,u,v,w) \in \Theta = ([0,\sigma^3(T)]_{\mathbb{T}}, \mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^n)$. Obviously,

$$G_0 \leqslant \frac{238}{27}, Q_0 \leqslant \frac{2}{3}, P_0 \leqslant \frac{14}{9}.$$

Thus, we obtain

$$\eta \leqslant 3 \times \frac{238}{27} \times \frac{1}{10^2} + \frac{2}{3} \times \frac{1}{10^2} + 2 \times \frac{14}{9} \times \frac{1}{10^2} + 3 \times \frac{238}{27} \times \frac{1}{10^2} \approx 0.57 < 1.$$

It follows from Theorem 3.1 that the system (4.1) has at least a solution.

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