

SOME NEW OSCILLATION CRITERIA FOR HIGHER-ORDER QUASI-LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. Oscillation criteria for the higher order quasi-linear neutral delay differential equations of the form

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' + \sum_{i=1}^m q_i(t)f_i(|u(\tau_i(t))|^{\alpha_i-1}u(\tau_i(t))) = 0,$$

$t \geq t_0$, $z(t) = u(t) + p(t)u(t - \sigma)$, $\alpha > 0$, $\alpha_i > 0$ ($i = 1, 2, 3, \dots, m$), are established under the condition:

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds = \infty \text{ or } \int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds < \infty \text{ respectively,}$$

where n is even. The obtained results improve and extend some known results in literature.

1. Introduction

In this paper, we are here concerned with the oscillation behavior of solution of higher order quasi-linear neutral delay differential equation of the form

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' + \sum_{i=1}^m q_i(t)f_i(|u(\tau_i(t))|^{\alpha_i-1}u(\tau_i(t))) = 0, \quad (1.1)$$

where $t \geq t_0$, $z(t) = u(t) + p(t)u(t - \sigma)$, $\alpha > 0$, $\alpha_i > 0$ ($i = 1, 2, 3, \dots, m$), $\sigma \geq 0$ are constants and n is even.

In this paper, we assume that

(B₁) $p(t) \in C([t_0, \infty), [0, 1])$, $q_i(t) \in C([t_0, \infty), [0, \infty))$ ($i = 1, 2, 3, \dots, m$), where $p(t)$ and $q_i(t)$ do not identify zero on $(a, b) \subset [0, 1]$, $a < b$;

(B₂) $r(t) \in C^1([t_0, \infty), (0, \infty))$, $R(t) = \int_{t_0}^t r^{-\frac{1}{\alpha}}(s)ds$;

(B₃) $\psi(u)$, $f_i(u) \in C^1(\mathbb{R}, \mathbb{R})$, $\psi(u) > 0$, $u f_i(u) > 0$ for $u \neq 0$, $i = 1, 2, 3, \dots, m$, there exists constants $L > 0$ and $\beta_i > 0$ ($i = 1, 2, 3, \dots, m$), such that $\psi(u) \leq L^{-1}$ and

$$\frac{f_i(u^{\alpha_i}(\tau_i(t)))}{|u(\tau_i(t))|^{\alpha-1}u(\tau_i(t))} \geq \beta_i > 0 \text{ for } u(\tau_i(t)) \neq 0;$$

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(B₄) $\tau(t) \in C^1([t_0, \infty), [0, \infty))$, $\tau(t) \leq \tau_i(t)$, $\tau(t) \leq t$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$, $\tau'(t) > 0$ for $t \in [t_0, \infty)$.

In what follows, we shall consider only the nonconstant solutions of (1.1) which are defined for all large t . The solutions of (1.1) mean a function $u \in C^1([T_u, \infty), \mathbb{R})$, $T_u \geq t_0$ such that u and $r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ are continuously differentiable and satisfy Eq.(1.1). A solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Eq.(1.1) is called oscillatory if all of its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for various classes of second order differential equations, e.g., see [1-6], [8-14], [16] and the references therein. Xu et al. [21] extended the results of Dzurina et al. [3] and Sun et al. [16] to the neutral delay differential equation

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.2}$$

where $y(t) = x(t) + p(t)x(t - \tau)$. More precisely, they proved that if

$$\lim_{t \rightarrow \infty} R(t) = \lim_{t \rightarrow \infty} \int_{t_0}^t \frac{ds}{r^{\frac{1}{\alpha}}(s)} = \infty \tag{1.3}$$

and

$$\int_a^\infty \left[kq(t)(1 - p(\sigma(t)))R^\alpha(\sigma(t)) - \left(\frac{\alpha}{\alpha + 1}\right)^{\alpha+1} \frac{\sigma'(t)}{R(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))} \right] dt = \infty,$$

then (1.3) is oscillatory.

Very recently, the oscillatory behavior of solutions of higher order neutral differential equations are of both theoretical and practical interest. There have been some results on the oscillatory and asymptotic behavior of even order neutral equations. we mention here [7], [17-19].

In [20], by using Riccati technique and averaging functions method, wang established some general oscillation criteria for even order neutral delay differential equation

$$[r(t) + c(t)x(t - \tau)]^{(n)} + \int_a^b p(t, \xi)x[g(t, \xi)]d\sigma(\xi) = 0, \quad t > t_0.$$

In this paper, we shall continue in this direction to study the oscillatory properties of (1.1). Motivated by [19], by using the Riccati technique and the integral averaging technique, under the condition: $\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)ds = \infty$ or $\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)ds < \infty$, we establish the oscillation criteria for (1.1), which extend and improve the main results in [3, 16, 21].

In order to prove our theorems, we will use the following lemmas.

LEMMA 1.1. (Kiguradze [8]) *Let $x(t)$ be a positive and n times differentiable function on R . If $x^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_1, +\infty)$*

for $(t_1 > 0)$, then there exists a $t_x > t_1$ and an integer $l(0 \leq l \leq n)$, with $n+l$ even for $x(t)x^{(n)}(t) \geq 0$ or $n+l$ odd for $x(t)x^{(n)}(t) \leq 0$; and for $t \geq t_x$,

$$x(t)x^{(k)}(t) > 0, \quad 0 \leq k \leq l; \quad (-1)^{k-l}x(t)x^{(k)}(t) > 0, \quad l \leq k \leq n.$$

LEMMA 1.2. (Philos [15]) Suppose that the condition of Lemma 1.1 is satisfied, and

$$x^{(n-1)}(t)x^{(n)}(t) \leq 0, \quad t \geq t_x,$$

then there exists a constant θ in $(0, 1)$ such that for sufficiently large t , satisfying

$$|x'(t/2)| \geq M_\theta t^{n-2} |x^{(n-1)}(t)|,$$

where $M_\theta = \frac{\theta}{(n-2)!}$ and n is even.

We define the following notations to be used through this paper

$$R(t) = \int_{t_0}^t \frac{ds}{r^{\frac{1}{\alpha}}(s)}, \quad Q(t) = \sum_{i=1}^m q_i(t)\beta_i(1 - p(\tau_i(t)))^\alpha, \quad \xi = \left(\frac{2}{M}\right)^\alpha \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \quad (1.4)$$

2. Main results

We first consider the oscillatory property of (1.1) under the condition

$$\lim_{t \rightarrow \infty} R(t) = \lim_{t \rightarrow \infty} \int_{t_0}^t \frac{ds}{r^{\frac{1}{\alpha}}(s)} = \infty. \quad (2.1)$$

THEOREM 2.1. Assume that (2.1) holds and

$$\int^\infty \left(R^\alpha[\tau(t)]Q(t) - \frac{\xi \tau'(t)}{LR[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))((\tau(t))^{n-2})^\alpha} \right) dt = \infty, \quad (2.2)$$

then Eq.(1.1) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.1) has a nonoscillatory solution $u(t)$. Without loss of generally, we assume that

$$u(t) > 0, u(\tau_i(t)) > 0, u(t - \tau) > 0, \quad t \geq t_1 \geq t_0.$$

Since the case $u(t) < 0$ can be treated similarly.

From (1.1) and (B_3) , we can get

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' \leq 0, \quad z(t) \geq u(t). \quad (2.3)$$

Therefore $r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is a decreasing function. We claim that

$$z^{(n-1)}(t) > 0, \quad t \geq t_1.$$

Otherwise, if there exist $t_2 \geq t_1$, such that $z^{(n-1)}(t) \leq 0$ for all $t \geq t_2$, from (2.3) we have

$$\begin{aligned} r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) \\ \leq r(t_2)\psi(u(t_2))|z^{(n-1)}(t_2)|^{\alpha-1}z^{(n-1)}(t_2) = -\beta \ (\beta > 0), \ t \geq t_2, \end{aligned}$$

which implies that

$$z^{(n-1)}(t) \leq -\left(\frac{\beta}{\psi(u(t))}\right)^{\frac{1}{\alpha}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \leq -(\beta L)^{\frac{1}{\alpha}} \frac{1}{r^{\frac{1}{\alpha}}(t)}.$$

Integrating the above inequality from t_2 to t , we have

$$z^{(n-2)}(t) \leq z^{(n-2)}(t_2) - (\beta L)^{\frac{1}{\alpha}} \int_{t_2}^t \frac{ds}{r^{\frac{1}{\alpha}}(s)}.$$

Letting $t \rightarrow \infty$, from (2.1) we get $\lim_{t \rightarrow \infty} z^{(n-2)}(t) = -\infty$, which implies $\lim_{t \rightarrow \infty} z(t) = -\infty$. This get a contradiction with $z(t) > 0$.

If there exists $t_3 > t_1$, such that $z^{(n-1)}(t_3) = 0$, since the function

$$r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$$

is decreasing, we can obtain that there exists $t_4 \geq t_3$, such that $z^{(n-1)}(t_4) < 0$, get the some contradiction as above.

If there exists $t_6 > t_5 > t_1$, such that $z^{(n-1)}(t_5) = z^{(n-1)}(t_6) = 0$, integrating (1.1) from t_5 to t_6 , we have

$$\int_{t_5}^{t_6} \sum_{i=1}^m q_i(t) f_i(|u(\tau_i(t))|^{\alpha_i-1} u(\tau_i(t))) = 0,$$

which arrive a contradiction with main assumption of the paper. Hence $z^{(n-1)}(t) > 0$ for $t \geq t_1$. From (2.3), we can get

$$\begin{aligned} [r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]' \\ = [r(t)\psi(u(t))]'(z^{(n-1)}(t))^{\alpha} + \alpha r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha-1}z^{(n)}(t) \leq 0, \end{aligned}$$

then

$$z^{(n)}(t) \leq \frac{-[r(t)\psi(u(t))]'(z^{(n-1)}(t))^{\alpha}}{\alpha r(t)\psi(u(t))} \leq 0.$$

From Lemma 1.1 and Lemma 1.2, we have

$$z^{(n-1)}(t) > 0, \ z^{(n)}(t) \leq 0, \ z'(t) > 0, \ t > t_1,$$

$$z' \left(\frac{\tau(t)}{2} \right) \geq M(\tau(t))^{n-2} z^{(n-1)}(\tau(t)). \tag{2.4}$$

Since $u(t) \leq z(t)$,

$$\begin{aligned} u(t) &= z(t) - p(t)u(t - \sigma) \geq z(t) - p(t)z(t - \sigma) \\ &\geq z(t) - p(t)z(t) = (1 - p(t))z(t). \end{aligned} \quad (2.5)$$

Therefore, we have

$$r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha \leq r(\tau(t))\psi(u(\tau(t)))(z^{(n-1)}(\tau(t)))^\alpha.$$

It follows that

$$\frac{z^{(n-1)}(\tau(t))}{z^{(n-1)}(t)} \geq \left(\frac{r(t)\psi(u(t))}{r(\tau(t))\psi(u(\tau(t)))} \right)^{\frac{1}{\alpha}}. \quad (2.6)$$

Following (B_3) and (2.5), we get

$$[r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha]' + \sum_{i=1}^m q_i(t)\beta_i(1 - p(\tau_i(t)))^\alpha z^\alpha(\tau_i(t)) \leq 0,$$

then Eq.(1.1) will become

$$[r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha]' + Q(t)z^\alpha(\tau(t)) \leq 0. \quad (2.7)$$

Define

$$w(t) = R^\alpha[\tau(t)] \frac{r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha}{z^\alpha\left(\frac{\tau(t)}{2}\right)}, \quad t \geq t_1. \quad (2.8)$$

Then $w(t) \geq 0$. From (2.4), (2.6), (2.7) and (2.8), we get

$$\begin{aligned} w'(t) &\leq \frac{\alpha\tau'(t)R^{\alpha-1}[\tau(t)]}{r^{\frac{1}{\alpha}}[\tau(t)]} \frac{r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha}{z^\alpha\left(\frac{\tau(t)}{2}\right)} - \frac{R^\alpha[\tau(t)]}{z^\alpha\left(\frac{\tau(t)}{2}\right)} (Q(t)z^\alpha(\tau(t))) \\ &\quad - R^\alpha[\tau(t)] [r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha] \frac{\alpha z'(\frac{\tau(t)}{2})^{\frac{1}{2}} \tau'(t)}{z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \\ &\leq \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]} w(t) - R^\alpha[\tau(t)]Q(t) \\ &\quad - \frac{\alpha\tau'(t)R^\alpha[\tau(t)] [r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha] M(\tau(t))^{n-2} z^{(n-1)}(\tau(t))}{2 z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \\ &\leq \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]} w(t) - R^\alpha[\tau(t)]Q(t) \\ &\quad - \frac{M\alpha\tau'(t)R^\alpha[\tau(t)] [r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha+1}] (\tau(t))^{n-2} z^{(n-1)}(\tau(t))}{2z^{\alpha+1}\left(\frac{\tau(t)}{2}\right) z^{(n-1)}(t)} \\ &\leq \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]} w(t) - R^\alpha[\tau(t)]Q(t) \end{aligned}$$

$$\begin{aligned}
 & - \frac{M\alpha\tau'(t)R^\alpha[\tau(t)][r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha+1}](\tau(t))^{n-2}}{2z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \left(\frac{r(t)\psi(u(t))}{r(\tau(t))\psi(u(\tau(t)))}\right)^{\frac{1}{\alpha}} \\
 \leq & \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}w(t) - R^\alpha[\tau(t)]Q(t) - \frac{M\alpha L^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2R[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))}w^{\frac{\alpha+1}{\alpha}}(t). \tag{2.9}
 \end{aligned}$$

Set

$$F(v) = \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}v - \frac{M\alpha L^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2R[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))}v^{\frac{\alpha+1}{\alpha}}, \quad v > 0.$$

By calculating, we have that when

$$v = \frac{(2\alpha)^\alpha}{L(M(\alpha + 1)(\tau(t))^{n-2})^\alpha},$$

$F(v)$ obtain its maximum. So

$$F(v) \leq F_{max} = \frac{\xi\tau'(t)}{LR[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))((\tau(t))^{n-2})^\alpha}.$$

Therefore,

$$w'(t) \leq -R^\alpha[\tau(t)]Q(t) + \frac{\xi\tau'(t)}{LR[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))((\tau(t))^{n-2})^\alpha}.$$

Integrating the above inequality from $T_0(T_0 \geq t_1)$ to t , we have

$$\begin{aligned}
 0 & < w(t) \\
 & < w(T_0) - \int_{T_0}^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi\tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))((\tau(s))^{n-2})^\alpha} \right] ds. \tag{2.10}
 \end{aligned}$$

Letting $t \rightarrow \infty$ in (2.10), from (2.2) we get a contradiction. This completes the proof of Theorem 2.1.

COROLLARY 2.1. Assume that (2.1) holds and for some $T \geq t_0$,

$$\liminf_{t \rightarrow \infty} \frac{1}{\ln R[\tau(t)]} \int_T^t R^\alpha[\tau(s)]Q(s)ds > \frac{\xi}{L}, \tag{2.11}$$

then Eq (1.1) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.1) has a nonoscillatory solution $u(t)$. Without of loss generally, we assume that $u(t) > 0$, $u(\tau_i(t)) > 0$, $u(t - \tau) > 0$ for $t \geq t_1 \geq t_0$. Since the case $u(t) < 0$ can be treated similarly.

It is easily prove that there exist $T \geq t_0$, such that $(\tau(t))^{n-2} > 1$. From (2.11), we can get that yields the existence $\varepsilon > 0$, such that for all large t,

$$\frac{1}{\ln R[\tau(t)]} \int_T^t R^\alpha[\tau(s)]Q(s)ds \geq \frac{\xi}{L} + \varepsilon,$$

which follows that

$$\begin{aligned} & \int_T^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi \tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))((\tau(s))^{n-2})^\alpha} \right] ds \\ & > \int_T^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi \tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))} \right] ds \\ & = \int_T^t R^\alpha[\tau(s)]Q(s)ds - \frac{\xi}{L} [\ln R(\tau(t)) - \ln R(\tau(T))] \\ & > \left(\frac{\xi}{L} + \varepsilon \right) \ln R(\tau(t)) - \frac{\xi}{L} [\ln R(\tau(t)) - \ln R(\tau(T))] \\ & = \varepsilon \ln R(\tau(t)) + \frac{\xi}{L} \ln R(\tau(T)). \end{aligned} \tag{2.12}$$

Now, it is obvious that (2.12) implies (2.2) and the assertion of corollary 2.1 follows from Theorem 2.1.

COROLLARY 2.2. Assume that (2.1) holds, and

$$\liminf_{t \rightarrow \infty} \frac{r^{\frac{1}{\alpha}}[\tau(t)]R^{\alpha+1}[\tau(t)]Q(t)}{\tau'(t)} > \frac{\xi}{L}, \tag{2.13}$$

then Eq.(1.1) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.1) has a nonoscillatory solution $u(t)$. Without of loss generally, we assume that $u(t) > 0$, $u(\tau_i(t)) > 0$, $u(t - \tau) > 0$ for $t \geq t_1 \geq t_0$. Since the case $u(t) < 0$ can be treated similarly.

It is easily prove that there exist $T > t_0$, such that $(\tau(t))^{n-2} > 1$. From (2.12), we can get that yields the existence $\varepsilon > 0$, such that for all large t,

$$\frac{r^{\frac{1}{\alpha}}[\tau(t)]R^{\alpha+1}[\tau(t)]Q(t)}{\tau'(t)} \geq \frac{\xi}{L} + \varepsilon, \tag{2.14}$$

multiplying $\frac{\tau'(t)}{r^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]}$ on both sides of (2.14), we obtain that

$$R^\alpha[\tau(t)]Q(t) - \frac{\xi}{L} \frac{\tau'(t)}{r^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]} \geq \varepsilon \frac{\tau'(t)}{r^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]}.$$

Therefore, we have

$$\begin{aligned} & \int_T^t \left(R^\alpha[\tau(s)]Q(s) - \frac{\xi}{L} \frac{\tau'(s)}{r^{\frac{1}{\alpha}}[\tau(s)]R[\tau(s)]} \right) ds \\ & \geq \varepsilon \int_T^t \frac{\tau'(s)}{r^{\frac{1}{\alpha}}[\tau(s)]R[\tau(s)]} ds \\ & = \varepsilon(\ln R(\tau(t)) - \ln R(\tau(T))). \end{aligned}$$

Then

$$\begin{aligned} & \int_T^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi \tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))((\tau(s))^{n-2})^\alpha} \right] ds \\ & > \int_T^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi}{L} \frac{\tau'(s)}{R[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))} \right] ds \\ & > \varepsilon(\ln R(\tau(t)) - \ln R(\tau(T))). \end{aligned} \quad (2.15)$$

It is obvious that (2.15) implies (2.2) and Corollary 2.2 is evident by Theorem 2.1.

If $p(t) = 0$, from Theorem 2.1 we obtain the following conclusions.

COROLLARY 2.3. *Assume that (2.1) holds and*

$$\int^\infty \left(R^\alpha[\tau(t)] \sum_{i=1}^m q_i(t)\beta_i - \frac{\xi \tau'(t)}{Lr^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]((\tau(t))^{n-2})^\alpha} \right) dt = \infty, \quad (2.16)$$

then Eq.(1.1) with $p(t) = 0$ is oscillatory.

COROLLARY 2.4. *Assume that (2.1) holds and for some $t_1 > t_0$,*

$$\liminf_{t \rightarrow \infty} \frac{1}{\ln R[\tau(t)]} \int_{t_1}^t R^\alpha[\tau(s)] \sum_{i=1}^m q_i(s)\beta_i ds > \frac{\xi}{L}, \quad (2.17)$$

then Eq (1.1) with $p(t) = 0$ is oscillatory.

COROLLARY 2.5. *Assume that (2.1) holds, and*

$$\liminf_{t \rightarrow \infty} \sum_{i=1}^m q_i(t)\beta_i \frac{r^{\frac{1}{\alpha}}[\tau(t)]R^{\alpha+1}[\tau(t)]}{\tau'(t)} > \frac{\xi}{L}, \quad (2.18)$$

then Eq.(1.1) is oscillatory.

THEOREM 2.2. *Let (2.1) holds. If there exists a function $h \in C^1([t_0, \infty), \mathbb{R}^+)$ such that*

$$\int^\infty \left(Q(t) - \frac{ML^{\frac{1}{\alpha}} \tau'(t) (\tau(t))^{n-2} h^{\alpha+1}(t)}{2r^{\frac{1}{\alpha}}(\tau(t))} \right) \exp \left(\frac{(\alpha + 1)ML^{\frac{1}{\alpha}}}{2} \int^t \frac{\tau'(s) (\tau(s))^{n-2} h(s)}{r^{\frac{1}{\alpha}}(\tau(s))} ds \right) dt = \infty, \quad (2.19)$$

then (1.1) is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, as in the proof of Theorem 2.1, we may assume that there exists a number $t_1 \geq t_0$ such that $u(t) > 0$, $u(\tau_i(t)) > 0$, $u(t - \tau) > 0$ for $t \geq t_1$. Since the case $u(t) < 0$ can be treated similarly.

From the proof of Theorem 2.1, we get that $z^{(n-1)}(t) > 0$ holds for $t \geq t_1$. Furthermore, (2.6) and (2.7) hold. Define

$$w(t) = \frac{r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha}{z^\alpha\left(\frac{\tau(t)}{2}\right)}, \quad t \geq t_1. \quad (2.20)$$

Obviously, $w(t) > 0$. Differentiating (2.20), in view of (2.6) and (2.7), we have

$$\begin{aligned} w'(t) &= \frac{[r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha]'}{z^\alpha\left(\frac{\tau(t)}{2}\right)} - [r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha] \frac{\alpha z'(\frac{\tau(t)}{2}) \frac{1}{2} \tau'(t)}{z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \\ &\leq -\frac{Q(t)z^\alpha(\tau(t))}{z^\alpha\left(\frac{\tau(t)}{2}\right)} - \frac{\alpha \tau'(t)}{2} \frac{[r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha]M(\tau(t))^{n-2}z^{(n-1)}(\tau(t))}{z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \\ &\leq -Q(t) - \frac{M\alpha \tau'(t)[r(t)\psi(u(t))(z^{(n-1)}(t))^\alpha](\tau(t))^{n-2}}{2z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \left(\frac{r(t)\psi(u(t))}{r(\tau(t))\psi(u(\tau(t)))} \right)^{\frac{1}{\alpha}} \\ &\leq -Q(t) - \frac{M\alpha L^{\frac{1}{\alpha}} \tau'(t) (\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))} w^{\frac{\alpha+1}{\alpha}}(t) \\ &= -\left[Q(t) - \frac{ML^{\frac{1}{\alpha}} \tau'(t) (\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))} h^{\alpha+1}(t) \right] \\ &\quad - \frac{ML^{\frac{1}{\alpha}} \tau'(t) (\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))} \left[\alpha w^{\frac{\alpha+1}{\alpha}}(t) + h^{\alpha+1}(t) \right]. \end{aligned}$$

By Holder’s inequality, $\alpha w^{\frac{\alpha+1}{\alpha}}(t) + h^{\alpha+1}(t) \geq (\alpha + 1)h(t)w(t)$. Thus,

$$w'(t) \leq -\left[Q(t) - \frac{ML^{\frac{1}{\alpha}} \tau'(t) (\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))} h^{\alpha+1}(t) \right] - \frac{(\alpha + 1)ML^{\frac{1}{\alpha}} \tau'(t) (\tau(t))^{n-2} h(t)w(t)}{2r^{\frac{1}{\alpha}}(\tau(t))},$$

which follows that

$$\begin{aligned} & \left[\exp \left(\frac{(\alpha + 1)ML^{\frac{1}{\alpha}}}{2} \int_T^t \frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))} ds \right) w(t) \right]' \\ & \leq - \left[Q(t) - \frac{ML^{\frac{1}{\alpha}} \tau'(t)(\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))} h^{\alpha+1}(t) \right] \times \\ & \quad \exp \left(\frac{(\alpha + 1)ML^{\frac{1}{\alpha}}}{2} \int_T^t \frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))} ds \right). \end{aligned} \tag{2.21}$$

Integrating (2.21) from T ($T \geq t_1$) to t , we have

$$\begin{aligned} 0 & < \exp \left(\frac{(\alpha + 1)ML^{\frac{1}{\alpha}}}{2} \int_T^t \frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))} ds \right) w(t) \\ & \leq w(T) - \int_T^t \left[Q(s) - \frac{ML^{\frac{1}{\alpha}} \tau'(s)(\tau(s))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(s))} h^{\alpha+1}(s) \right] \\ & \quad \exp \left(\frac{(\alpha + 1)ML^{\frac{1}{\alpha}}}{2} \int_T^s \frac{\tau'(u)(\tau(u))^{n-2}h(u)}{r^{\frac{1}{\alpha}}(\tau(u))} du \right) ds. \end{aligned}$$

Let $t \rightarrow \infty$ in the above inequality, which contradicts(2.19). This completes the proof of Theorem 2.3.

Now, let us consider the case when

$$\lim_{t \rightarrow \infty} R(t) = \lim_{t \rightarrow \infty} \int_{t_0}^t \frac{ds}{r^{\frac{1}{\alpha}}(s)} < \infty. \tag{2.22}$$

THEOREM 2.3. *Assume that (2.2) and (2.22) hold. Suppose that there exist a continuously differentiable $\varphi(t)$, such that $\varphi(t) > 0, \varphi'(t) > 0, p'(t) \geq 0$ for $t \geq t_0$, $\lim_{t \rightarrow \infty} p(t) = A$, if*

$$\int^{\infty} \left(\frac{L}{\varphi(t)r(t)} \int^t \left(\sum_{i=1}^m q_i(s)\beta_i\varphi(s) \right) ds \right)^{\frac{1}{\alpha}} dt = \infty, \tag{2.23}$$

then every solution $u(t)$ of (1.1) oscillates or $\lim_{t \rightarrow \infty} u(t) = 0$.

Proof. Suppose to the contrary that $u(t)$ is an eventually positive solution of (1.1), without loss of generally, we assume that $u(t) > 0, t > t_1$, then $z(t) > 0$.

It is easy to conclude that $r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is a decreasing function. There exist two possible case of the sign of $z^{(n-1)}(t)$, $z^{(n-1)}(t) > 0$ or $z^{(n-1)}(t) < 0$.

CASE (1). Suppose $z^{(n-1)}(t) > 0$ for sufficiently large t , then we are back to the case of Theorem 2.1. Thus the proof of Theorem 2.1 goes through, and we may get contradiction by (2.2).

CASE (2). Suppose $z^{(n-1)}(t) < 0$ for sufficiently large t , from Lemma 1.1 and Lemma 1.2 we can easily get that $z'(t) < 0$, since $p'(t) \geq 0$, $z'(t) = u'(t) + p'(t)u(t - \sigma) + p(t)u'(t - \sigma)$, then $u'(t) \leq 0$. It follows that $\lim_{t \rightarrow \infty} z(t) = a \geq 0$, now we claim that $a = 0$. Otherwise, $\lim_{t \rightarrow \infty} z(t) = a > 0$, so $\lim_{t \rightarrow \infty} u(t) = \frac{a}{1+A} > 0$, there exist constants $M_i > 0$ ($i = 1, 2, 3, \dots, m$) and $M > 0$ such that $u^\alpha(\tau_i(t)) \geq M_i$ and $M_i \geq M$ for all $t \geq t_1 \geq t_0$. From (1.1) we get

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' + \sum_{i=1}^m q_i(t)f_i(u^{\alpha_i}(\tau_i(t))) = 0.$$

Then

$$\begin{aligned} [r(t)\psi(u(t))(-z^{(n-1)}(t))^\alpha]' &= \sum_{i=1}^m q_i(t)f_i(u^{\alpha_i}(\tau_i(t))) \\ &\geq \sum_{i=1}^m q_i(t)\beta_i u^\alpha(\tau_i(t)) \\ &\geq \sum_{i=1}^m q_i(t)\beta_i M_i \geq M \sum_{i=1}^m q_i(t)\beta_i \end{aligned}$$

for $t \geq t_1$. Define $v(t) = \varphi(t)r(t)\psi(u(t))(-z^{(n-1)}(t))^\alpha$, then $v(t) \geq 0$,

$$\begin{aligned} v'(t) &= \varphi(t)[r(t)\psi(u(t))(-z^{(n-1)}(t))^\alpha]' + \varphi'(t)[r(t)\psi(u(t))(-z^{(n-1)}(t))^\alpha] \\ &\geq \varphi(t)[r(t)\psi(u(t))(-z^{(n-1)}(t))^\alpha]' \\ &\geq M\varphi(t) \sum_{i=1}^m q_i(t)\beta_i. \end{aligned} \tag{2.24}$$

Integrating (2.24) from t_1 to t , we have

$$v(t) \geq v(t_1) + M \int_{t_1}^t \left(\sum_{i=1}^m q_i(s)\beta_i\varphi(s) \right) ds \geq M \int_{t_1}^t \left(\sum_{i=1}^m q_i(s)\beta_i\varphi(s) \right) ds,$$

that is

$$\varphi(t)r(t)\psi(u(t))(-z'(t))^\alpha \geq M \int_{t_1}^t \left(\sum_{i=1}^m q_i(s)\beta_i\varphi(s) \right) ds,$$

so that

$$(-z'(t)) \geq M^{\frac{1}{\alpha}} \left(\frac{L}{\varphi(t)r(t)} \int_{t_1}^t \left(\sum_{i=1}^m q_i(s)\beta_i\varphi(s) \right) ds \right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t_1 to t , we obtain

$$z(t) \leq z(t_1) - M^{\frac{1}{\alpha}} \int_{t_1}^t \left(\frac{L}{\varphi(s)r(s)} \int_{t_1}^s \left(\sum_{i=1}^m q_i(\xi)\beta_i\varphi(\xi) \right) d\xi \right)^{\frac{1}{\alpha}} ds.$$

We can easily obtain a contradiction. So that $\lim_{t \rightarrow \infty} z(t) = 0$, then $\lim_{t \rightarrow \infty} u(t) = 0$. This completes the proof of Theorem 2.4.

3. Example

Consider the following high-order differential equation

$$\left(\frac{1}{t^\alpha(1+u^2(t))} |z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t) \right)' + t|u(t)|^{\alpha-1}u(t) = 0, \quad t \geq 1, \quad (3.1)$$

where

$$p(t) = \frac{1}{2}, \quad r(t) = t^{-\alpha}, \quad \sigma(t) = t - 1, \\ \psi(u) = \frac{1}{1+u^2}, \quad q(t) = t, \quad f(u) = u, \quad \tau(t) = t.$$

If we take $\beta = L = 1$, then

$$R(\tau(t)) = \int_1^t s ds = \frac{1}{2}(t^2 - 1).$$

For any constant $T > t_0$, we have

$$\liminf_{t \rightarrow \infty} \frac{1}{\ln R(\tau(t))} \int_1^t R^\alpha(\tau(s)) Q(s) ds \\ = \liminf_{t \rightarrow \infty} \frac{1}{\ln(\frac{1}{2}(t^2 - 1))} \int_1^t \frac{1}{4^\alpha} (s^2 - 1)^\alpha s ds \\ = \frac{1}{2^{2\alpha+1}} \liminf_{t \rightarrow \infty} (t^2 - 1)^{\alpha+1} = \infty,$$

from Corollary 2.1, we can see that Eq.(3.1) is oscillatory.

REFERENCES

- [1] R.P. AGARWAL, S.-L. SHIEH, C.-C. YEH, *Oscillation criteria for second order retarded differential equations*, Math. Comput. Model., **26** (1997), 1–11.
- [2] J.-L. CHERN, W.-C. LIAN, C.-C. YEH, *Oscillation criteria for second Order half-linear differential equations with functional arguments*, Publ. Math. Debrecen., **48**, 3-4 (1996), 209–216.
- [3] J. DZURINA, I. P. STAVROULAKIS, *Oscillatory criteria for second order delay differential equations*, Appl. Math. Comput., **140** (2003), 445–453.
- [4] A. ELBERT, *A half-linear second order differential equation*, Colloquia Math. Soc. Janos Bolyai, Qualitat. Theor. Diff. Equat., **30** (1979), 153–180.
- [5] A. ELBERT, *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations*, In Ordinary and Partial Differential Equations, Lecture Notes in Mathematics, **964** (1982), 187–212.
- [6] J. K. HALE, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
- [7] G. G. HAMEDANI, *Oscillation behavior of nth-order forced functional differential equations*, J. Math. Anal. Appl., **195** (1995), 123–134.
- [8] I. T. KIGURADZE, *On the oscillatory of solutions of the equation $d^m u/dt^m + a(t)|u^n|sgnu = 0$* , Mat. Sb., **65** (1964), 172–187.
- [9] T. KUSANO, Y. NAITO, *Oscillation and nonoscillation criteria for second order quasilinear differential equations*, Acta Math. Hungar., **76** (1997), 81–99.

- [10] T. KUSANO, Y. NAITO, A. OGATA, *Strong oscillation and nonoscillation of quasilinear differential equations of second order*, *Diff. Equat. Dyn. Syst.*, **2** (1994), 1–10.
- [11] T. KUSANO, Y. YOSHIDA, *Nonoscillation theorems for a class of quasilinear differential equations of second order*, *J. Math. Anal. Appl.*, **189** (1995), 115–127.
- [12] D. D. MIRZOV, *On the oscillation of a system of nonlinear differential equations*, *Diferencijal'nye Uravnenija*, **9** (1973), 581–583.
- [13] D. D. MIRZOV, *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, *J. Math. Anal. Appl.*, **53** (1976), 418–425.
- [14] D. D. MIRZOV, *On the oscillation of solutions of a system of differential equations*, *Mat. Zametki*, **23** (1978), 401–404.
- [15] CH. G. PHILOS, *A new criterion for the oscillatory and asymptotic behavior of delay differential equations*, *Bull. Acad. Pol. Sci. Ser. Mat.*, **39** (1981), 61–64.
- [16] Y. G. SUN, F. W. MENG, *Note on the paper of Dzurina and Stavroulakis*, *Appl. Math. Comput.*, **174** (2006), 1634–1641.
- [17] S. G. SHUN, *Oscillation of n th-order functional differential equations*, *Appl. Math. Comput.*, **21**, 3 (1991), 95–102.
- [18] P. WANG, X. FU, Y. YU, *Oscillations of solution for a class of higher order neutral differential equation*, *Appl. Math.*, Ser. B, **13**, 4 (1998), 397–402.
- [19] P. WANG, W. GE, *Oscillation of a class of higher order functional differential equations with damped term*, *Appl. Math. Comput.*, **148** (2004), 351–358.
- [20] P. G. WANG, K. L. TEO, Y. LIU, *Oscillation properties for even order neutral equations with distributed deviating arguments*, *Comput. Math. Appl.*, **182** (2005), 290–303.
- [21] R. XU, F.W. MENG, *Some new Oscillation criteria for second order quasi-linear neutral delay differential equations*, *Appl. Math. Comput.*, **182** (2006), 797–803.

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