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SOME NEW OSCILLATION CRITERIA FOR HIGHER–ORDER QUASI–LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. Oscillation criteria for the higher order quasi-linear neutral delay differential equations of the form

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' + \sum_{i=1}^{m} q_i(t)f_i(|u(\tau_i(t))|^{\alpha_i-1}u(\tau_i(t))) = 0,$$

$t \geqslant t_0,\ z(t) = u(t)+p(t)u(t-\sigma), \alpha > 0, \alpha_i > 0\ (i = 1,2,3,...,m)$, are established under the condition:

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds = \infty \text{ or } \int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds < \infty \text{ respectively,}$$

where n is even. The obtained results improve and extend some known results in literature.

1. Introduction

In this paper, we are here concerned with the oscillation behavior of solution of higher order quasi-linear neutral delay differential equation of the form

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' + \sum_{i=1}^{m} q_i(t)f_i(|u(\tau_i(t))|^{\alpha_i-1}u(\tau_i(t))) = 0, \quad (1.1)$$

where $t \geqslant t_0,\ z(t) = u(t)+p(t)u(t-\sigma),\ \alpha > 0,\ \alpha_i > 0\ (i = 1,2,3,...,m),\ \sigma \geqslant 0$ are constants and n is even.

In this paper, we assume that

(B_1) $p(t) \in C([t_0,\infty),[0,1]),\ \ q_i(t) \in C([t_0,\infty),[0,\infty))(i = 1,2,3,...,m)$, where $p(t)$ and $q_i(t)$ do not identify zero on $(a,b) \subset [0,1], a < b$;

(B_2) $r(t) \in C^1([t_0,\infty),(0,\infty)),\ \ R(t) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)ds$;

(B_3) $\psi(u),\ f_i(u) \in C^1(\mathbb{R},\mathbb{R}),\ \psi(u) > 0,\ uf_i(u) > 0$ for $u \neq 0$, $i = 1,2,3,...,m$, there exists constants $L > 0$ and $\beta_i > 0(i = 1,2,3,...,m)$, such that $\psi(u) \leqslant L^{-1}$ and

$$\frac{f_i(u^{\alpha_i}(\tau_i(t)))}{|u(\tau_i(t))|^{\alpha-1}u(\tau_i(t))} \geqslant \beta_i > 0 \text{ for } u(\tau_i(t)) \neq 0;$$

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(B_4) $\tau(t) \in C^1([t_0,\infty),[0,\infty))$, $\tau(t) \leqslant \tau_i(t)$, $\tau(t) \leqslant t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\tau'(t) > 0$ for $t \in [t_0,\infty)$.

In what follows, we shall consider only the nonconstant solutions of (1.1) which are defined for all large t. The solutions of (1.1) mean a function $u \in C^1([T_u,\infty),\mathbb{R}), T_u \geqslant t_0$ such that u and $r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ are continuously differentiable and satisfy Eq.(1.1). A solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Eq.(1.1) is called oscillatory if all of its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for various classes of second order differential equations, e.g., see [1-6], [8-14], [16] and the references therein. Xu et al. [21] extended the results of Dzurina et al. [3] and Sun et al. [16] to the neutral delay differential equation

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + q(t)f(x(\sigma(t))) = 0, \ t \geqslant t_0, \tag{1.2}$$

where $y(t) = x(t) + p(t)x(t-\tau)$. More precisely, they proved that if

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r^{\frac{1}{\alpha}}(s)} = \infty \tag{1.3}$$

and

$$\int^{\infty} \left[kq(t)(1-p(\sigma(t)))R^{\alpha}(\sigma(t)) - (\frac{\alpha}{\alpha+1})^{\alpha+1} \frac{\sigma'(t)}{R(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))} \right] dt = \infty,$$

then (1.3) is oscillatory.

Very recently, the oscillatory behavior of solutions of higher order neutral differential equations are of both theoretical and practical interest. There have been some results on the oscillatory and asymptotic behavior of even order neutral equations. we mention here [7], [17-19].

In [20], by using Riccati technique and averaging functions method, wang established some general oscillation criteria for even order neutral delay differential equation

$$[r(t) + c(t)x(t-\tau)]^{(n)} + \int_a^b p(t,\xi)x[g(t,\xi)]d\sigma(\xi) = 0, \ t > t_0.$$

In this paper, we shall continue in this direction to study the oscillatory properties of (1.1). Motivated by [19], by using the Riccati technique and the integral averaging technique, under the condition: $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds = \infty$ or $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds < \infty$, we establish the oscillation criteria for (1.1), which extend and improve the main results in [3, 16, 21].

In order to prove our theorems, we will use the following lemmas.

LEMMA 1.1. (Kiguradze [8]) *Let $x(t)$ be a positive and n times differentiable function on R. If $x^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_1,+\infty)$*

for $(t_1 > 0)$, *then there exists a* $t_x > t_1$ *and an integer* $l(0 \leqslant l \leqslant n)$, *with* $n+l$ *even for* $x(t)x^{(n)}(t) \geqslant 0$ *or* $n+l$ *odd for* $x(t)x^{(n)}(t) \leqslant 0$*; and for* $t \geqslant t_x$,

$$x(t)x^{(k)}(t) > 0, \ \ 0 \leqslant k \leqslant l; \ \ (-1)^{k-l}x(t)x^{(k)}(t) > 0, \ \ l \leqslant k \leqslant n.$$

LEMMA 1.2. (Philos [15]) *Suppose that the condition of Lemma 1.1 is satisfied, and*

$$x^{(n-1)}(t)x^{(n)}(t) \leqslant 0, \ \ t \geqslant t_x,$$

then there exists a constant θ *in* $(0,1)$ *such that for sufficiently large* t, *satisfying*

$$|x'(t/2)| \geqslant M_\theta t^{n-2}|x^{(n-1)}(t)|,$$

where $M_\theta = \frac{\theta}{(n-2)!}$ *and* n *is even.*

We define the following notations to be used through this paper

$$R(t) = \int_{t_0}^{t} \frac{ds}{r^{\frac{1}{\alpha}}(s)}, \ \ Q(t) = \sum_{i=1}^{m} q_i(t)\beta_i(1-p(\tau_i(t)))^{\alpha}, \ \ \xi = \left(\frac{2}{M}\right)^{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \quad (1.4)$$

2. Main results

We first consider the oscillatory property of (1.1) under the condition

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\int_{t_0}^{t} \frac{ds}{r^{\frac{1}{\alpha}}(s)} = \infty. \quad (2.1)$$

THEOREM 2.1. *Assume that* (2.1) *holds and*

$$\int^{\infty}\left(R^{\alpha}[\tau(t)]Q(t) - \frac{\xi\tau'(t)}{LR[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))((\tau(t))^{n-2})^{\alpha}}\right)dt = \infty, \quad (2.2)$$

then Eq.(1.1) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.1) has a nonoscillatory solution $u(t)$. Without loss of generally, we assume that

$$u(t) > 0, u(\tau_i(t)) > 0, u(t-\tau) > 0, \ \ t \geqslant t_1 \geqslant t_0.$$

Since the case $u(t) < 0$ can be treated similarly.

From (1.1) and (B_3), we can get

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' \leqslant 0, \ \ z(t) \geqslant u(t). \quad (2.3)$$

Therefore $r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is a decreasing function. We claim that

$$z^{(n-1)}(t) > 0, \ \ t \geqslant t_1.$$

Otherwise, if there exist $t_2 \geqslant t_1$, such that $z^{(n-1)}(t) \leqslant 0$ for all $t \geqslant t_2$, from (2.3) we have

$$r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) \\ \leqslant r(t_2)\psi(u(t_2))|z^{(n-1)}(t_2)|^{\alpha-1}z^{(n-1)}(t_2) = -\beta\ (\beta > 0),\ \ t \geqslant t_2,$$

which implies that

$$z^{(n-1)}(t) \leqslant -\left(\frac{\beta}{\psi(u(t))}\right)^{\frac{1}{\alpha}} \frac{1}{r^{\frac{1}{\alpha}}(t)} \leqslant -(\beta L)^{\frac{1}{\alpha}} \frac{1}{r^{\frac{1}{\alpha}}(t)}.$$

Integrating the above inequality from t_2 to t, we have

$$z^{(n-2)}(t) \leqslant z^{(n-2)}(t_2) - (\beta L)^{\frac{1}{\alpha}} \int_{t_2}^{t} \frac{ds}{r^{\frac{1}{\alpha}}(s)}.$$

Letting $t \to \infty$, from (2.1) we get $\lim_{t\to\infty} z^{(n-2)}(t) = -\infty$, which implies $\lim_{t\to\infty} z(t) = -\infty$. This get a contradiction with $z(t) > 0$.

If there exists $t_3 > t_1$, such that $z^{(n-1)}(t_3) = 0$, since the function

$$r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$$

is decreasing, we can obtain that there exists $t_4 \geqslant t_3$, such that $z^{(n-1)}(t_4) < 0$, get the some contradiction as above.

If there exists $t_6 > t_5 > t_1$, such that $z^{(n-1)}(t_5) = z^{(n-1)}(t_6) = 0$, integrating (1.1) from t_5 to t_6, we have

$$\int_{t_5}^{t_6} \sum_{i=1}^{m} q_i(t) f_i(|u(\tau_i(t))|^{\alpha_i-1}u(\tau_i(t))) = 0,$$

which arrive a contradiction with main assumption of the paper. Hence $z^{(n-1)}(t) > 0$ for $t \geqslant t_1$. From (2.3), we can get

$$[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]' \\ = [r(t)\psi(u(t))]'(z^{(n-1)}(t))^{\alpha} + \alpha r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha-1}z^{(n)}(t) \leqslant 0,$$

then

$$z^{(n)}(t) \leqslant \frac{-[r(t)\psi(u(t))]'z^{(n-1)}(t)}{\alpha r(t)\psi(u(t))} \leqslant 0.$$

From Lemma 1.1 and Lemma 1.2, we have

$$z^{(n-1)}(t) > 0,\ \ z^{(n)}(t) \leqslant 0,\ \ z'(t) > 0,\ \ t > t_1,$$

$$z'\left(\frac{\tau(t)}{2}\right) \geqslant M(\tau(t))^{n-2} z^{(n-1)}(\tau(t)). \tag{2.4}$$

Since $u(t) \leqslant z(t)$,

$$
\begin{aligned}
u(t) = z(t) - p(t)u(t-\sigma) &\geqslant z(t) - p(t)z(t-\sigma) \\
&\geqslant z(t) - p(t)z(t) = (1-p(t))z(t).
\end{aligned} \tag{2.5}
$$

Therefore, we have

$$
r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha} \leqslant r(\tau(t))\psi(u(\tau(t)))(z^{(n-1)}(\tau(t)))^{\alpha}.
$$

It follows that

$$
\frac{z^{(n-1)}(\tau(t))}{z^{(n-1)}(t)} \geqslant \left(\frac{r(t)\psi(u(t))}{r(\tau(t))\psi(u(\tau(t)))}\right)^{\frac{1}{\alpha}}. \tag{2.6}
$$

Following (B_3) and (2.5), we get

$$
[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]' + \sum_{i=1}^{m} q_i(t)\beta_i(1-p(\tau_i(t)))^{\alpha} z^{\alpha}(\tau_i(t)) \leqslant 0,
$$

then Eq.(1.1) will become

$$
[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]' + Q(t)z^{\alpha}(\tau(t)) \leqslant 0. \tag{2.7}
$$

Define

$$
w(t) = R^{\alpha}[\tau(t)]\frac{r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}}{z^{\alpha}\left(\frac{\tau(t)}{2}\right)}, \quad t \geqslant t_1. \tag{2.8}
$$

Then $w(t) \geqslant 0$. From (2.4), (2.6), (2.7) and (2.8), we get

$$
\begin{aligned}
w'(t) \leqslant{} & \frac{\alpha\tau'(t)R^{\alpha-1}[\tau(t)]}{r^{\frac{1}{\alpha}}[\tau(t)]}\frac{r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}}{z^{\alpha}\left(\frac{\tau(t)}{2}\right)} - \frac{R^{\alpha}[\tau(t)]}{z^{\alpha}\left(\frac{\tau(t)}{2}\right)}(Q(t)z^{\alpha}(\tau(t))) \\
& - R^{\alpha}[\tau(t)][r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]\frac{\alpha z'(\frac{\tau(t)}{2})\frac{1}{2}\tau'(t)}{z^{\alpha+1}(\frac{\tau(t)}{2})} \\
\leqslant{} & \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}w(t) - R^{\alpha}[\tau(t)]Q(t) \\
& - \frac{\alpha\tau'(t)}{2}\frac{R^{\alpha}[\tau(t)][r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]M(\tau(t))^{n-2}z^{(n-1)}(\tau(t))}{z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)} \\
\leqslant{} & \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}w(t) - R^{\alpha}[\tau(t)]Q(t) \\
& - \frac{M\alpha\tau'(t)R^{\alpha}[\tau(t)][r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha+1}](\tau(t))^{n-2}}{2z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)}\frac{z^{(n-1)}(\tau(t))}{z^{(n-1)}(t)} \\
\leqslant{} & \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}w(t) - R^{\alpha}[\tau(t)]Q(t)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{M\alpha\tau'(t)R^{\alpha}[\tau(t)][r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha+1}](\tau(t))^{n-2}}{2z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)}\left(\frac{r(t)\psi(u(t))}{r(\tau(t))\psi(u(\tau(t)))}\right)^{\frac{1}{\alpha}}\\
&\leqslant \frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}w(t)-R^{\alpha}[\tau(t)]Q(t)-\frac{M\alpha L^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2R[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))}w^{\frac{\alpha+1}{\alpha}}(t). \qquad (2.9)
\end{aligned}
$$

Set

$$
F(\upsilon)=\frac{\alpha\tau'(t)}{R[\tau(t)]r^{\frac{1}{\alpha}}[\tau(t)]}\upsilon-\frac{M\alpha L^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2R[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))}\upsilon^{\frac{\alpha+1}{\alpha}},\ \ \upsilon>0.
$$

By calculating, we have that when

$$
\upsilon=\frac{(2\alpha)^{\alpha}}{L(M(\alpha+1)(\tau(t))^{n-2})^{\alpha}},
$$

$F(\upsilon)$ obtain its maximum. So

$$
F(\upsilon)\leqslant F_{max}=\frac{\xi\tau'(t)}{LR[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))((\tau(t))^{n-2})^{\alpha}}.
$$

Therefore,

$$
w'(t)\leqslant -R^{\alpha}[\tau(t)]Q(t)+\frac{\xi\tau'(t)}{LR[\tau(t)]r^{\frac{1}{\alpha}}(\tau(t))((\tau(t))^{n-2})^{\alpha}}.
$$

Integrating the above inequality from $T_0 (T_0 \geqslant t_1)$ to t, we have

$$
\begin{aligned}
&0<w(t)\\
&\quad< w(T_0)-\int_{T_0}^{t}\left[R^{\alpha}[\tau(s)]Q(s)-\frac{\xi\tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))((\tau(s))^{n-2})^{\alpha}}\right]ds. \qquad (2.10)
\end{aligned}
$$

Letting $t\to\infty$ in (2.10), from (2.2) we get a contradiction. This completes the proof of Theorem 2.1.

COROLLARY 2.1. *Assume that* (2.1) *holds and for some* $T \geqslant t_0$,

$$
\liminf_{t\to\infty}\frac{1}{\ln R[\tau(t)]}\int_{T}^{t}R^{\alpha}[\tau(s)]Q(s)ds>\frac{\xi}{L}, \qquad (2.11)
$$

then Eq (1.1) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.1) has a nonoscillatory solution $u(t)$. Without of loss generally, we assume that $u(t)>0$, $u(\tau_i(t))>0$, $u(t-\tau)>0$ for $t\geqslant t_1\geqslant t_0$. Since the case $u(t)<0$ can be treated similarly.

It is easily prove that there exist $T \geqslant t_0$, such that $(\tau(t))^{n-2} > 1$. From (2.11), we can get that yields the existence $\varepsilon > 0$, such that for all large t,

$$\frac{1}{\ln R[\tau(t)]}\int_T^t R^\alpha[\tau(s)]Q(s)ds \geqslant \frac{\xi}{L}+\varepsilon,$$

which follows that

$$\begin{aligned}
&\int_T^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi\tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))((\tau(s))^{n-2})^\alpha}\right]ds \\
&\qquad > \int_T^t \left[R^\alpha[\tau(s)]Q(s) - \frac{\xi\tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))}\right]ds \\
&\qquad = \int_T^t R^\alpha[\tau(s)]Q(s)ds - \frac{\xi}{L}[\ln R(\tau(t)) - \ln R(\tau(T))] \\
&\qquad > \left(\frac{\xi}{L}+\varepsilon\right)\ln R(\tau(t)) - \frac{\xi}{L}[\ln R(\tau(t)) - \ln R(\tau(T))] \\
&\qquad = \varepsilon \ln R(\tau(t)) + \frac{\xi}{L}\ln R(\tau(T)).
\end{aligned} \tag{2.12}$$

Now, it is obvious that (2.12) implies (2.2) and the assertion of corollary 2.1 follows from Theorem 2.1.

COROLLARY 2.2. *Assume that* (2.1) *holds, and*

$$\liminf_{t\to\infty}\frac{r^{\frac{1}{\alpha}}[\tau(t)]R^{\alpha+1}[\tau(t)]Q(t)}{\tau'(t)} > \frac{\xi}{L}, \tag{2.13}$$

then Eq.(1.1) is oscillatory.

Proof. Suppose to the contrary that Eq.(1.1) has a nonoscillatory solution $u(t)$. Without of loss generally, we assume that $u(t) > 0$, $u(\tau_i(t)) > 0$, $u(t-\tau) > 0$ for $t \geqslant t_1 \geqslant t_0$. Since the case $u(t) < 0$ can be treated similarly.

It is easily prove that there exist $T > t_0$, such that $(\tau(t))^{n-2} > 1$. From (2.12), we can get that yields the existence $\varepsilon > 0$, such that for all large t,

$$\frac{r^{\frac{1}{\alpha}}[\tau(t)]R^{\alpha+1}[\tau(t)]Q(t)}{\tau'(t)} \geqslant \frac{\xi}{L}+\varepsilon, \tag{2.14}$$

multiplying $\frac{\tau'(t)}{r^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]}$ on both sides of (2.14), we obtain that

$$R^\alpha[\tau(t)]Q(t) - \frac{\xi}{L}\frac{\tau'(t)}{r^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]} \geqslant \varepsilon\frac{\tau'(t)}{r^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]}.$$

Therefore, we have

$$\begin{aligned}\int_T^t \left(R^{\alpha}[\tau(s)]Q(s) - \frac{\xi}{L}\frac{\tau'(s)}{r^{\frac{1}{\alpha}}[\tau(s)]R[\tau(s)]}\right)ds \\ \geqslant \varepsilon \int_T^t \frac{\tau'(s)}{r^{\frac{1}{\alpha}}[\tau(s)]R[\tau(s)]}ds \\ = \varepsilon(\ln R(\tau(t)) - \ln R(\tau(T))).\end{aligned}$$

Then

$$\begin{aligned}\int_T^t \left[R^{\alpha}[\tau(s)]Q(s) - \frac{\xi\tau'(s)}{LR[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))((\tau(s))^{n-2})^{\alpha}}\right]ds \\ > \int_T^t \left[R^{\alpha}[\tau(s)]Q(s) - \frac{\xi}{L}\frac{\tau'(s)}{R[\tau(s)]r^{\frac{1}{\alpha}}(\tau(s))}\right]ds \\ > \varepsilon(\ln R(\tau(t)) - \ln R(\tau(T))). \qquad (2.15)\end{aligned}$$

It is obvious that (2.15) implies (2.2) and Corollary 2.2 is evident by Theorem 2.1.

If $p(t) = 0$, from Theorem 2.1 we obtain the following conclusions.

COROLLARY 2.3. *Assume that* (2.1) *holds and*

$$\int^{\infty} \left(R^{\alpha}[\tau(t)]\sum_{i=1}^{m} q_i(t)\beta_i - \frac{\xi\tau'(t)}{Lr^{\frac{1}{\alpha}}[\tau(t)]R[\tau(t)]((\tau(t))^{n-2})^{\alpha}}\right)dt = \infty, \qquad (2.16)$$

then Eq.(1.1) with $p(t) = 0$ *is oscillatory.*

COROLLARY 2.4. *Assume that* (2.1) *holds and for some* $t_1 > t_0$*,*

$$\liminf_{t\to\infty} \frac{1}{\ln R[\tau(t)]}\int_{t_1}^{t} R^{\alpha}[\tau(s)]\sum_{i=1}^{m} q_i(s)\beta_i ds > \frac{\xi}{L}, \qquad (2.17)$$

then Eq (1.1) with $p(t) = 0$ *is oscillatory.*

COROLLARY 2.5. *Assume that* (2.1) *holds, and*

$$\liminf_{t\to\infty} \sum_{i=1}^{m} q_i(t)\beta_i \frac{r^{\frac{1}{\alpha}}[\tau(t)]R^{\alpha+1}[\tau(t)]}{\tau'(t)} > \frac{\xi}{L}, \qquad (2.18)$$

then Eq.(1.1) is oscillatory.

THEOREM 2.2. *Let* (2.1) *holds. If there exists a function* $h \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that*

$$\int^{\infty}\left(Q(t)-\frac{ML^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}h^{\alpha+1}(t)}{2r^{\frac{1}{\alpha}}(\tau(t))}\right)\exp\left(\frac{(\alpha+1)ML^{\frac{1}{\alpha}}}{2}\int^{t}\frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))}ds\right)dt=\infty, \quad (2.19)$$

then (1.1) is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, as in the proof of Theorem 2.1, we may assume that there exists a number $t_1 \geqslant t_0$ such that $u(t)>0,\ \ u(\tau_i(t))>0,\ \ u(t-\tau)>0$ for $t \geqslant t_1$. Since the case $u(t)<0$ can be treated similarly.

From the proof of Theorem 2.1, we get that $z^{(n-1)}(t)>0$ holds for $t \geqslant t_1$. Furthermore, (2.6) and (2.7) hold. Define

$$w(t)=\frac{r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}}{z^{\alpha}\left(\frac{\tau(t)}{2}\right)},\ \ t \geqslant t_1. \quad (2.20)$$

Obviously, $w(t)>0$. Differentiating (2.20), in view of (2.6) and (2.7), we have

$$\begin{aligned}
w'(t)&=\frac{[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]'}{z^{\alpha}\left(\frac{\tau(t)}{2}\right)}-[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]\frac{\alpha z'(\frac{\tau(t)}{2})\frac{1}{2}\tau'(t)}{z^{\alpha+1}(\frac{\tau(t)}{2})}\\
&\leqslant-\frac{Q(t)z^{\alpha}(\tau(t))}{z^{\alpha}\left(\frac{\tau(t)}{2}\right)}-\frac{\alpha\tau'(t)}{2}\frac{[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}]M(\tau(t))^{n-2}z^{(n-1)}(\tau(t))}{z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)}\\
&\leqslant-Q(t)-\frac{M\alpha\tau'(t)[r(t)\psi(u(t))(z^{(n-1)}(t))^{\alpha}](\tau(t))^{n-2}}{2z^{\alpha+1}\left(\frac{\tau(t)}{2}\right)}\left(\frac{r(t)\psi(u(t))}{r(\tau(t))\psi(u(\tau(t)))}\right)^{\frac{1}{\alpha}}\\
&\leqslant-Q(t)-\frac{M\alpha L^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))}w^{\frac{\alpha+1}{\alpha}}(t)\\
&=-\left[Q(t)-\frac{ML^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))}h^{\alpha+1}(t)\right]\\
&\qquad-\frac{ML^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))}\left[\alpha w^{\frac{\alpha+1}{\alpha}}(t)+h^{\alpha+1}(t)\right].
\end{aligned}$$

By Holder's inequality, $\alpha w^{\frac{\alpha+1}{\alpha}}(t)+h^{\alpha+1}(t)\geqslant(\alpha+1)h(t)w(t)$. Thus,

$$w'(t)\leqslant-\left[Q(t)-\frac{ML^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))}h^{\alpha+1}(t)\right]-\frac{(\alpha+1)ML^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}h(t)w(t)}{2r^{\frac{1}{\alpha}}(\tau(t))},$$

which follows that

$$\left[\exp\left(\frac{(\alpha+1)ML^{\frac{1}{\alpha}}}{2}\int_T^t \frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))}ds\right)w(t)\right]'$$
$$\leqslant -\left[Q(t)-\frac{ML^{\frac{1}{\alpha}}\tau'(t)(\tau(t))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(t))}h^{\alpha+1}(t)\right]\times$$
$$\exp\left(\frac{(\alpha+1)ML^{\frac{1}{\alpha}}}{2}\int_T^t \frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))}ds\right). \quad (2.21)$$

Integrating (2.21) from T $(T \geqslant t_1)$ to t, we have

$$0<\exp\left(\frac{(\alpha+1)ML^{\frac{1}{\alpha}}}{2}\int_T^t \frac{\tau'(s)(\tau(s))^{n-2}h(s)}{r^{\frac{1}{\alpha}}(\tau(s))}ds\right)w(t)$$
$$\leqslant w(T)-\int_T^t\left[Q(s)-\frac{ML^{\frac{1}{\alpha}}\tau'(s)(\tau(s))^{n-2}}{2r^{\frac{1}{\alpha}}(\tau(s))}h^{\alpha+1}(s)\right]$$
$$\exp\left(\frac{(\alpha+1)ML^{\frac{1}{\alpha}}}{2}\int_T^s \frac{\tau'(u)(\tau(u))^{n-2}h(u)}{r^{\frac{1}{\alpha}}(\tau(u))}du\right)ds.$$

Let $t \to \infty$ in the above inequality, which contradicts(2.19). This completes the proof of Theorem 2.3.

Now, let us consider the case when

$$\lim_{t\to\infty} R(t)=\lim_{t\to\infty}\int_{t_0}^t \frac{ds}{r^{\frac{1}{\alpha}}(s)}<\infty. \quad (2.22)$$

THEOREM 2.3. *Assume that* (2.2) *and* (2.22) *hold. Suppose that there exist a continuously differentiable* $\varphi(t)$*, such that* $\varphi(t)>0, \varphi'(t)>0. p'(t) \geqslant 0$ *for* $t \geqslant t_0$*,* $\lim_{t\to\infty} p(t)=A$*, if*

$$\int^\infty \left(\frac{L}{\varphi(t)r(t)}\int^t\left(\sum_{i=1}^m q_i(s)\beta_i\varphi(s)\right)ds\right)^{\frac{1}{\alpha}}dt=\infty, \quad (2.23)$$

then every solution $u(t)$ *of* (1.1) *oscillates or* $\lim_{t\to\infty} u(t)=0$.

Proof. Suppose to the contrary that $u(t)$ is an eventually positive solution of (1.1), without loss of generally , we assume that $u(t)>0$, $t>t_1$, then $z(t)>0$.

It is easy to conclude that $r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is a decreasing function. There exist two possible case of the sign of $z^{(n-1)}(t)$, $z^{(n-1)}(t)>0$ or $z^{(n-1)}(t)<0$.

CASE (1). Suppose $z^{(n-1)}(t)>0$ for sufficiently large t, then we are back to the case of Theorem 2.1. Thus the proof of Theorem 2.1 goes through, and we may get contradition by (2.2).

CASE (2). Suppose $z^{(n-1)}(t) < 0$ for sufficiently large t, from Lemma 1.1 and Lemma 1.2 we can easily get that $z'(t) < 0$, since $p'(t) \geqslant 0$, $z'(t) = u'(t) + p'(t)u(t-\sigma) + p(t)u'(t-\sigma)$, then $u'(t) \leqslant 0$. It follows that $\lim_{t\to\infty} z(t) = a \geqslant 0$, now we claim that $a = 0$. Otherwise, $\lim_{t\to\infty} z(t) = a > 0$, so $\lim_{t\to\infty} u(t) = \frac{a}{1+A} > 0$, there exist constants $M_i > 0$ $(i = 1,2,3,...,m)$ and $M > 0$ such that $u^{\alpha}(\tau_i(t)) \geqslant M_i$ and $M_i \geqslant M$ for all $t \geqslant t_1 \geqslant t_0$. From (1.1) we get

$$[r(t)\psi(u(t))|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]' + \sum_{i=1}^{m} q_i(t)f_i(u^{\alpha_i}(\tau_i(t))) = 0.$$

Then

$$\begin{aligned}
[r(t)\psi(u(t))(-z^{(n-1)}(t))^{\alpha}]' &= \sum_{i=1}^{m} q_i(t)f_i(u^{\alpha_i}(\tau_i(t))) \\
&\geqslant \sum_{i=1}^{m} q_i(t)\beta_i u^{\alpha}(\tau_i(t)) \\
&\geqslant \sum_{i=1}^{m} q_i(t)\beta_i M_i \geqslant M\sum_{i=1}^{m} q_i(t)\beta_i
\end{aligned}$$

for $t \geqslant t_1$. Define $v(t) = \varphi(t)r(t)\psi(u(t))(-z^{(n-1)}(t))^{\alpha}$, then $v(t) \geqslant 0$,

$$\begin{aligned}
v'(t) &= \varphi(t)[r(t)\psi(u(t))(-z^{(n-1)}(t))^{\alpha}]' + \varphi'(t)[r(t)\psi(u(t))(-z^{(n-1)}(t))^{\alpha}] \\
&\geqslant \varphi(t)[r(t)\psi(u(t))(-z^{(n-1)}(t))^{\alpha}]' \\
&\geqslant M\varphi(t)\sum_{i=1}^{m} q_i(t)\beta_i. \qquad (2.24)
\end{aligned}$$

Integrating (2.24) from t_1 to t, we have

$$v(t) \geqslant v(t_1) + M\int_{t_1}^{t}\left(\sum_{i=1}^{m} q_i(s)\beta_i\varphi(s)\right)ds \geqslant M\int_{t_1}^{t}\left(\sum_{i=1}^{m} q_i(s)\beta_i\varphi(s)\right)ds,$$

that is

$$\varphi(t)r(t)\psi(u(t))(-z'(t))^{\alpha} \geqslant M\int_{t_1}^{t}\left(\sum_{i=1}^{m} q_i(s)\beta_i\varphi(s)\right)ds,$$

so that

$$(-z'(t)) \geqslant M^{\frac{1}{\alpha}}\left(\frac{L}{\varphi(t)r(t)}\int_{t_1}^{t}\left(\sum_{i=1}^{m} q_i(s)\beta_i\varphi(s)\right)ds\right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality form t_1 to t, we obtain

$$z(t) \leqslant z(t_1) - M^{\frac{1}{\alpha}}\int_{t_1}^{t}\left(\frac{L}{\varphi(s)r(s)}\int_{t_1}^{s}\left(\sum_{i=1}^{m} q_i(\xi)\beta_i\varphi(\xi)\right)d\xi\right)^{\frac{1}{\alpha}}ds.$$

We can easily obtain a contradiction. So that $\lim_{t\to\infty} z(t) = 0$, then $\lim_{t\to\infty} u(t) = 0$. This completes the proof of Theorem 2.4.

3. Example

Consider the following high-order differential equation

$$\left(\frac{1}{t^{\alpha}(1+u^2(t))}|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\right)' + t|u(t)|^{\alpha-1}u(t) = 0, \ \ t \geqslant 1, \tag{3.1}$$

where

$$p(t) = \frac{1}{2}, \ \ r(t) = t^{-\alpha}, \ \ \sigma(t) = t-1,$$
$$\psi(u) = \frac{1}{1+u^2}, \ \ q(t) = t, \ \ f(u) = u, \ \ \tau(t) = t.$$

If we take $\beta = L = 1$, then

$$R(\tau(t)) = \int_1^t s ds = \frac{1}{2}(t^2-1).$$

For any constant $T > t_0$, we have

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{\ln R(\tau(t))}\int_1^t R^{\alpha}(\tau(s))Q(s)ds \\
&= \liminf_{t\to\infty} \frac{1}{\ln(\frac{1}{2}(t^2-1))}\int_1^t \frac{1}{4^{\alpha}}(s^2-1)^{\alpha} s ds \\
&= \frac{1}{2^{2\alpha+1}} \liminf_{t\to\infty}(t^2-1)^{\alpha+1} = \infty,
\end{aligned}$$

from Corollary 2.1, we can see that Eq.(3.1) is oscillatory.

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