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EXISTENCE OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS AT FULL RESONANCE

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Abstract. The focus of this paper is the study of nonlinear differential equations of the form

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\varepsilon, t, x_1(t), \cdots, x_n(t)), \;\; i = 1, 2, \cdots, n,$$

subject to two-point boundary conditions

$$b_i x_i(0) + d_i x_i(1) = 0, \;\; i = 1, 2, \cdots, n.$$

We formulate sufficient conditions for the existence of solutions based on the dimension of the solution space of the corresponding linear, homogeneous equation and the properties of the non-linear term when $\varepsilon = 0$. We focus on the case when the solution space of the corresponding linear, homogeneous equation is n-dimensional; that is, when the system is at full resonance. The argument we use relies on the Lyapunov-Schmidt procedure and the Schauder fixed point theorem.

1. Introduction

In this paper, we establish criteria for the existence of solutions to the parameter dependent vector equation

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\varepsilon, t, x_1(t), \cdots, x_n(t)), \;\; i = 1, 2, \cdots, n, \tag{1}$$

subject to the boundary conditions

$$b_i x_i(0) + d_i x_i(1) = 0, \;\; i = 1, 2, \cdots, n. \tag{2}$$

We focus on the case where the solution space of the corresponding linear, homogeneous vector equation

$$\dot{x}_i(t) - a_i(t)x_i(t) = 0, \;\; i = 1, 2, \cdots, n, \tag{3}$$

subject to boundary conditions (2) is n-dimensional. We will provide sufficient conditions for existence of solutions to (1), (2). The asymptotic behavior of the function

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$f_i(0,t,x_1(t),\cdots,x_n(t))$ and the solution space of the linear, homogeneous boundary value problem (3), (2) will play crucial roles in establishing sufficient conditions.

Our analysis depends, in a fundamental way, on the nature of the boundary conditions. It is evident that periodic boundary value problems represent an important special case of (1), (2). In fact, the study of periodic behavior and its generalizations motivate the approach presented in this paper.

Our technique used to establish existence of solutions to (1), (2) relies on the Lyapunov-Schmidt Procedure. Ideas and techniques similar to the ones used in this paper were successfully applied to the study of discrete and continuous dynamical systems. A general abstract approach appears in [1], [3], [4]; [2], [9], [10], [21] present applications to periodic solutions of ordinary differential equations; discrete-time periodic solutions are considered in [5], [7]; existence results for boundary value problems in both differential and difference equations are established in [6], [13], [15], [16], [18], [19]; a functional analytic approach to the study of strongly nonlinear boundary value problems appears in [11], [12], [14], [17].

2. Preliminaries

Before we establish solvability criteria for (1), (2), we will first analyze the linear, homogeneous boundary value problem (3), (2). It is easily verified that solutions to (3), (2) are of the form

$$\phi(t) = \Phi(t)v,$$

where $v \in \mathbb{R}^n$ and $\Phi(t)$ is the matrix

$$\Phi(t) = \begin{bmatrix} e^{\int_0^t a_1(s)ds} & 0 & 0 & \cdots & 0 \\ 0 & e^{\int_0^t a_2(s)ds} & 0 & \cdots 0 & \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\int_0^t a_n(s)ds} \end{bmatrix}.$$

Note that solutions to the nonhomogeneous equation

$$\dot{x}_i(t) = a_i(t)x_i(t) + h_i(t), \;\; i = 1,2,\cdots,n \tag{4}$$

have the form

$$x_i(t) = e^{\int_0^t a_i(s)ds}x_i(0) + e^{\int_0^t a_i(s)ds}\int_0^t e^{-\int_0^s a_i(r)dr}h_i(s)ds.$$

Thus, the boundary value problem (4), (2) is solvable when

$$d_i\int_0^1 e^{-\int_0^s a_i(r)dr}h_i(s)ds = 0$$

for each $i = 1,2,\cdots,n$.

We now wish to analyze the solvability of (1), (2). In order to do this, we will introduce notation that allows us to proceed using functional analysis tools.

We define $L : D(L) \to \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$ by

$$Lx = \dot{x} - Ax,$$

where

$$D(L) = \mathscr{C}^1([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \cap X$$

and

$$X = \{x \in \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \mid Bx(0) + Dx(1) = 0\}.$$

Here, we use the notation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}.$$

The spaces $\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$ and $\mathscr{C}^1([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$ will denote respectively $\{\phi : [0,1] \to \mathbb{R}^n : \phi \text{ is continuous}\}$ and $\{\phi : [0,1] \to \mathbb{R}^n : \phi \text{ is continuously differentiable}\}$. The norm used on these spaces is the sup norm; that is, $\|\phi\|_\infty = \sup\{|\phi(t)| : 0 \leqslant t \leqslant 1\}$ where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

We let $F : \mathbb{R} \times \mathbb{R} \times \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \to \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$ be given by

$$F(\varepsilon,x)(t) = \begin{bmatrix} f_1(\varepsilon,t,x(t)) \\ f_2(\varepsilon,t,x(t)) \\ \vdots \\ f_n(\varepsilon,t,x(t)) \end{bmatrix}.$$

For simplicity, we will write $F(0,x)(t) = F(x)(t)$. We assume f_i for $i = 1,\cdots,n$ is continuous and $\sup\{|f_i(0,u)| : u \in \mathbb{R}^{n+1}\} \leqslant m$ for some $m \in \mathbb{R}$. Hence, F is continuous and, for all $x \in X$, $\|F(x)\|_\infty \leqslant m$.

With this notation, the problem (1),(2) is equivalent to $Lx = F(\varepsilon,x)$. We first consider the particular case when $\varepsilon = 0$. This is equivalent to the operator equation $Lx = Fx$.

The fact that L is not invertible makes it impossible to establish the solvability of $Lx = Fx$ by a direct use of the Schauder Fixed Point Theorem. Instead, we will analyze this operator equation with a projection scheme usually referred to as the Lyapunov-Schmidt Procedure. For the reader's convenience, we provide all the necessary background. We will exploit the structure of the linear system, discussed above, in the construction of the projections. For an abstract formulation of the methods used below and for a vast number of applications of these methods, we refer the interested reader to [3],[4],[8].

For

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix},$$

if follows that

$$Lx = h \quad \text{if and only if} \quad d_i \int_0^1 e^{-\int_0^s a_i(r)dr} h_i(s)ds = 0 \ \text{ for each } i = 1, 2, \cdots, n.$$

We see that this simplifies to $Lx = h$ if and only if $\int_0^1 \Phi^{-1}(t)h(t)dt = 0$.

The projections we define below are familiar to the Lyapunov-Schmidt Procedure. We will now provide a self-contained presentation of the Lyapunov-Schmidt Procedure for the reader's convenience.

By direct computation, we can verify that the maps $P : X \to X$ defined by

$$Px_i(t) = \frac{e^{\int_0^t a_i(s)ds}}{\int_0^1 e^{2\int_0^t a_i(s)ds}dt} \int_0^1 e^{\int_0^s a_i(l)dl} x_i(s)ds$$

and $E : \mathscr{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) \to \mathscr{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty)$ defined by

$$Ex_i(t) = x_i(t) - \frac{e^{-\int_0^t a_i(s)ds}}{\int_0^1 e^{-2\int_0^t a_i(s)ds}dt} \int_0^1 e^{-\int_0^s a_i(l)dl} x_i(s)ds$$

are projections and $\operatorname{Im}(P) = \ker(L)$ and $\operatorname{Im}(E) = \operatorname{Im}(L)$. This allows us to write $X = \ker(L) \oplus \operatorname{Im}(I - P)$ and $\mathscr{C}([0,1], \mathbb{R}^n, \|\cdot\|_\infty) = \operatorname{Im}(L) \oplus \operatorname{Im}(I - E)$.

REMARK 2.1. If $\tilde{L}$ is the restriction of L to $D(L) \cap \operatorname{Im}(I - P)$ then $\operatorname{Im}(\tilde{L}) = \operatorname{Im}(L)$. $\tilde{L}$, viewed as a map from $D(L) \cap \operatorname{Im}(I - P)$ into $\operatorname{Im}(L)$ is invertible. We denote $(\tilde{L})^{-1}$ by M and note that $MLx = (I - P)x$. Later, we will use the fact that M is compact.

PROPOSITION 2.2. *$Lx = F(x)$ is equivalent to*

$$\begin{cases} x = Px + MEF(x) \\ \textit{and} \\ (I - E)F(Px + MEF(x)) = 0. \end{cases}$$

Proof. Using the fact that E is a projection, we have $Lx = Fx$ if and only if

$$\begin{cases} E(Lx - Fx) = 0 \\ \text{and} \\ (I - E)(Lx - Fx) = 0. \end{cases}$$

Since $(I-E)L=0$ and $EL=L$, this is equivalent to

$$\begin{cases} Lx = EF(x) \\ \text{and} \\ (I-E)F(x)=0. \end{cases}$$

Applying M to the first equation, we obtain

$$\begin{cases} (I-P)x = MEF(x) \\ \text{and} \\ (I-E)F(x)=0. \end{cases}$$

From this, we conclude that $Lx=F(x)$ is equivalent to

$$\begin{cases} x = Px+MEF(x) \\ \text{and} \\ (I-E)F(Px+MEF(x))=0. \end{cases}$$

3. Main Results

According to Proposition 2.2, $Lx=Fx$ if and only if

$$\begin{cases} x = \beta_1\Phi_1(t)+\cdots+\beta_n\Phi_n(t)+MEF(x) \\ 0=\int_0^1(\Phi_1(t))^{-1}f_1(0,t,\beta_1\Phi_1(t)+\cdots+\beta_n\Phi_n(t)+MEF(x)(t))dt \\ \vdots \\ 0=\int_0^1(\Phi_n(t))^{-1}f_n(0,t,\beta_1\Phi_1(t)+\cdots+\beta_n\Phi_n(t)+MEF(x)(t))dt), \end{cases} \tag{5}$$

where $\Phi_i(t)=e^{\int_0^t a_i(s)ds}$.

LEMMA 3.1. *Suppose that:*
(i) $b_i+d_ie^{\int_0^1 a_i(s)ds}=0$ *for all* $i=1,2,\cdots,n$;
(ii) $f_i:\mathbb{R}^{n+2}\to\mathbb{R}$ *is continuous for all* $i=1,\cdots,n$;
(iii) *for each* $i=1,\cdots,n$, *there exists* $\gamma_i\in\mathbb{R}$ *such that*

$$f_i(0,t,\beta_1,\cdots,\beta_i,\cdots,\beta_n)f_i(0,t,\beta_1,\cdots,-\beta_i,\cdots,\beta_n)<0$$

whenever $\beta_i\geqslant\gamma_i$.
Then (5) *has a solution.*

Proof. Assume, without loss of generality, that for

$$\beta_i\geqslant\gamma_i,\ \ (\Phi_i(t))^{-1}f_i(0,t,\beta_1,\cdots,\beta_i,\cdots,\beta_n)>0.$$

We define mappings

$$H_1:\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R}^n\to\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty),$$

$$H_{i+1} : \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R}^n \to \mathbb{R}, \text{ for } i=1,\cdots,n,$$

by

$$H_1(x,\beta_1,\cdots,\beta_n) = \beta_1\Phi_1(t)+\cdots+\beta_n\Phi_n(t)+MEF(x),$$

$$\begin{aligned}&H_{i+1}(x,\beta_1,\cdots,\beta_n)\\&\qquad = \beta_i - \int_0^1 (\Phi_1(t))^{-1} f_i(0,t,\beta_1\Phi_1(t)+\cdots+\beta_n\Phi_n(t)+MEF(x)(t))dt,\end{aligned}$$

and

$$H(x,\beta_1,\cdots,\beta_n) = (H_1(x,\beta_1,\cdots,\beta_n),\cdots,H_{n+1}(x,\beta_1,\cdots,\beta_n)).$$

If β_i is sufficiently large, we have

$$\Phi_i(t)^{-1} f_i(0,t,\beta_1\Phi_1(t)+\cdots+\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)+MEF(x)(t)) > 0$$

and

$$\Phi_i(t)^{-1} f_i(0,t,\beta_1\Phi_1(t)+\cdots-\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)+MEF(x)(t)) < 0$$

for all $t\in[0,1]$ and every $x\in\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$. Therefore there is some $\alpha_i \geqslant m\|(\Phi_i)^{-1}\|_\infty$ such that for all $\beta_i\geqslant\alpha_i,\ x\in\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$,

$$H_{i+1}(x,\beta_1,\cdots,\beta_i,\cdots,\beta_n) < \beta_i \quad\text{and}\quad H_{i+1}(x,\beta_1,\cdots,-\beta_i,\cdots,\beta_n) > -\beta_i.$$

Letting $\delta = \max\{\alpha_i + m\|(\Phi_i)^{-1}\|_\infty\}$, define

$$\begin{aligned}\mathscr{B} = \{&(x,\beta_1,\cdots,\beta_n)\in\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R}^n :\\&\|x\|_\infty \leqslant \delta(\|\Phi_1\|_\infty+\cdots+\|\Phi_n\|_\infty)+m|||ME|||,\ |\beta_i|\leqslant\delta \text{ for } i=1,\cdots,n\}.\end{aligned}$$

Here, $|||ME|||$ denotes the operator norm of the bounded, linear map ME. Since M is compact, we will show that the completely continuous function H maps the non-empty, closed, bounded, convex set $\mathscr{B}$ into itself. Then the Schauder Fixed Point Theorem will guarantee the existence of a fixed point, $(x,\beta_1,\cdots,\beta_n)$, of H in $\mathscr{B}$. This fixed point is a solution of (5).

Note that $\|MEF(x)\|_\infty \leqslant m|||ME|||$ for every $x\in\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$. Now if $\beta_i\in[\alpha_i,\delta]$, for all $x\in\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$, we have

$$\begin{aligned}&H_{i+1}(x,\beta_1,\cdots,\beta_i,\cdots,\beta_n)\\&\qquad = \beta_i - \int_0^1 (\Phi_i(t))^{-1} f_i\big(0,t,\beta_1\Phi_1(t)+\cdots+\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)\\&\qquad\qquad\qquad +MEF(x)(t)\big)dt\\&\qquad \geqslant \beta_i - \int_0^1 |(\Phi_i(t))^{-1}||f_i\big(0,t,\beta_1\Phi_1(t)+\cdots+\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)\\&\qquad\qquad\qquad +MEF(x)(t)\big)|dt\end{aligned}$$

$$\geqslant \beta_i - m\|(\Phi_i)^{-1}\|_\infty \geqslant \beta_i - \alpha_i \geqslant 0$$

and

$$\begin{aligned}
&H_{i+1}(x,\beta_1,\cdots,-\beta_i,\cdots,\beta_n)\\
&\quad = -\beta_i - \int_0^1 (\Phi_i(t))^{-1} f_i\big(0,t,\beta_1\Phi_1(t)+\cdots-\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + MEF(x)(t)\big)\,dt\\
&\quad \leqslant -\beta_i + \int_0^1 |(\Phi_i(t))^{-1}|\Big|f_i\big(0,t,\beta_1\Phi_1(t)+\cdots-\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + MEF(x)(t)\big)\Big|\,dt\\
&\quad \leqslant -\beta_i + m\|(\Phi_i)^{-1}\|_\infty\\
&\quad \leqslant -\beta_i + \alpha_i \leqslant 0.
\end{aligned}$$

Thus, for all $x \in \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$, $\beta_i \in [\alpha_i,\delta]$, and $i = 1,\cdots,n$,

$$H_{i+1}(x,\beta_1,\cdots,\beta_i,\cdots,\beta_n),\ H_{i+1}(x,\beta_1,\cdots,-\beta_i,\cdots,\beta_n) \in [-\beta_i,\beta_i] \subseteq [-\delta,\delta].$$

Furthermore, if $0 \leqslant \beta_i < \alpha_i$, for all $x \in \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)$,

$$\begin{aligned}
&|H_{i+1}(x,\beta_1,\cdots,\pm\beta_i,\cdots,\beta_n)|\\
&\quad \leqslant |\pm\beta_i| + \int_0^1 |(\Phi_i(t))^{-1}|\big|f_i(0,t,\beta_1\Phi_1(t)+\cdots\pm\beta_i\Phi_i(t)+\cdots+\beta_n\Phi_n(t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + MEF(x)(t))\big|\,dt\\
&\quad \leqslant \alpha_i + m\|(\Phi_i)^{-1}\|_\infty \leqslant \delta
\end{aligned}$$

for $i = 1,\cdots,n$.

We have shown that H_{i+1} maps $\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times[-\delta,\delta]\times\mathbb{R}^{n-1}$ into $[-\delta,\delta]$. From this it follows that $H(\mathscr{B}) \subseteq \mathscr{B}$. For if $(x,\beta_1,\cdots,\beta_n) \in \mathscr{B}$, then

$$H_{i+1}(x,\beta_1,\cdots,\beta_n) \in [-\delta,\delta] \ \text{ for } i = 1,\cdots,n,$$

while

$$\begin{aligned}
|H_1(x,\beta_1,\cdots,\beta_n)| &\leqslant |\beta_1|\|\Phi_1\|_\infty + \cdots + |\beta_n|\|\Phi_n\|_\infty + \|MEF(x)\|_\infty\\
&\leqslant \delta(\|\Phi_1\|_\infty + \cdots + \|\Phi_n\|_\infty) + m|\|ME\||.
\end{aligned}$$

We now establish existence of solutions of (1), (2) for values of ε different from zero. It is significant to observe that the nonlinearities $f_i(\varepsilon,t,x_1(t),\cdots,x_n(t))$ are allowed to be unbounded.

THEOREM 3.2. *Suppose that:*

(i) $b_i + d_i e^{\int_0^1 a_i(s)ds} = 0$ *for all* $i = 1,2,\cdots,n$;

(ii) $f_i : \mathbb{R}^{n+2} \to \mathbb{R}$ *is continuous for all* $i = 1, \cdots, n$;

(iii) *for each* $i = 1, \cdots, n$, *there exists* $\gamma_i \in \mathbb{R}$ *such that*

$$f_i(0,t,\beta_1,\cdots,\beta_i,\cdots,\beta_n) f_i(0,t,\beta_1,\cdots,-\beta_i,\cdots,\beta_n) < 0$$

whenever $\beta_i \geqslant \gamma_i$.

Then, there exists an ε_0 *such that for* $\varepsilon \in [0,\varepsilon_0]$, *there is at least one solution of*

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\varepsilon,t,x_1(t),\cdots,x_n(t)), \ \ i = 1,2,\cdots,n,$$

that satisfies

$$b_i x_i(0) + d_i x_i(1) = 0, \ \ i = 1,2,\cdots,n.$$

Proof. As above, we define mappings

$$\begin{aligned} &H_1 : \mathbb{R} \times \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \times \mathbb{R}^n \to \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \\ &H_{i+1} : \mathbb{R} \times \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \times \mathbb{R}^n \to \mathbb{R} \\ &H : \mathbb{R} \times \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \times \mathbb{R}^n \to \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \times \mathbb{R}^n \end{aligned}$$

by

$$H_1(\varepsilon,x,\beta_1,\cdots,\beta_n) = \beta_1\Phi_1 + \cdots + \beta_n\Phi_n + MEF(\varepsilon,x),$$

$$\begin{aligned} &H_{i+1}(\varepsilon,x,\beta_1,\cdots,\beta_n) \\ &\qquad = \beta_i - (\int_0^1 \Phi_i(t)^{-1} f_i(\varepsilon,t,\beta_1\Phi_1(t) + \cdots + \beta_n\Phi_n(t) + MEF(\varepsilon,x)(t), \end{aligned}$$

and

$$H(\varepsilon,x,\beta_1,\cdots,\beta_n) = (H_1(\varepsilon,x,\beta_1,\cdots,\beta_n),\cdots,H_{n+1}(\varepsilon,x,\beta_1,\cdots,\beta_n)).$$

By the proof of Lemma 3.1, redefining

$$\alpha_i \geqslant \|(\Phi_i)^{-1}\|(m+K) \text{ and } \delta = \max\{\alpha_i + \|\Phi_i^{-1}\|_\infty(m+K) : i = 1,\cdots,n\}$$

for some fixed real number K, we can create a nonempty, convex set

$$\begin{aligned} \mathscr{B} = \big\{ &(x,\beta_1,\cdots,\beta_n) \in \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) \times \mathbb{R}^n : \\ &\|x\|_\infty \leqslant \delta(\|\Phi_1\|_\infty + \cdots + \|\Phi_n\|_\infty) + \||ME\||(m+K) \text{ and} \\ &\hfill |\beta_i| \leqslant \delta \text{ for } i = 1,\cdots,n \big\} \end{aligned}$$

such that, when $\varepsilon = 0$, the following hold true:

1. for all $\beta_i \geqslant \alpha_i \geqslant \|(\Phi_i)^{-1}\|(m+K)$,

$$\begin{aligned} &H_{i+1}(0,x,\beta_1,\cdots,\beta_i,\cdots,\beta_n) \leqslant \beta_i - K \ \text{ and} \\ &H_{i+1}(0,x,\beta_1,\cdots,-\beta_i,\cdots,\beta_n) \geqslant -\beta_i + K; \end{aligned}$$

2. for $\beta_i \in [\alpha_i, \delta]$,

$$H_{i+1}(0,x,\beta_1,\cdots,\beta_i,\cdots,\beta_n) \geqslant -K \text{ and } H_{i+1}(0,x,\beta_1,\cdots,-\beta_i,\cdots,\beta_n) \leqslant K;$$

3. for $0 \leqslant \beta_i < \alpha_i$, $|H_{i+1}(0,x,\beta_1,\cdots,\pm\beta_i,\cdots,\beta_n)| \leqslant \delta + K$; and
4. $|H_1(0,x,\beta_1,\cdots,\beta_n)| \leqslant \delta(\|\Phi_1\| + \cdots + \|\Phi_n\|) + \||ME\||(m+K)$.

It is evident that

$$\inf_{(x,\beta_1,\cdots,\beta_n)\in\mathscr{B}} \text{dist}(H(0,x,\beta_1,\cdots,\beta_n),\partial\mathscr{B}) > 0;$$

that is, when $\varepsilon = 0$, there is a positive distance between the boundary of the set $\mathscr{B}$ and the set of $H(0,x,\beta_1,\cdots,\beta_n)$ for $(x,\beta_1,\cdots,\beta_n) \in \mathscr{B}$. Since $\{\beta_1\Phi_1 + \cdots + \beta_n\Phi_n + MEF(x)|(\beta_1,\cdots,\beta_n,x) \in \mathscr{B}\}$ is equicontinuous and uniformly bounded, it is compact by Arzela-Ascoli's Theorem. This implies that if we choose a positive value, $\tilde{\varepsilon}$, so that we restrict ε to the interval $[0,\tilde{\varepsilon}]$, the map $(\varepsilon,\beta_1,\cdots,\beta_n,x) \mapsto H(\varepsilon,\beta_1,\cdots,\beta_n,x)$ is uniformly continuous on $\mathscr{B}$. From this it follows that there exists ε_0 such that if $|\varepsilon| \leqslant \varepsilon_0$,

$$H(\varepsilon,\beta_1,\cdots,\beta_n,x) \in \mathscr{B}$$

for all $(\beta_1,\cdots,\beta_n,x) \in \mathscr{B}$. The solvability of the parameter dependent vector equation

$$\dot{x}_i(t) = a_i(t)x_i(t) + f_i(\varepsilon,t,x_1(t),\cdots,x_n(t)),\ \ i = 1,2,\cdots,n,$$

that satisfies

$$b_i x_i(0) + d_i x_i(1) = 0,\ \ i = 1,2,\cdots,n,$$

is now a consequence of Schauder's fixed point theorem.

4. Example

We close with an example to better illustrate the applicability of the results of this paper. Consider the two-dimensional parameter-dependent equation

$$\begin{aligned} \dot{x}_1(t) &= a_1(t)x_1(t) + f_1(\varepsilon,t,x_1(t),x_2(t)) \\ \dot{x}_2(t) &= a_2(t)x_2(t) + f_2(\varepsilon,t,x_1(t),x_2(t)) \end{aligned} \tag{6}$$

subject to the periodic boundary conditions

$$x_1(0) - x_1(1) = 0x_2(0) - x_2(1) = 0. \tag{7}$$

Here, we assume $f_i : \mathbb{R}^4 \to \mathbb{R}$ to be of the form $f_i(\varepsilon,t,\beta_1,\beta_2) = w_i(\beta_i)h_i(t,\beta_1,\beta_2) + \varepsilon g_i(t,\beta_1,\beta_2)$, where $g_i : \mathbb{R}^3 \to \mathbb{R}$ is continuous, $h_i : \mathbb{R}^3 \to \mathbb{R}$ is continuous with the conditions $\sup\{|h_i(t,\beta_1,\beta_2)| : t \in [0,1] \text{ and } (\beta_1,\beta_2) \in \mathbb{R}^2\}$ is finite and $h_i(t,\beta_1,\beta_2) \geqslant 0$ for all $t \in [0,1]$, $(\beta_1,\beta_2) \in \mathbb{R}^2$, and $w_i(\beta_i) : \mathbb{R} \to \mathbb{R}$ with the asymptotic behavior

$$\lim_{\beta_i \to \infty} w_i(\beta_i) = w_i(\infty) > 0$$

$$\lim_{\beta_i \to -\infty} w_i(\beta_i) = w_i(-\infty) < 0$$

for $i = 1,2$. It is evident that if $\int_0^1 a_i(t)dt = 0$ for $i = 1,2$, then using Theorem 3.2, we see that the boundary value problem (6), (7) has a solution.

REFERENCES

[1] S. BANCROFT, J.K. HALE AND D. SWEET, *Alternative problems for nonlinear functional equations*, J. Differential Equations, **4** (1968), 40–56.
[2] L. CESARI, *Functional analysis and periodic solutions of nonlinear differential equations*, Contributions to Differential Equations, **1** (1963), 149–187.
[3] L. CESARI, *Functional analysis and Galerkin's method*, Michigan Math. J., **11** (1964), 385–414.
[4] S. CHOW AND J.K. HALE, *Methods of Bifurcation Theory*, Springer, Berlin, 1982.
[5] D.L. ETHERIDGE AND J. RODRÍGUEZ, *Periodic solutions of nonlinear discrete-time systems*, Appl. Anal., **62** (1996), 119–137.
[6] D.L. ETHERIDGE AND J. RODRÍGUEZ, *Scalar discrete nonlinear two-point boundary value problems*, J. Differ. Equations Appl., **4** (1998), 127–144.
[7] A. HALANAY, *Solutions periodiques et presque-periodiques des systemes d'equations aux differences finies*, Arch. Rational Mech. Anal., **12** (1963), 134–149.
[8] J.K. HALE, *Applications of alternative problems*, Lecture Notes 71-1, Brown University, Providence, 1971.
[9] J.K. HALE, *Ordinary Differential Equations*, Second edition, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.
[10] D.C. LEWIS, *On the role of first integrals in the pertubation of periodic solutions*, The Annals of Mathematics, Second Series, **63** (1956), 535–548.
[11] J. RODRÍGUEZ, *An alternative method for boundary value problems with large nonlinearities*, J. Differential Equations, **43** (1982), 157–167.
[12] J. RODRÍGUEZ, *Nonlinear differential equations under Stieltjes boundary conditions*, Nonlinear Anal., **7** (1983), 107–116.
[13] J. RODRÍGUEZ, *On resonant discrete boundary value problems*, Applicable Anal., **19** (1985), 265–274.
[14] J. RODRÍGUEZ, *Galerkin's method for ordinary differential equations subject to generalized nonlinear boundary conditions*, J. Differential Equations, **97** (1992), 112–126.
[15] J. RODRÍGUEZ, *Nonlinear discrete Sturm-Liouville problems*, J. Math. Anal. Appl., **308** (2005), 380–391.
[16] J. RODRÍGUEZ AND K. ABERNATHY, *On the solvability of nonlinear boundary value problems*, Differ. Equ. Appl., **2** (2010), 487–501.
[17] J. RODRÍGUEZ AND D. SWEET, *Projection methods for nonlinear boundary value problems*, J. Differential Equations, **58** (1985), 282–293.
[18] J. RODRÍGUEZ AND P. TAYLOR, *Weakly nonlinear discrete multipoint boundary value problems*, J. Math. Anal. Appl., **329** (2007), 77–91.
[19] J. RODRÍGUEZ AND P. TAYLOR, *Scalar discrete nonlinear multipoint boundary value problems*, J. Math. Anal. Appl., **330** (2007), 876–890.
[20] J. RODRÍGUEZ AND P. TAYLOR, *Multipoint boundary value problems for nonlinear ordinary differential equations*, Nonlinear Anal., **68** (2008), 3465–3474.
[21] N. ROUCHE AND J. MAWHIN, *Ordinary Differential Equations*, Surveys Reference Works Math. 5, Advanced Publishing Program, Pitman, Boston MA, 1980.

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