

## POSITIVE SOLUTIONS TO A TWO POINT SINGULAR BOUNDARY VALUE PROBLEM

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*Abstract.* We employ fixed point index theory to establish existence results for positive solutions to the singular boundary value problem

$$\begin{cases} -(au')'(t) = b(t)f(t, u(t)), & t \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases}$$

where  $a \in C^1((0, 1), (0, \infty))$ ,  $1/a$  is integrable on any compact subset of  $(0, 1]$ ,  $b \in C((0, 1), [0, +\infty))$  does not vanish identically and is integrable on any compact subset of  $[0, 1)$ , and  $f : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous with  $f(t, u) > 0$  for all  $(t, u) \in [0, 1] \times (0, \infty)$ . As applications, existence and nonexistence criteria for positive radial solutions to some elliptic equations are deduced.

### 1. Introduction

In this paper, we present some new results guaranteeing the existence of positive solutions to the singular boundary value problem (BVP)

$$\begin{cases} -(au')'(t) = b(t)f(t, u(t)), & t \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \quad (1.1)$$

where  $\mathbb{R}^+ = [0, \infty)$ ,  $a \in C^1((0, 1), (0, \infty))$ ,  $1/a$  is integrable on any compact subset of  $(0, 1]$ ,  $b \in C((0, 1), \mathbb{R}^+)$  does not vanish identically and is integrable on any compact subset of  $[0, 1)$ , and  $f : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous with  $f(t, u) > 0$  for all  $(t, u) \in [0, 1] \times (0, \infty)$ . By a *positive solution* to BVP (1.1), we mean a function  $u \in C([0, 1], \mathbb{R}^+) \cap C^1((0, 1), \mathbb{R}^+)$  with  $u(t) > 0$  on  $[0, 1)$  satisfying both the differential equation and the boundary conditions in (1.1).

It is worth pointing out the difference between the boundary conditions in (1.1) and the conditions

$$(au')(0) = u(1) = 0 \quad (1.2)$$

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that are often considered in the literature. If  $u$  is a positive solution of (1.1), then  $u'$  may be positive on some subinterval of  $(0, 1)$ . For example,

$$u(t) = -\frac{1}{15}(1-t\sqrt{t}) + \frac{2}{5}(1-t^2\sqrt{t})$$

is a positive solution of

$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{t}}\right)' = 1, \\ u'(0) = u(1) = 0, \end{cases}$$

and  $u'(t) = \sqrt{t}\left(\frac{1}{10} - t\right)$  is positive on  $(0, \frac{1}{10})$ . On the other hand,

$$v(t) = \frac{2}{5}(1-t^2\sqrt{t})$$

is a positive solution of

$$\begin{cases} -\left(\frac{v'(t)}{\sqrt{t}}\right)' = 1, \\ \left(\frac{v'}{\sqrt{t}}\right)(0) = v(1) = 0, \end{cases}$$

and  $v'(t) = -t\sqrt{t} < 0$  on  $(0, 1)$ . This difference, together with the possible singularities of  $a$  and  $b$ , makes the study of problem (1.1) harder than using the boundary conditions (1.2).

Throughout this paper, we assume that  $a$  and  $b$  satisfy the following conditions:

$$\lim_{x \rightarrow 0} \frac{1}{a(x)} \int_0^x b(t) dt = 0, \tag{1.3}$$

$$\int_0^1 \left( \frac{1}{a(s)} \int_0^s b(t) dt \right) ds < \infty, \tag{1.4}$$

and

$$\int_0^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt < \infty. \tag{1.5}$$

Typical examples of  $a$  and  $b$  satisfying (1.3)–(1.5) are  $a(t) = b(t) = t^\alpha(1-t)^\beta$  with  $\alpha > -1$  and  $\beta < 1$ . Note that a solution of (1.1) in the case  $a(t) = b(t) = t^{n-1}$  with  $n \in \mathbb{N}$  is a radial solution to the elliptic equation

$$\begin{cases} -\Delta u(x) = f(|x|, u(x)), & x \in \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

where  $\Omega$  is the unit ball in  $\mathbb{R}^n$ . Moreover, in the case  $\Omega = \mathbb{R}^n$  ( $n \geq 3$ ), by means of a change of variables, we can see that a radial solution of (1.6) is a solution of (1.1) (see Section 4 for details).

Since singular BVPs arise in many physical problems, existence and multiplicity results for positive solutions of such problems have been studied by many authors in

recent years; see, for example, [1, 4, 12, 15, 18, 21, 28] and references therein. For some studies on radial positive solutions of elliptic equations, the reader may refer to [6, 8, 13, 16, 17, 23] and the included references.

Different variations of the regular version of BVP (1.1) have been studied extensively in the literature, for example, in [2, 7, 9, 10, 11, 19, 26, 24, 25, 22, 30]. In this paper, as in the papers [7, 10, 19, 22, 24, 25, 30], our criteria for the existence of positive solutions are determined by the relationship between the behavior of the term  $f(t,x)/x$  near 0 and  $\infty$  when compared with the smallest eigenvalue of an associated linear problem subject to the same boundary conditions. However, our existence results can not be proved with the same arguments as in those papers since the weights  $a$  and  $b$  here are singular. To overcome the difficulty caused by the singularity of  $a$  and  $b$ , many new ideas and techniques are developed in this paper.

In what follows, we let  $E$  be the Banach space of all continuous functions defined on  $[0, 1]$  endowed with the usual sup-norm denoted by  $\|\cdot\|$ ,  $K$  be the cone of nonnegative functions in  $E$ , and  $K^* = K \setminus \{0\}$ .

We recall the following convergence result due to Brezis and Lieb; see [3, Theorem 1].

LEMMA 1.1. *Assume that  $0 < p < \infty$  and  $\Omega$  is a measurable set in  $\mathbb{R}^N$ . If  $\{f_n\}$  is a bounded sequence in  $L^p(\Omega)$  with  $f_n \rightarrow f$  a.e. in  $\Omega$ , then*

$$\|f\|_p^p = \lim_{n \rightarrow \infty} (\|f_n\|_p^p - \|f - f_n\|_p^p).$$

The following corollary is an immediate consequence of Lemma 1.1.

COROLLARY 1.1. *Assume that  $0 < p < \infty$  and  $\Omega$  is a measurable set in  $\mathbb{R}^N$ . If  $\{f_n\}$  is a sequence in  $L^p(\Omega)$  with  $f_n \rightarrow f$  a.e. in  $\Omega$  and  $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$ , then*

$$\lim_{n \rightarrow \infty} \|f - f_n\|_p = 0.$$

In the remainder of this section, for the sake of completeness, we recall some basic facts from fixed point index theory. Let  $X$  be a real Banach space and  $K$  be a closed subset of  $X$ . Then  $K$  is called a *cone* if

- ◊  $K$  is convex;
- ◊  $tx \in K$  for all  $t \geq 0$  and  $x \in K$ ;
- ◊ if  $x \in K$  and  $-x \in K$ , then  $x = 0$ .

The set  $K$  is called a *retract* of  $X$  if there exists a continuous mapping  $r : X \rightarrow K$  such that  $r(x) = x$  for all  $x \in K$ . The mapping  $r$  is called a *retraction*. From a theorem by Dugundji, every nonempty closed convex subset of  $X$  is a retract of  $X$ . In particular, every cone of  $X$  is a retract of  $X$ .

Let  $K$  be a retract of  $X$ ,  $U$  be an open bounded subset of  $K$ , and  $B(0, R)$  be the ball of radius  $R$  in  $X$  such that  $U \subset B(0, R)$ . For any completely continuous mapping  $f : \overline{U} \rightarrow K$  with  $f(x) \neq x$  for all  $x \in \partial U$ , the integer given by

$$i(f, U, K) = \deg(I - f \circ r, B(0, R) \cap r^{-1}(U), 0)$$

where  $\deg$  is the Leray-Schauder degree, is well defined and is called the *fixed point index*.

PROPERTIES OF THE FIXED POINT INDEX:

1. *Normality*:  $i(f, U, K) = 1$  if  $f(x) = x_0 \in \bar{U}$  for all  $x \in \bar{U}$ ;
2. *Homotopy invariance*: Let  $H : [0, 1] \times \bar{U} \rightarrow K$  be a completely continuous mapping such that  $H(t, x) \neq x$  for all  $(t, x) \in [0, 1] \times \partial U$ . Then, the integer  $i(H(t, \cdot), U, K)$  is independent of  $t$ ;

3. *Additivity*:

$$i(f, U, K) = i(f, U_1, K) + i(f, U_2, K)$$

whenever  $U_1$  and  $U_2$  are two disjoint open subsets of  $U$  such that  $f$  has no fixed point in  $\bar{U} \setminus (U_1 \cup U_2)$ ;

4. *Permanence*: If  $K'$  is a retract of  $K$  with  $f(\bar{U}) \subset K'$ , then

$$i(f, U, K) = i(f, U \cap K', K');$$

5. *Solution property*: If  $i(f, U, K) \neq 0$ , then  $f$  admits a fixed point in  $U$ .

Now, assume that  $K$  is a cone, and for all  $R > 0$ , we set  $K_R = B(0, R) \cap K$ . The following lemmas and their proofs can be found in [14].

LEMMA 1.2. *If  $f(x) \neq \lambda x$  for all  $x \in \partial K_R = \partial B(0, R) \cap K$  and  $\lambda \geq 1$ , then*

$$i(f, K_R, K) = 1.$$

LEMMA 1.3. *If*

- ◇  $f(x) \neq \lambda x$  for all  $x \in \partial K_R = \partial B(0, R) \cap K$  and  $\lambda \in (0, 1]$ , and
- ◇  $\inf \{\|f(x)\| : x \in \partial K_R\} > 0$ ,

then

$$i(f, K_R, K) = 0.$$

LEMMA 1.4. *If  $\|f(x)\| \geq \|x\|$  for all  $x \in \partial K_R = \partial B(0, R) \cap K$ , then*

$$i(f, K_R, K) = 0.$$

## 2. Preliminary results

In this section, we focus our attention on the linear eigenvalue problem associated with BVP (1.1), namely,

$$\begin{cases} -(au')'(t) = \lambda b(t)u(t), & t \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \quad (2.1)$$

where  $\lambda$  is a real parameter.

DEFINITION 2.1. We say that  $\lambda$  is a positive eigenvalue of (2.1) if  $\lambda > 0$  and there exists  $\phi \in K^*$  such that  $(\lambda, \phi)$  satisfies (2.1).

Consider the linear operator  $L : E \rightarrow E$  defined by

$$Lu(x) = \int_x^1 \frac{1}{a(s)} \left( \int_0^s b(t)u(t)dt \right) ds, \quad u \in E.$$

LEMMA 2.1. Assume that (1.4) and (1.5) hold. Then the function  $H(x)$  defined by

$$H(x) = \int_0^x b(t)dt \int_x^1 \frac{1}{a(t)}dt$$

satisfies

$$\lim_{x \rightarrow 0} H(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} H(x) = 0.$$

Moreover, for all  $u \in E$ ,

$$Lu(x) = \int_0^1 G(x,t)b(t)u(t)dt,$$

where

$$G(x,t) = \begin{cases} \int_x^1 \frac{ds}{a(s)}, & \text{if } 0 < t \leq x < 1, \\ \int_t^1 \frac{ds}{a(s)}, & \text{if } 0 < x \leq t < 1. \end{cases}$$

*Proof.* It is easy to see that

$$H(x) \leq \int_0^x b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt$$

and

$$H(x) \leq \int_x^1 \frac{1}{a(t)} \left( \int_0^t b(s)ds \right) dt.$$

Thus,

$$\lim_{x \rightarrow 0} H(x) = \lim_{x \rightarrow 1} H(x) = 0.$$

Now, let  $x \in (0, 1)$  and  $\varepsilon \in (0, 1 - x)$ . For any  $u \in E$ , an integration by parts yields

$$\begin{aligned} \int_x^{1-\varepsilon} \frac{1}{a(t)} \left( \int_0^t b(s)u(s)ds \right) dt &= - \int_{1-\varepsilon}^1 \frac{ds}{a(s)} \int_0^{1-\varepsilon} b(s)u(s)ds \\ &+ \int_x^1 \frac{ds}{a(s)} \int_0^x b(s)u(s)ds + \int_x^{1-\varepsilon} b(t) \left( \int_t^1 \frac{1}{a(s)} ds \right) u(t)dt. \end{aligned} \quad (2.2)$$

Taking into account that

$$\int_{1-\varepsilon}^1 \frac{ds}{a(s)} \int_0^{1-\varepsilon} b(s)u(s)ds \leq H(1 - \varepsilon) \|u\|$$

and letting  $\varepsilon \rightarrow 0$  in (2.2), we obtain

$$\begin{aligned} Lu(x) &= \int_x^1 \frac{ds}{a(s)} \int_0^x b(s)u(s)ds + \int_x^1 b(t) \left( \int_t^1 \frac{1}{a(s)} ds \right) u(t)dt \\ &= \int_0^1 G(x,t)b(t)u(t)dt. \end{aligned}$$

This completes the proof of the lemma.

LEMMA 2.2. *Assume that (1.3) and (1.4) hold and  $u \in E$ . Then  $v = Lu$  is a solution of the BVP*

$$\begin{cases} -(av')'(t) = b(t)u(t), & t \in (0, 1), \\ v'(0) = v(1) = 0. \end{cases} \quad (2.3)$$

*Proof.* If, for all  $x \in [0, 1]$ ,

$$v(x) = Lu(x) = \int_x^1 \frac{1}{a(t)} \left( \int_0^t b(s)u(s)ds \right) dt,$$

then  $v(1) = 0$  and

$$-(av')'(x) = b(x)u(x), \quad x \in (0, 1).$$

Moreover,

$$\begin{aligned} \left| \frac{v(x) - v(0)}{x} \right| &= \frac{1}{x} \int_0^x \frac{1}{a(t)} \left( \int_0^t b(s)u(s)ds \right) dt \\ &\leq \|u\| \frac{1}{x} \int_0^x \frac{1}{a(t)} \left( \int_0^t b(s)ds \right) dt. \end{aligned}$$

Applying L'Hôpital's rule and (1.3), we have

$$|v'(0)| = \left| \lim_{x \rightarrow 0} \frac{v(x) - v(0)}{x} \right| \leq \lim_{x \rightarrow 0} \frac{\|u\|}{a(x)} \int_0^x b(s)ds = 0.$$

Thus,  $v(t)$  is a solution of (2.3), completing the proof of the lemma.

LEMMA 2.3. *Assume that (1.4) holds. Then  $L$  is completely continuous.*

*Proof.* This follows from the uniform continuity of the function

$$x \rightarrow \int_x^1 \frac{1}{a(t)} \left( \int_0^t b(s)ds \right) dt$$

on  $[0, 1]$  and the Ascoli-Arzelà theorem.

Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be two sequences such that, for all  $n \in \mathbb{N}$ ,  $0 < \alpha_{n+1} \leq \alpha_n < 1/2$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\beta_n = 1 - \alpha_n$ . Let  $I_n = [\alpha_n, \beta_n]$ . For each  $n \in \mathbb{N}$ , it is well-known that the eigenvalue problem

$$\begin{cases} -(au')'(t) = \lambda b(t)u(t), & t \in (\alpha_n, \beta_n), \\ u'(\alpha_n) = u(\beta_n) = 0, \end{cases} \tag{2.4}$$

has a smallest positive eigenvalue  $\lambda_1^n = \lambda_1(a, b, I_n)$  and the eigenfunction  $\phi_n$  associated with  $\lambda_1^n$  has no zero in  $(0, 1)$ ; see, for example, [2, Theorem 9] or [32, Theorem 4.3.2].

For each  $n \in \mathbb{N}$ , let the functions  $\varphi_n$ ,  $\psi_n$ , and  $v_n$  be defined by

$$\begin{aligned} \varphi_n &: [0, 1] \rightarrow I_n \text{ with } \varphi_n(s) = (\beta_n - \alpha_n)s + \alpha_n, \\ \psi_n &: I_n \rightarrow [0, 1] \text{ with } \psi_n(s) = \frac{s - \alpha_n}{\beta_n - \alpha_n}, \text{ and} \\ v_n &: [0, 1] \rightarrow \mathbb{R} \text{ with } v_n(t) = \phi_n(\varphi_n(t)). \end{aligned}$$

From (2.4), we see that  $v_n$  satisfies the BVP

$$\begin{cases} -(a_nv_n')'(t) = \lambda_1^n (\beta_n - \alpha_n)^2 b_n(t)v_n(t), & t \in (0, 1), \\ v_n'(0) = v_n(1) = 0, \end{cases} \tag{2.5}$$

where  $a_n(t) = a(\varphi_n(t))$  and  $b_n(t) = b(\varphi_n(t))$ . Thus,  $v_n = \lambda_1^n L_n(v_n)$ , where

$$L_n(u)(x) = (\beta_n - \alpha_n)^2 \int_x^1 \frac{1}{a_n(s)} \left( \int_0^s b_n(t)u(t)dt \right) ds, \quad u \in E.$$

LEMMA 2.4. *Assume that (1.4) and (1.5) hold. Then  $\lim_{n \rightarrow \infty} L_n = L$ .*

*Proof.* As in the case of the operator  $L$ , it is easy to see that for each  $n \in \mathbb{N}$ ,

$$L_n u(x) = \int_0^1 G_n(x, t) b_n(t) u(t) dt,$$

where

$$G_n(x, t) = c_n^2 \begin{cases} \int_x^1 \frac{ds}{a_n(s)}, & \text{if } 0 \leq t \leq x, \\ \int_t^1 \frac{ds}{a_n(s)}, & \text{if } x \leq t \leq 1, \end{cases}$$

with  $c_n = (\beta_n - \alpha_n)$ . Then, for any  $u \in E$  with  $\|u\| = 1$ , we have

$$\begin{aligned} |Lu(x) - L_n u(x)| &= \left| \int_0^1 [G(x, t)b(t) - G_n(x, t)b_n(t)]u(t)dt \right| \\ &\leq \int_0^1 |G(x, t)b(t) - G_n(x, t)b_n(t)| dt. \end{aligned}$$

Thus, we have to show that

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [0, 1]} \int_0^1 |G(x, t)b(t) - G_n(x, t)b_n(t)| dt \right) = 0.$$

Note that

$$\int_0^1 |G_n(x, t)b_n(t) - G(x, t)b(t)| dt = K_n(x) + M_n(x),$$

where

$$K_n(x) = \int_x^1 \left| c_n^2 b_n(t) \int_t^1 \frac{ds}{a_n(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt$$

and

$$M_n(x) = \int_0^x \left| c_n^2 b_n(t) \int_x^1 \frac{ds}{a_n(s)} - b(t) \int_x^1 \frac{ds}{a(s)} \right| dt.$$

We need to show that  $K_n$  and  $M_n$  converge uniformly to 0 on  $[0, 1]$ . Clearly, we have

$$K_n(x) \leq K_n(0) = \int_0^1 \left| c_n^2 b_n(t) \int_t^1 \frac{ds}{a_n(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt,$$

and for all  $t \in (0, 1)$ ,

$$\lim_{n \rightarrow \infty} c_n^2 b_n(t) \int_t^1 \frac{ds}{a_n(s)} = b(t) \int_t^1 \frac{ds}{a(s)}.$$

Since

$$\begin{aligned} & \left| \int_0^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt - \int_0^1 c_n^2 b_n(t) \left( \int_t^1 \frac{ds}{a_n(s)} \right) dt \right| \\ &= \left| \int_0^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt - \int_{\alpha_n}^{\beta_n} b(t) \left( \int_t^{\beta_n} \frac{ds}{a(s)} \right) dt \right| \\ &= \int_{\beta_n}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt + \int_{\alpha_n}^{\beta_n} b(t) \left( \int_{\beta_n}^1 \frac{ds}{a(s)} \right) dt + \int_0^{\alpha_n} b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt \\ &\leq \int_{\beta_n}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt + \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)} + \int_0^{\alpha_n} b(t) \int_{\alpha_n}^1 \frac{ds}{a(s)} dt \\ &= \int_{\beta_n}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt + H(\beta_n) + H(\alpha_n), \end{aligned}$$

from Lemma 2.1 and (1.5), it follows that

$$\lim_{n \rightarrow \infty} \left| \int_0^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt - \int_0^1 c_n^2 b_n(t) \left( \int_t^1 \frac{ds}{a_n(s)} \right) dt \right| = 0.$$

In view of the fact that

$$\begin{aligned} \int_0^1 c_n^2 b_n(t) \left( \int_t^1 \frac{ds}{a_n(s)} \right) dt &= \int_{\alpha_n}^{\beta_n} b(t) \left( \int_t^{\beta_n} \frac{ds}{a(s)} \right) dt \\ &< \int_0^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt < \infty, \end{aligned}$$



then from Corollary 1.1, we conclude that

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [0,1]} K_n(x) \right) = \lim_{n \rightarrow \infty} K_n(0) = 0.$$

Next, for  $x \in [0, 1/2]$ , we have  $\varphi_n(x) \geq x$  and

$$\begin{aligned} M_n(x) &\leq \int_0^x \left| c_n b_n(t) \int_{\varphi_n(x)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(x)}^1 \frac{ds}{a(s)} \right| dt \\ &\quad + \int_0^x b(t) dt \int_x^{\varphi_n(x)} \frac{ds}{a(s)} + \int_0^x b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)} \\ &\leq \int_0^x \left| c_n b_n(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right| dt + H(x) - H(\varphi_n(x)) \\ &\quad + \int_x^{\varphi_n(x)} b(t) dt \int_{\varphi_n(x)}^1 \frac{ds}{a(s)} + \int_0^x b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)}, \end{aligned}$$

which implies that

$$\begin{aligned} M_n(x) &\leq \int_0^{\frac{1}{2}} \left| c_n b_n(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right| dt \\ &\quad + H(x) - H(\varphi_n(x)) \\ &\quad + \int_x^{\varphi_n(x)} b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt + \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)}. \end{aligned}$$

Note that if  $I$  is the function defined by  $I(x) = x$ , then  $\varphi_n$  converges uniformly to  $I$  and the functions  $H$  and  $x \rightarrow \int_0^x b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt$  are uniformly continuous on  $[0, 1]$ . Hence,  $H(x) - H(\varphi_n(x))$  and  $\int_x^{\varphi_n(x)} b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt$  converge to 0 uniformly on  $[0, 1/2]$ . Moreover, it is clear that

$$\lim_{n \rightarrow \infty} \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)} = 0$$

and

$$\int_0^{\frac{1}{2}} \left( c_n b_n(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) dt \leq 2 \int_0^{\frac{1}{2}} \left( b(t) \int_t^1 \frac{ds}{a(s)} \right) dt < \infty.$$

It remains to show that

$$\lim_{n \rightarrow \infty} \int_0^{\frac{1}{2}} \left| c_n b_n(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right| dt = 0. \tag{2.6}$$

For all  $t \in (0, 1/2]$ ,

$$\lim_{n \rightarrow \infty} \left( c_n b_n(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) = 0.$$

By straightforward computations,

$$\begin{aligned} & \left| \int_0^{\frac{1}{2}} \left( c_n b_n(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} - b(t) \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) dt \right| \\ &= \left| \int_{\alpha_n}^{\frac{1}{2}} b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt - \int_0^{\frac{1}{2}} b(t) \left( \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) dt \right| \\ &\leq \int_0^{\alpha_n} b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt + \int_0^{\frac{1}{2}} b(t) \left( \int_t^{\varphi_n(t)} \frac{ds}{a(s)} \right) dt. \end{aligned}$$

It is clear that  $\lim_{n \rightarrow \infty} \int_0^{\alpha_n} b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt = 0$ ,  $\lim_{n \rightarrow \infty} b(t) \left( \int_t^{\varphi_n(t)} \frac{ds}{a(s)} \right) = 0$ , and for all  $n \in \mathbb{N}$  and  $t \in (0, 1/2]$ ,  $b(t) \left( \int_t^{\varphi_n(t)} \frac{ds}{a(s)} \right) \leq b(t) \left( \int_t^1 \frac{ds}{a(s)} \right)$ . Then, by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^{\frac{1}{2}} b(t) \left( \int_t^{\varphi_n(t)} \frac{ds}{a(s)} \right) dt = 0.$$

Thus, from Corollary 1.1, (2.6) holds, and so

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [0, \frac{1}{2}]} M_n(x) \right) = 0.$$

Now, for  $x \in [1/2, 1]$ , we have  $\varphi_n(x) \leq x$  and

$$\begin{aligned} M_n(x) &\leq \int_0^x \left| c_n b_n(t) \int_x^1 \frac{ds}{a(s)} - b(t) \int_x^1 \frac{ds}{a(s)} \right| dt + \int_0^x b(t) dt \int_{\varphi_n(x)}^x \frac{ds}{a(s)} \\ &\quad + \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)} \\ &\leq \int_0^x \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt + H(\varphi_n(x)) - H(x) \\ &\quad + \int_{\varphi_n(x)}^x b(t) dt \int_{\varphi_n(x)}^1 \frac{ds}{a(s)} + \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)} \\ &\leq \int_0^x \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt + H(\varphi_n(x)) - H(x) \\ &\quad + \int_{\varphi_n(x)}^x b(t) \left( \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) dt + \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)}. \end{aligned}$$

As above,  $\{\varphi_n\}$  converges uniformly to  $I$  and the functions  $H$  and

$$x \rightarrow \int_0^x b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt$$

are uniformly continuous on  $[0, 1]$ . Then,  $H(x) - H(\varphi_n(x))$  and

$$\int_{\varphi_n(x)}^x b(t) \left( \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) dt$$

converge uniformly to 0 on  $[1/2, 1]$ . Moreover, it is clear that

$$\lim_{n \rightarrow \infty} \int_0^{\beta_n} b(t) dt \int_{\beta_n}^1 \frac{ds}{a(s)} = 0.$$

Then, it remains to show that

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [1/2, 1]} \int_0^x \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt \right) = 0. \tag{2.7}$$

For  $x \in [1/2, 1]$ , we have

$$\begin{aligned} \int_0^x \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt \\ = \int_0^{1/2} \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt \\ + \int_{1/2}^x \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt. \end{aligned}$$

Note that

$$\int_0^{1/2} \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt \leq \int_0^{1/2} |c_n b_n(t) - b(t)| dt \int_t^1 \frac{ds}{a(s)}.$$

Then, from Corollary 1.1,  $\lim_{n \rightarrow \infty} \int_0^{1/2} |c_n b_n(t) - b(t)| dt = 0$ , and so

$$\lim_{n \rightarrow \infty} \int_0^{1/2} \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt = 0.$$

We need to show that

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [1/2, 1]} \int_{1/2}^x \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt \right) = 0.$$

To this end, let us prove

$$\lim_{n \rightarrow \infty} \int_{1/2}^1 \left| c_n b_n(t) \int_t^1 \frac{ds}{a(s)} - b(t) \int_t^1 \frac{ds}{a(s)} \right| dt = 0. \tag{2.8}$$

For all  $t \in [1/2, 1)$ ,  $\lim_{n \rightarrow \infty} c_n b_n(t) \int_t^1 \frac{ds}{a(s)} = b(t) \int_t^1 \frac{ds}{a(s)}$  and

$$\int_{1/2}^1 c_n b_n(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt = \int_{1/2}^{\beta_n} b(t) \left( \int_{\psi_n(t)}^1 \frac{ds}{a(s)} \right) dt$$

$$\leq \int_{\frac{1}{2}}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt < \infty.$$

Moreover,

$$\begin{aligned} & \left| \int_{\frac{1}{2}}^1 c_n b_n(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt - \int_{\frac{1}{2}}^1 b(t) \int_t^1 \frac{ds}{a(s)} dt \right| \\ &= \left| \int_{\frac{1}{2}}^{\beta_n} b(t) \left( \int_{\varphi_n(t)}^1 \frac{ds}{a(s)} \right) dt - \int_{\frac{1}{2}}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt \right| \\ &\leq \int_{\frac{1}{2}}^1 b(t) \left( \int_{\varphi_n(t)}^t \frac{ds}{a(s)} \right) dt + \int_{\beta_n}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt. \end{aligned}$$

It is clear that  $\lim_{n \rightarrow \infty} \int_{\beta_n}^1 b(t) \left( \int_t^1 \frac{ds}{a(s)} \right) dt = 0$ ,  $\lim_{n \rightarrow \infty} b(t) \left( \int_{\varphi_n(t)}^t \frac{ds}{a(s)} \right) = 0$ , and for all  $t \in [1/2, 1)$ ,  $b(t) \left( \int_{\varphi_n(t)}^t \frac{ds}{a(s)} \right) \leq b(t) \left( \int_t^1 \frac{ds}{a(s)} \right)$ , so by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\frac{1}{2}}^1 b(t) \left( \int_{\varphi_n(t)}^t \frac{ds}{a(s)} \right) dt = 0.$$

Hence, by Corollary 1.1, (2.8) holds, which in turn implies (2.7).

Finally, from the above discussion, we conclude that

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [0,1]} \int_0^x \left| c_n^2 b_n(t) \int_x^1 \frac{ds}{a(s)} - b(t) \int_x^1 \frac{ds}{a(s)} \right| dt \right) = 0$$

and

$$\lim_{n \rightarrow \infty} \left( \sup_{x \in [0,1]} \int_0^1 |b_n(t) G_n(x, t) - b(t) G_n(x, t)| dt \right) = 0.$$

This completes the proof of the lemma.

LEMMA 2.5. Assume that (1.3)–(1.5) hold. Then, the spectral radius,  $r_L$ , of  $L$  satisfies  $r_L > 0$ , and there exists  $\varphi \in K^*$  such that  $L\varphi(t) = r_L\varphi(t)$ . Consequently,  $\lambda_1 := 1/r_L$  is the smallest positive eigenvalue of (2.1), i.e.,  $\varphi(t) = \lambda_1 L\varphi(t)$ .

*Proof.* By Lemma 2.3,  $L$  is completely continuous and it is clear that  $L$  maps  $K$  into  $K$ . Since the weight  $b$  does not vanish identically on  $(0, 1)$ , there exists  $[\gamma, \chi] \subset (0, 1)$  such that  $b > 0$  on  $[\gamma, \chi]$ . Choose  $u \in E$  such that  $u(t) \geq 0$  on  $[0, 1]$ ,  $u(t^*) > 0$  for some  $t^* \in [\gamma, \chi]$ , and  $u(t) = 0$  on  $[0, 1] \setminus [\gamma, \chi]$ . Then for  $t \in [\gamma, \chi]$ , we have

$$Lu(t) = \int_t^1 \frac{1}{a(s)} \left( \int_0^s b(\tau) u(\tau) d\tau \right) ds \geq \int_\chi^1 \frac{1}{a(s)} \left( \int_\gamma^s b(\tau) u(\tau) d\tau \right) ds > 0.$$

Thus, there exists  $c > 0$  such that  $cLu(t) \geq u(t)$  for  $t \in [0, 1]$ . Now, from [20, Chapter 5, Theorem 2.1], it follows that  $r_L > 0$ . Finally, since  $r_L > 0$  and  $K$  is a total cone, the conclusion of the lemma readily follows from the well known Krein–Rutman theorem (see, for example, [31, Proposition 7.26]). This completes the proof of the lemma.

We need the following fundamental result of Nussbaum [27] as quoted by Webb [29].

LEMMA 2.6. *Let  $\{L_n\}$  be a sequence of compact linear operators on a Banach space  $X$  and suppose that  $L_n \rightarrow L$  in operator norm as  $n \rightarrow \infty$ . Then  $r(L_n) \rightarrow r(L)$  where  $r(L_n)$  and  $r(L)$  denote the spectral radii of  $L_n$  and  $L$ , respectively.*

REMARK 2.1. For each  $n \in \mathbb{N}$ , let  $r_L^n$  be the spectral radius of  $L_n$ . As in the proof of Lemma 2.5, we see that  $r_L^n > 0$ . Then, by the Krein–Rutman theorem, there exists a normalized eigenfunction  $\phi_n$  (i.e.,  $\|\phi_n\| = 1$ ) such that  $L_n\phi_n(t) = r_L^n\phi_n(t)$  and  $\lambda_1^n = 1/r_L^n$ . Thus, Lemmas 2.4 and 2.6 imply that  $\lambda_1 = \lim_{n \rightarrow \infty} \lambda_1^n$ .

### 3. Nonexistence and existence results

Below, we let  $\lambda_1, \lambda_1^n, \phi_n, v_n, L_n, L$ , and  $I_n$  be as defined in Section 2.

THEOREM 3.1. *Assume that (1.3)-(1.5) hold. If*

$$\inf \left\{ \frac{f(t, u)}{u} : t \in [0, 1) \text{ and } u > 0 \right\} > \lambda_1,$$

then BVP (1.1) has no positive solution.

*Proof.* Assume, to the contrary, that BVP (1.1) has a positive solution  $u(t)$ . Integrating the equation in (1.1) on  $[\varepsilon, \zeta]$  with  $\varepsilon, \zeta \in (0, 1)$ , we have

$$a(\varepsilon)u'(\varepsilon) - a(\zeta)u'(\zeta) = \int_{\varepsilon}^{\zeta} b(s)(f(s, u(s))ds.$$

This implies that  $\lim_{\varepsilon \rightarrow 0} a(\varepsilon)u'(\varepsilon)$  exists; denote this value by  $\ell$ . We will consider two cases.

If  $\ell \leq 0$  and  $\phi_n$  is the solution of (2.4) corresponding to  $\lambda_1^n$ , we have

$$\begin{aligned} & a(\beta_n)u(\beta_n)\phi_n'(\beta_n) + a(\alpha_n)u'(\alpha_n)\phi_n(\alpha_n) \\ &= \int_{\alpha_n}^{\beta_n} [-(au')'(t)\phi_n(t) + (a\phi_n')'(t)u(t)]dt \\ &= \int_{\alpha_n}^{\beta_n} b(t)\phi_n(t)(f(t, u) - \lambda_1^n u(t))dt. \end{aligned}$$

This equality is impossible since  $a(\beta_n)u(\beta_n)\phi_n'(\beta_n) + a(\alpha_n)u'(\alpha_n)\phi_n(\alpha_n) \leq 0$ , but from Remark 2.1,  $\frac{f(t, u)}{u} > \lambda_1^n$  for large  $n$  implying that

$$\int_{\alpha_n}^{\beta_n} b(t)\phi_n(t)(f(t, u(t)) - \lambda_1^n u(t))dt > 0.$$

Now suppose  $\ell > 0$ . Since  $(au')'(t) = -b(t)f(t, u(t)) \leq 0$ , there exists a unique  $t_1 \in (0, 1)$  such that  $u'(t_1) = 0$  and  $u(t_1) = \|u\| = \max_{t \in [0,1]} u(t)$ . Moreover, there exists an interval  $I = (t_2, t_3) \subset (t_1, 1)$  on which  $b(t)$  does not vanish identically. For sufficiently large  $n$ , we then have

$$\begin{aligned} 0 &\geq a(t_3)u(t_3)\phi'_n(t_3) = \int_{t_1}^{t_3} [-(au')'(t)\phi_n(t) + (a\phi'_n)'(t)u(t)]dt \\ &= \int_{t_1}^{t_3} b(t)\phi_n(t)(f(t, u(t)) - \lambda_1^n u(t))dt > 0, \end{aligned}$$

which is a contradiction. This completes the proof of the theorem.

REMARK 3.1. Notice that  $\ell \neq 0$  holds only if  $\int_0^1 \frac{dt}{a(t)} < \infty$  and  $\lim_{x \rightarrow \infty} \frac{1}{a(x)} = 0$ .

THEOREM 3.2. Assume that (1.3)-(1.5) hold. If

$$\sup \left\{ \frac{f(t, u)}{u} : t \in [0, 1) \text{ and } u > 0 \right\} < \lambda_1,$$

then BVP (1.1) has no positive solution.

Proof. Assume, to the contrary, that BVP (1.1) has a positive solution  $u(t)$ . Then,

$$\begin{cases} -(au')'(t) = b(t)m(t)u(t), & t \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases}$$

where  $m(t) = \frac{f(t, u(t))}{u(t)}$ . From Property 3 of [2, Theorem 9], it follows that

$$\begin{aligned} 1 &= \lambda_1(a, bm, [0, 1]) = \lim_{n \rightarrow \infty} \lambda_1(a, bm, I_n) \\ &\geq \lim_{n \rightarrow \infty} \frac{\lambda_1(a, b, I_n)}{\alpha} = \frac{\lambda_1(a, b, [0, 1])}{\alpha} = \frac{\lambda_1}{\alpha} > 1, \end{aligned}$$

where  $\alpha = \sup \left\{ \frac{f(t, u)}{u} : t \in [0, 1) \text{ and } u > 0 \right\}$ . This is impossible and thus completes the proof of the theorem.

REMARK 3.2. In the above proof,  $\lambda_1(a, bm, [0, 1])$  is well defined because  $m$  is bounded and the weights  $a$  and  $bm$  satisfy conditions (1.3)-(1.5).

In the following, we will give some existence results for positive solutions of BVP (1.1). To this end, we introduce the following notations:

$$\begin{aligned} f^0 &= \limsup_{u \rightarrow 0} \left( \max_{t \in [0,1]} \frac{f(t, u)}{u} \right), & f^\infty &= \limsup_{u \rightarrow +\infty} \left( \max_{t \in [0,1]} \frac{f(t, u)}{u} \right), \\ f_0 &= \liminf_{u \rightarrow 0} \left( \min_{t \in [0,1]} \frac{f(t, u)}{u} \right), & f_\infty &= \liminf_{u \rightarrow +\infty} \left( \min_{t \in [0,1]} \frac{f(t, u)}{u} \right). \end{aligned}$$

Let  $F : E \rightarrow E$  be the Nemyckii operator defined by  $Fu(t) = f(t, u(t))$ . It is well known that  $F$  is continuous and maps bounded sets into bounded sets. For any  $n \in \mathbb{N}$ , we write  $T_n = L_n \circ F$  and  $T = L \circ F$ .

As was the case for  $L$  (see the proof of Lemma 2.3),  $L_n$  is compact. Then,  $T$  and  $T_n$  are completely continuous operators. Moreover, if  $\Omega$  is an open bounded set in  $E$ , and  $T_\Omega$  and  $T_{n,\Omega}$  are the restrictions of  $T$  and  $T_n$  to  $\Omega$ , respectively, then it follows from Lemma 2.4 that  $\lim_{n \rightarrow \infty} T_{n,\Omega} = T_\Omega$ . Furthermore, for any retract  $X$  of  $E$  and retraction  $r : E \rightarrow X$ , we have  $\lim_{n \rightarrow \infty} I - T_{n,X_\Omega} \circ r = I - T_{X_\Omega} \circ r$ , where  $X_\Omega = X \cap \Omega$ . Thus,  $\lim_{n \rightarrow \infty} i(T_{n,X_\Omega}, X_\Omega, X) = i(T_{X_\Omega}, X_\Omega, X)$ . It is clear that  $u$  is a fixed point of  $T$  in  $E$  if and only if  $u$  is a solution of BVP (1.1).

LEMMA 3.1. *Assume that (1.3)-(1.5) hold. If  $f^0 < \lambda_1$ , then there exists  $q^0 > 0$  such that  $i(T, K \cap B(0, q), K) = 1$  for all  $q \in (0, q^0]$ .*

*Proof.* In view of Remark 2.1, there exist  $q^0 > 0$  and  $N \in \mathbb{N}$  such that  $f(t, u) < \lambda_1^n u$  for all  $u \in [0, q^0]$  and  $n \geq N$ .

For any  $n \geq N$ , assume that for  $q \in (0, q^0]$ , there exist  $u \in K \cap \partial B(0, q)$  and  $\lambda \geq 1$  such that  $T_n u = \lambda u$ . Then, we have

$$\begin{cases} -(a_n u')'(t) = \lambda^{-1} (\beta_n - \alpha_n)^2 b_n(t) f(t, u(t)), & t \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \tag{3.1}$$

Multiplying the differential equation in (3.1) by a solution  $v_n$  of (2.5) and integrating over  $[0, 1]$ , we reach the contradiction

$$0 = \int_0^1 [-(a_n u')' v_n + (a_n v_n')' u] = (\beta_n - \alpha_n)^2 \int_0^1 b_n v_n (\lambda^{-1} f(t, u) - \lambda_1^n u) < 0.$$

Thus, from Lemma 1.2,  $i(T_n, K \cap B(0, q), K) = 1$  for any  $n \geq N$ . Hence,

$$i(T, K \cap B(0, q), K) = \lim_{n \rightarrow \infty} i(T_n, K \cap B(0, q), K) = 1.$$

This completes the proof of the lemma.

LEMMA 3.2. *Assume that (1.3)-(1.5) hold. If  $f_0 > \lambda_1$ , then there exists  $q_0 > 0$  such that  $i(T, K \cap B(0, q), K) = 0$  for all  $q \in (0, q_0]$ .*

*Proof.* Let  $\varepsilon > 0$  be such that  $f_0 > \lambda_1 + \varepsilon$ . Then there exists  $n_0 \in \mathbb{N}$  such that  $n \geq n_0$  implies  $f_0 > \lambda_1 + \varepsilon > \lambda_1^n$ . Thus, there exists  $q_0 > 0$  such that  $f(t, u) > \lambda_1^n u$  for all  $u \in [0, q_0]$  and  $n \geq n_0$ . Let

$$K_n = \{u \in K : u(x) \geq \rho_n(x) \|u\| \text{ for all } x \in [0, 1]\},$$

where  $\rho_n(x) = \frac{1}{\bar{\rho}_n} \int_x^1 \frac{ds}{a_n(s)}$  with  $\bar{\rho}_n = \int_0^1 \frac{ds}{a_n(s)}$ .

We claim that  $T_n(K) \subset K_n$ . In fact, from the definition of  $G_n(t, s)$ , we see that  $G_n(x, t) \leq G_n(t, t)$  for  $x, t \in [0, 1]$  and

$$\begin{aligned} \frac{G_n(x, t)}{G_n(t, t)} &= \begin{cases} \frac{\int_x^1 \frac{ds}{a_n(s)}}{\int_t^1 \frac{ds}{a_n(s)}} & \text{if } 0 \leq t \leq x \leq 1, \\ 1, & \text{if } 0 \leq x \leq t \leq 1, \end{cases} \\ &\geq \frac{\int_x^1 \frac{ds}{a_n(s)}}{\int_0^1 \frac{ds}{a_n(s)}} = \rho_n(x). \end{aligned}$$

Thus,  $G_n(x, t) \geq \rho_n(x)G_n(t, t)$  for  $x, t \in [0, 1]$ . Hence, for  $x \in [0, 1]$ , we have

$$\begin{aligned} T_n u(x) &= \int_0^1 G_n(x, t) b_n(t) f(t, u(t)) dt \\ &\geq \rho_n(x) \int_0^1 G_n(t, t) b_n(t) f(t, u(t)) dt \geq \rho_n(x) \|T_n u\|, \end{aligned}$$

i.e.,  $T_n(K) \subset K_n$ .

For any  $n \geq n_0$ , assume that, for  $q \in (0, q_0]$ , there exist  $u \in K_n \cap \partial B(0, q)$  and  $0 < \lambda \leq 1$  such that  $T_n u = \lambda u$ . Multiplying the differential equation in (3.1) by  $v_n$  and integrating over  $[0, 1]$ , we obtain the contradiction

$$0 = \int_0^1 [-(a_n u')' v_n + (a_n v_n')' u] = (\beta_n - \alpha_n)^2 \int_0^1 b_n v_n (\lambda^{-1} f(t, u) - \lambda_1^n u) > 0.$$

Moreover, for  $u \in K_n \cap \partial B(0, q)$ ,

$$\begin{aligned} \|T_n u\| &= T_n u(0) = (\beta_n - \alpha_n)^2 \int_0^1 \frac{1}{a(s)} \left( \int_0^s b(t) f(t, u(t)) dt \right) ds \\ &> \lambda_1^n (\beta_n - \alpha_n)^2 \int_0^1 \frac{1}{a_n(s)} \left( \int_0^s b_n(t) u(t) dt \right) ds \\ &\geq \lambda_1^n q (\beta_n - \alpha_n)^2 \int_0^1 \frac{1}{a_n(s)} \left( \int_0^s b_n(t) \rho_n(t) dt \right) ds > 0. \end{aligned}$$

Hence,

$$\begin{aligned} \inf \{ \|T_n u\| : u \in K_n \cap \partial B(0, q) \} \\ \geq \lambda_1^n q (\beta_n - \alpha_n)^2 \int_0^1 \frac{1}{a_n(s)} \left( \int_0^s b_n(t) \rho_n(t) dt \right) ds > 0. \end{aligned}$$

Then, by Lemma 1.3,  $i(T_n, K_n \cap B(0, q), K_n) = 0$  for all  $n \geq n_0$ . This, together with the permanence property of the fixed point index, implies that  $i(T_n, K \cap B(0, q), K) = 0$  for all  $n \geq n_0$ . Thus,  $i(T, K \cap B(0, q), K) = \lim_{n \rightarrow \infty} i(T_n, K \cap B(0, q), K) = 0$ . This completes the proof of the lemma.

LEMMA 3.3. Assume that (1.3)-(1.5) hold. If  $f^\infty < \lambda_1$ , then there exists  $q^\infty > 0$  such that  $i(T, K \cap B(0, q^\infty), K) = 1$ .



*Proof.* Let  $\alpha > 0$  be such that  $\alpha < \lambda_1$  and let us compute  $i(\alpha L, K \cap B(0, q), K)$  for any  $q > 0$ . Arguing as in the proof of Lemma 3.1, we obtain

$$i(\alpha L_n, K \cap B(0, q), K) = 1,$$

so

$$i(\alpha L, K \cap B(0, q), K) = \lim_{n \rightarrow \infty} i(\alpha L_n, K \cap B(0, q), K) = 1.$$

Now consider the BVP

$$\begin{cases} -(au')'(s) = b(s)(t\alpha u(s) + (1-t)f(s, u(s))), & s \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \tag{3.2}$$

where  $t \in [0, 1]$ . We claim that there exists  $q^\infty > 0$  such that (3.2) has no solution in  $K \cap \partial B(0, q^\infty)$  for all  $t \in [0, 1]$ . Assume, to the contrary, that for all  $q > 0$  there exist  $t_q \in [0, 1]$  and  $u_q \in K \cap \partial B(0, q)$  such that  $t_q \alpha L u + (1 - t_q) T u_q = u_q$ . Now, if  $\{q_n\}$  is a sequence such that

$$\lim_{n \rightarrow \infty} q_n = +\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \left( \max_{s \in [0, 1]} \frac{f(s, q_n)}{q_n} \right) = f^\infty,$$

then there exist  $\{u_n\} \subset K$  and  $\{t_n\}$  such that  $\|u_n\| = q_n$ ,  $t_n \in [0, 1]$ , and  $t_n \alpha L u_n + (1 - t_n) T u_n = u_n$ . Clearly, the sequence  $\{w_n\}$  given by  $w_n = u_n / \|u_n\|$  is bounded ( $\|w_n\| = 1$ ) and satisfies

$$w_n(x) = \int_x^1 \frac{1}{a(s)} \left( \int_0^s b(\sigma) \left( t_n \alpha w_n(\sigma) + (1 - t_n) \frac{f(\sigma, u_n(\sigma))}{\|u_n\|} \right) d\sigma \right) ds. \tag{3.3}$$

For  $n \in \mathbb{N}$ , let  $\theta_n(\sigma) = t_n \alpha w_n(\sigma) + (1 - t_n) \frac{f(\sigma, u_n(\sigma))}{\|u_n\|}$ . The condition  $f^\infty < \lambda_1$  implies that there exists  $C > 0$  such that  $f(\sigma, u) < \lambda_1 u + C$  for all  $(\sigma, u) \in [0, 1] \times [0, +\infty)$ . Thus,

$$\theta_n(\sigma) \leq \left( \lambda_1 w_n(\sigma) + \frac{C}{\|u_n\|} \right) \leq \lambda_1 + C.$$

Since  $w_n = L(\theta_n)$  and  $L$  is compact, there exists a subsequence of  $\{w_n\}$ , also denoted by  $\{w_n\}$ , that converges to  $w$  in  $E$  with  $\|w\| = 1$ . Without loss generality, we may assume that  $t_\infty = \lim_{n \rightarrow \infty} t_n$ . Then, taking the limit as  $n \rightarrow \infty$  in (3.3), we see that  $w$  satisfies

$$\begin{aligned} w(x) &\leq \int_x^1 \frac{1}{a(s)} \left( \int_0^s b(\sigma) (t_\infty \alpha + (1 - t_\infty) f^\infty) w(\sigma) d\sigma \right) ds \\ &= (t_\infty \alpha + (1 - t_\infty) f^\infty) Lw(x). \end{aligned}$$

Let  $\xi = (t_\infty \alpha + (1 - t_\infty) f^\infty)^{-1}$  and  $\tilde{L} = \frac{L}{\xi}$ . Then, we have

$$w \leq \tilde{L}w \leq \tilde{L}^2 w \leq \dots \leq \tilde{L}^n w \leq \dots$$

Hence,

$$1 = \|w\|^{\frac{1}{n}} \leq \|\tilde{L}^n w\|^{\frac{1}{n}} \leq \|\tilde{L}^n\|^{\frac{1}{n}} = \frac{\|L^n\|^{\frac{1}{n}}}{\xi}.$$

Passing to the limit, we obtain that  $\xi \leq r(L)$ , where  $r(L)$  is the spectral radius of  $L$ .

Since  $\frac{1}{\lambda_1} = r(L)$ , we obtain the contradiction

$$\frac{1}{\lambda_1} = r(L) \geq \xi = \frac{1}{(t_\infty \alpha + (1 - t_\infty) f^\infty)} > \frac{1}{\lambda_1}.$$

Finally, by the homotopy property of the fixed point index,

$$\begin{aligned} i(T, K \cap B(0, q^\infty), K) &= i(t\alpha L + (1 - t)T, K \cap B(0, q^\infty), K) \\ &= i(\alpha L, K \cap B(0, q^\infty), K) = 1. \end{aligned}$$

This complete the proof of the lemma.

LEMMA 3.4. Assume that (1.3)–(1.5) hold. If  $\lambda_1 < f_\infty \leq f^\infty < \infty$ , then there exists  $q_\infty > 0$  such that  $i(T, K \cap B(0, q_\infty), K) = 0$ .

*Proof.* Let  $\alpha > 0$  be such that  $\alpha > \lambda_1$  and let us compute  $i(\alpha L, K \cap B(0, q), K)$  for any  $q > 0$ . Using the cone  $K_n$  and arguing as in the proof of Lemma 3.2, we obtain

$$i(\alpha L_n, K_n \cap B(0, q), K_n) = 0.$$

Then, by the permanence property of the fixed point index, we have

$$\begin{aligned} i(\alpha L, K \cap B(0, q), K) &= \lim_{n \rightarrow \infty} i(\alpha L_n, K \cap B(0, q), K) \\ &= \lim_{n \rightarrow \infty} i(\alpha L_n, K_n \cap B(0, q), K_n) = 0. \end{aligned}$$

Consider the BVP

$$\begin{cases} -(au')'(s) = b(s)(t\alpha u(s) + (1 - t)f(s, u(s))), & s \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \tag{3.4}$$

where  $t \in [0, 1]$ . We claim that there exists  $q_\infty > 0$  such that (3.4) has no solution in  $K \cap \partial B(0, q_\infty)$  for all  $t \in [0, 1]$ . Assume, to the contrary, that for all  $q > 0$  there exist  $t_q \in [0, 1]$  and  $u_q \in K \cap \partial B(0, q)$  such that  $t_q \alpha L u_q + (1 - t_q) T u_q = u_q$ . Then, if  $\{q_n\}$  is the sequence such that

$$\lim_{n \rightarrow \infty} q_n = \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \left( \min_{s \in [0, 1]} \frac{f(s, q_n)}{q_n} \right) = f_\infty,$$

there exist  $\{u_n\}$  and  $\{t_n\}$  such that  $\|u_n\| = q_n$ ,  $t_n \in [0, 1]$ , and  $t_n \alpha L u_n + (1 - t_n) T u_n = u_n$ . Clearly, the sequence  $\{w_n\}$  given by  $w_n = u_n / \|u_n\|$  is bounded ( $\|w_n\| = 1$ ) and satisfies

$$w_n(x) = \int_x^1 \frac{1}{a(s)} \left( \int_0^s b(\sigma) \left( t_n \alpha w_n(\sigma) + (1 - t_n) \frac{f(\sigma, u_n(\sigma))}{\|u_n\|} \right) d\sigma \right) ds.$$

Taking into consideration the additional hypothesis in Lemma 3.4 and arguing as in the proof of Lemma 3.3, we see that the sequence  $\{w_n\}$  has a subsequence, also denoted by  $\{w_n\}$ , which converges to  $w$  in  $E$  with  $\|w\| = 1$  and

$$w(x) \geq \int_x^1 \frac{1}{a(s)} \left( \int_0^s b(\sigma) (\bar{t}\alpha + (1-\bar{t})f_\infty) w(\sigma) d\sigma \right) ds.$$

Then,  $\omega = Lw$  satisfies

$$\begin{cases} -(a\omega)'(s) \geq b(s) (\bar{t}\alpha + (1-\bar{t})f_\infty) \omega(s), & s \in (0, 1), \\ \omega'(0) = \omega(1) = 0. \end{cases} \tag{3.5}$$

Multiplying (3.5) by a solution  $\phi_n$  of (2.4) and integrating over  $I_n$  yields

$$\begin{aligned} a(\beta_n) \omega(\beta_n) \phi_n'(\beta_n) + a(\alpha_n) \omega'(\alpha_n) \phi_n(\alpha_n) &= \int_{\alpha_n}^{\beta_n} [-(a\omega)'\phi_n + (a\phi_n')'\omega] \\ &\geq (\bar{t}\alpha + (1-\bar{t})f_\infty - \lambda_1^n) \int_{\alpha_n}^{\beta_n} b\omega\phi_n. \end{aligned}$$

But this inequality is impossible since

$$a(\beta_n) \omega(\beta_n) \phi_n'(\beta_n) + a(\alpha_n) \omega'(\alpha_n) \phi_n(\alpha_n) < 0$$

and  $\bar{t}\alpha + (1-\bar{t})f_\infty > \lambda_1^n$  for  $n$  large enough.

Finally, by the homotopy property of the fixed point index, we have

$$\begin{aligned} i(T, K \cap B(0, q_\infty), K) &= i(t\alpha L + (1-t)T, K \cap B(0, q_\infty), K) \\ &= i(\alpha L, K \cap B(0, q_\infty), K) = 0. \end{aligned}$$

This completes the proof of the lemma.

LEMMA 3.5. Assume that (1.3) and (1.4) hold. If  $f_\infty = \infty$  and  $\int_0^1 \frac{ds}{a(s)} < \infty$ , then there exist  $q_\infty > 0$  such that  $i(T, K \cap B(0, q), K) = 0$  for all  $q > q_\infty$ .

*Proof.* Define

$$\bar{K} = \{u \in K : u(x) \geq p(x) \|u\| \text{ for all } x \in [0, 1]\},$$

where

$$p(x) = \frac{1}{\bar{a}} \int_x^1 \frac{ds}{a(s)} \quad \text{and} \quad \bar{a} = \int_0^1 \frac{ds}{a(s)}.$$

Then, using an argument similar to the one in the proof of Lemma 3.2 to show  $T_n(K) \subset K_n$ , we can prove that  $T(K) \subset \bar{K}$ . By the permanence property of the fixed point index,

$$i(T, K \cap B(0, q), K) = i(T, \bar{K} \cap B(0, q), \bar{K}) \quad \text{for all } q > 0.$$

We will compute  $i(T, \bar{K} \cap B(0, q), \bar{K})$  for  $q$  large enough.

Let  $\alpha$  be such that

$$\alpha \int_0^1 \frac{1}{a(t)} \int_0^t b(s)p(s)dsdt > 1.$$

Since  $f_\infty = \infty$ , there exists  $C > 0$  such that

$$f(s, u) \geq \alpha u - C \quad \text{for all } (s, u) \in [0, 1] \times [0, +\infty).$$

Let

$$A = C \int_0^1 \frac{1}{a(t)} \int_0^t b(s)dsdt.$$

Then, for any  $q \geq q_\infty$  with

$$q_\infty = \left( \alpha \int_0^1 \frac{1}{a(t)} \int_0^t b(s)p(s)dsdt - 1 \right)^{-1} CA \quad \text{and } u \in \overline{K} \cap \partial B(0, q),$$

we have

$$\begin{aligned} \|Tu\| &= Tu(0) \geq \alpha \int_0^1 \frac{1}{a(t)} \int_0^t b(s)u(s)dsdt - CA \\ &\geq \alpha \|u\| \int_0^1 \frac{1}{a(t)} \int_0^t b(s)p(s)dsdt - CA \\ &= \alpha q \int_0^1 \frac{1}{a(t)} \int_0^t b(s)p(s)dsdt - CA \\ &\geq q = \|u\|. \end{aligned}$$

Hence, by Lemma 1.4,  $i(T, K \cap B(0, q), K) = i(T, \overline{K} \cap B(0, q), \overline{K}) = 0$ . This complete the proof of the lemma.

**THEOREM 3.3.** *Assume that (1.3)-(1.5) hold. If  $f^\infty < \lambda_1 < f_0$ , then BVP (1.1) has at least one positive solution.*

*Proof.* From Lemmas 3.2 and 3.3, we have that for  $0 < q < q_0$ ,

$$\begin{aligned} i(T, K \cap (B(0, q^\infty) \setminus B(0, q)), K) \\ = i(T, K \cap B(0, q^\infty), K) - i(T, K \cap B(0, q), K) = 1. \end{aligned}$$

Then, from the solution property of the fixed point index, it follows that  $T$  has a fixed point  $u$  with  $q < \|u\| < q^\infty$  and which is a positive solution to BVP (1.1).

**THEOREM 3.4.** *Assume that (1.3)-(1.5) hold. If  $f^0 < \lambda_1 < f_\infty \leq f^\infty < \infty$ , then BVP (1.1) has at least one positive solution.*

*Proof.* From Lemmas 3.1 and 3.4, we have that for  $0 < q < q^0$ ,

$$\begin{aligned} i(T, K \cap (B(0, q_\infty) \setminus B(0, q)), K) \\ = i(T, K \cap B(0, q_\infty), K) - i(T, K \cap B(0, q), K) = -1. \end{aligned}$$

Then, from the solution property of the fixed point index, it follows that  $T$  has a fixed point  $u$  with  $q < \|u\| < q_\infty$  and which is a positive solution to BVP (1.1).

**THEOREM 3.5.** *Assume that (1.3)-(1.5) hold. If  $f^0 < \lambda_1$ ,  $f_\infty = \infty$  and  $\int_0^1 \frac{ds}{a(s)} < \infty$ , then BVP (1.1) has at least one positive solution.*

*Proof.* From Lemmas 3.1 and 3.5, we have that for  $0 < q_1 < q^0 < q_\infty < q_2$ ,

$$\begin{aligned} i(T, K \cap (B(0, q_2) \setminus B(0, q_1)), K) \\ = i(T, K \cap B(0, q_2), K) - i(T, K \cap B(0, q_1), K) = -1. \end{aligned}$$

Then, from the solution property of the fixed point index, it follows that  $T$  has a fixed point  $u$  with  $q_1 < \|u\| < q_2$  and which is a positive solution to BVP (1.1).

#### 4. Existence of positive radial solutions to an elliptic equation

Consider the BVP

$$\begin{cases} -\Delta u(x) = q(x)f(u(x)), & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where  $\Omega$  is an open set in  $\mathbb{R}^n$  and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous. The corresponding linear eigenvalue problem to BVP (4.1) is

$$\begin{cases} -\Delta u(x) = \lambda q(x)u(x), & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this section, we will provide existence and nonexistence results for positive radial solutions to BVP (4.1) in both the cases  $\Omega = B(0, 1)$  (the unit ball) or  $\Omega = \mathbb{R}^n$  with  $n \geq 3$ . Below, we assume that  $q : \Omega \setminus \{0\} \rightarrow \mathbb{R}^+$  is continuous, and for all  $x \in \Omega \setminus \{0\}$ ,  $q(x) = q(|x|)$ .

We will use the following notations.

$$\begin{aligned} f^0 &= \limsup_{u \rightarrow 0} \frac{f(u)}{u}, & f^\infty &= \limsup_{u \rightarrow +\infty} \frac{f(u)}{u}, \\ f_0 &= \liminf_{u \rightarrow 0} \frac{f(u)}{u}, & f_\infty &= \liminf_{u \rightarrow +\infty} \frac{f(u)}{u}. \end{aligned}$$

#### 4.1. $\Omega = B(0, 1)$ , the unit ball in $\mathbb{R}^n$

In this case, a radial solution to BVP (4.1) is a solution of the BVP

$$\begin{cases} -(t^{n-1}u')' = t^{n-1}q(t)f(u(t)), & t \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \quad (4.2)$$

The corresponding linear problem to BVP (4.2) is

$$\begin{cases} -(t^{n-1}u')' = \lambda t^{n-1}q(t)u(t), & t \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \quad (4.3)$$

**COROLLARY 4.1.** *Assume that  $0 < \int_0^1 (1-t)q(t)dt < \infty$ . Then BVP (4.3) has a smallest positive eigenvalue  $\lambda_1$  such that*

$$\text{if either } \inf \left\{ \frac{f(x)}{x} : x > 0 \right\} > \lambda_1 \text{ or } \sup \left\{ \frac{f(x)}{x} : x > 0 \right\} < \lambda_1,$$

*then BVP (4.2) has no positive solution. Moreover,*

$$\text{if either } f^\infty < \lambda_1 < f_0 \text{ or } f^0 < \lambda_1 < f_\infty \leq f^\infty < \infty,$$

*then BVP (4.2) has a positive solution.*

*Proof.* This corollary follows from Lemma 2.5 and Theorems 3.1, 3.2, 3.3, and 3.4 once conditions (1.3), (1.4), and (1.5) are shown to hold. Let  $a(t) = t^{n-1}$  and  $b(t) = t^{n-1}q(t)$ . Then, for  $t \in (0, 1/2)$ , we have

$$\begin{aligned} \frac{1}{a(t)} \int_0^t b(s)ds &= \int_0^t \left(\frac{s}{t}\right)^{n-1} q(s)ds \\ &\leq \int_0^t q(s)ds = \int_0^t (1-s)q(s)ds + \int_0^t sq(s)ds \\ &\leq 2 \int_0^t (1-s)q(s)ds. \end{aligned}$$

Thus,

$$\lim_{t \rightarrow 0} \frac{1}{a(t)} \int_0^t b(s)ds = \lim_{t \rightarrow 0} 2 \int_0^t (1-s)q(s)ds = 0,$$

so (1.3) holds. For all  $x \in (0, 1)$ , we have

$$\begin{aligned} \int_0^x \frac{1}{t^{n-1}} \left( \int_0^t s^{n-1} q(s)ds \right) dt &= \int_0^x \left( \int_0^t \left(\frac{s}{t}\right)^{n-1} q(s)ds \right) dt \\ &\leq \int_0^x \left( \int_0^t q(s)ds \right) dt \\ &= \int_0^x (x-t)q(t)dt \end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 (x-t)q(t)dt \\ &\leq \int_0^1 (1-t)q(t)dt < \infty, \end{aligned}$$

which shows that (1.4) holds. Finally, it is clear that

$$\begin{aligned} \int_0^1 t^{n-1}q(t) \left( \int_t^1 \frac{ds}{s^{n-1}} \right) dt &= \int_0^1 q(t) \left( \int_t^1 \frac{t^{n-1}ds}{s^{n-1}} \right) dt \\ &\leq \int_0^1 (1-t)q(t)dt. \end{aligned}$$

This shows that (1.5) holds and completes the proof of the corollary.

**4.2.**  $\Omega = \mathbb{R}^n, n \geq 3$

In this case, a radial solution to BVP (4.1) is a solution of the BVP

$$\begin{cases} -(t^{n-1}u')' = t^{n-1}q(t)f(u(t)), & t \in (0, \infty), \\ u'(0) = u(\infty) = 0. \end{cases} \tag{4.4}$$

The corresponding linear eigenvalue problem to BVP (4.4) is

$$\begin{cases} -(t^{n-1}u')' = \lambda t^{n-1}q(t)u(t), & t \in (0, \infty), \\ u'(0) = u(\infty) = 0. \end{cases} \tag{4.5}$$

**COROLLARY 4.2.** Assume that  $q \in C^1((0, \infty), (0, \infty))$ ,

$$0 < \int_0^\infty q(t)dt, \quad \int_0^\infty tq^2(t)dt < \infty,$$

and

$$\lim_{t \rightarrow 0} \frac{1}{q(t)} \int_0^t q^2(s)ds = 0.$$

Then BVP (4.5) has a smallest positive eigenvalue  $\lambda_1$  such that

$$\text{if either } \inf \left\{ \frac{f(x)}{x} : x > 0 \right\} > \lambda_1 \text{ or } \sup \left\{ \frac{f(x)}{x} : x > 0 \right\} < \lambda_1,$$

then BVP (4.4) has no positive solution. Moreover,

$$\text{if either } f^\infty < \lambda_1 < f_0 \text{ or } f^0 < \lambda_1 < f_\infty \leq f^\infty < \infty,$$

then BVP (4.4) has a positive solution.

*Proof.* Let  $\varphi : \mathbb{R}^+ \rightarrow [0, 1)$  be given by

$$\varphi(t) = \frac{1}{|q|_1} \int_0^t q(s) ds$$

and  $\psi = \varphi^{-1}$ , where  $|q|_1 = \int_0^\infty |q(s)| ds$ . Consider the singular BVP

$$\begin{cases} -(av')'(x) = a(x)|q|_1 f(v(x)), & x \in (0, 1), \\ v'(0) = v(1) = 0, \end{cases} \quad (4.6)$$

where

$$a(x) = \frac{\psi^{n-1}(x)}{\psi'(x)} \quad \text{for } x \in (0, 1).$$

The corresponding linear eigenvalue problem to BVP (4.6) is

$$\begin{cases} -(av')'(x) = \lambda a(x)|q|_1 v(x), & x \in (0, 1), \\ v'(0) = v(1) = 0. \end{cases} \quad (4.7)$$

It is easy to see that  $v$  is a positive solution to BVP (4.6) if and only if  $u(t) = v(\varphi(t))$  is a positive solution to BVP (4.4) and  $\lambda$  is the positive eigenvalue of (4.5) if and only if  $\lambda$  is the positive eigenvalue of (4.7). Therefore, the desired conclusion is obtained from Lemma 2.5 and Theorems 3.1, 3.2, 3.3, and 3.4 once conditions (1.3), (1.4), and (1.5) are verified.

Using the change of variables  $x = \varphi(y)$ , we obtain

$$\begin{aligned} \frac{1}{a(x)} \int_0^x a(t) dt &= \frac{\psi'(x)}{\psi^{n-1}(x)} \int_0^x \frac{\psi^{n-1}(t)}{\psi'(t)} dt \\ &= \frac{|q|_1}{y^{n-1}q(y)} \int_0^{\varphi(y)} \frac{\psi^{n-1}(t)}{\psi'(t)} dt. \end{aligned}$$

Substituting  $t = \varphi(s)$  into the last integral, we have

$$\begin{aligned} \frac{1}{a(x)} \int_0^x a(t) dt &= \frac{1}{|q|_1 y^{n-1} q(y)} \int_0^y s^{n-1} q^2(s) ds \\ &\leq \frac{1}{|q|_1 q(y)} \int_0^y q^2(s) ds. \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{1}{a(x)} \int_0^x a(t) dt = \lim_{y \rightarrow 0} \frac{1}{|q|_1 q(y)} \int_0^y q^2(s) ds = 0,$$

so (1.3) holds. Moreover, we have

$$\begin{aligned} \int_0^1 \frac{1}{a(x)} \left( \int_0^x a(t) dt \right) dx &= \int_0^1 \frac{\psi'(x)}{\psi^{n-1}(x)} \left( \int_0^x \frac{\psi^{n-1}(t)}{\psi'(t)} dt \right) dx \\ &= \int_0^\infty \frac{1}{y^{n-1}} \left( \int_0^{\varphi(y)} \frac{\psi^{n-1}(t)}{\psi'(t)} dt \right) dy \end{aligned}$$



$$= \frac{1}{|q|_1^2} \int_0^{+\infty} \frac{1}{y^{n-1}} \left( \int_0^y s^{n-1} q^2(s) ds \right) dy.$$

For  $H$  large and  $\varepsilon$  small, an integration by parts gives

$$\begin{aligned} & \int_\varepsilon^H \frac{1}{y^{n-1}} \left( \int_0^y s^{n-1} q^2(s) ds \right) dy \\ &= -\frac{1}{n-2} \int_0^H \left( \frac{y}{H} \right)^{n-2} y q^2(y) dy \\ & \quad + \frac{1}{n-2} \frac{1}{\varepsilon^{n-2}} \int_0^\varepsilon y^{n-1} q^2(y) dy + \frac{1}{n-2} \int_\varepsilon^H y q^2(y) dy \\ &\leq -\frac{1}{n-2} \int_0^H \left( \frac{y}{H} \right)^{n-2} y q^2(y) dy \\ & \quad + \frac{1}{n-2} \int_0^\varepsilon y q^2(y) dy + \frac{1}{n-2} \int_\varepsilon^H y q^2(y) dy. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\begin{aligned} \int_0^H \frac{1}{y^{n-1}} \left( \int_0^y s^{n-1} q^2(s) ds \right) dy &\leq \frac{1}{n-2} \int_0^H \left( 1 - \left( \frac{y}{H} \right)^{n-2} \right) y q^2(y) dy \\ &\leq \frac{1}{n-2} \int_0^H y q^2(y) dy. \end{aligned}$$

Thus,

$$\int_0^\infty \frac{1}{y^{n-1}} \left( \int_0^y s^{n-1} q^2(s) ds \right) dy \leq \frac{1}{n-2} \int_0^\infty y q^2(y) dy < \infty,$$

and so (1.4) holds. Finally, we have

$$\begin{aligned} \int_0^1 a(x) \left( \int_x^1 \frac{1}{a(t)} dt \right) dx &= \int_0^1 \frac{\psi^{n-1}(x)}{\psi'(x)} \left( \int_x^1 \frac{\psi'(t)}{\psi^{n-1}(t)} dt \right) dx \\ &= \frac{1}{|q|_1^2} \int_0^\infty y^{n-1} q^2(y) \left( \int_{\phi(y)}^1 \frac{\psi'(t)}{\psi^{n-1}(t)} dt \right) dy \\ &= \frac{1}{|q|_1^2} \int_0^\infty y^{n-1} q^2(y) \left( \int_y^1 \frac{1}{s^{n-1}} ds \right) dy \\ &= \frac{1}{(n-2)|q|_1^2} \int_0^\infty y q^2(y) dy < \infty, \end{aligned}$$

so (1.5) holds, and this completes the proof of the corollary.

In conclusion, we wish to point out that in Corollaries 4.1 and 4.2 we state that  $\lambda_1$  is the smallest positive eigenvalue, and this is true because in each case 0 is not an eigenvalue. In the case where  $\Omega$  is the unit ball in  $\mathbb{R}^n$ ,

$$a(x) = x^{n-1} \text{ so } \int_0^1 \frac{ds}{a(s)} = \infty \text{ if } n \geq 2, \text{ and } \lim_{s \rightarrow 0} \frac{1}{a(s)} \neq 0 \text{ if } n = 1.$$

In the case  $\Omega$  is  $\mathbb{R}^n$  with  $n \geq 3$ ,

$$a(x) = \frac{\psi^{n-1}(x)}{\psi'(x)} \quad \text{and} \quad \int_0^1 \frac{ds}{a(s)} = \int_0^\infty \frac{dt}{t^{n-1}} = +\infty \quad \text{since } n \geq 3.$$

Moreover, if  $\lambda_0 > 0$  is a positive eigenvalue, then  $\frac{1}{\lambda_0} \leq r(L) = \frac{1}{\lambda_1}$ , that is  $\lambda_1 \leq \lambda_0$ .

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