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ON THE ASYMPTOTIC BEHAVIOURS OF SOLUTIONS OF THIRD ORDER NON–LINEAR AUTONOMOUS DIFFERENTIAL EQUATION GOVERNING THE MHD FLOW

B. B. SINGH AND I. M. CHANDARKI

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Abstract. This paper deals with the asymptotic behaviour as $t \to \infty$ of the solutions for a steady laminar incompressible boundary layer equations governing the MHD flow near the forward stagnation point of two-dimensional and axisymmetric bodies. The asymptotic behaviour of the solutions is based on the method of asymptotic integration of second order linear differential equations. The results pertaining to the asymptotic behaviour of the solutions are also expressed in the form of Theorems 4.1 and 4.2.

1. Introduction

Boundary layer flow of electrically conducting fluid over moving surfaces emerges in a large variety of industrial and technological applications. As a result, it has been investigated by many researchers. Wu [47] has studied the effects of suction or injection on a steady two-dimensional MHD boundary layer flow on a flat plate. Takhar et al. [44] studied a MHD asymmetric flow over a semi-infinite moving surface and numerically obtained the solutions. An analysis of heat and mass transfer characteristics in an electricity conducting fluid over a linearly stretching sheet with variable wall temperature was investigated by Vajruvelu and Rollins [48]. Mahapatra and Gupta [28] treated the steady two-dimensional stagnation point flow of an incompressible viscous electrically conducting fluid towards a stretching surface; the flow being permeated by a uniform transverse magnetic field. For more details, see also [9], [30], [31], [43] and the references therein.

Motivated by the above works, we aim here to give the asymptotic behaviours of the solutions $f = f(t)$ of the third order non-linear autonomous differential equation governing the magnetohydrodynamic (MHD) flow near the forward stagnation point of two-dimensional and axisymmetric bodies:

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) + M(1 - f') = 0 \text{ on } [0, \infty) \tag{1}$$

accompanied by the boundary conditions

$$f(0) = a, \quad f'(0) = b \quad \text{and} \quad f'(\infty) = 1, \tag{2}$$

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where $a,b,m,M \in \mathbb{R}$, $f' = \frac{df}{dt}$ and $f'(\infty) = \lim_{t\to\infty} f'(t)$.

The equation (1) is very interesting because it contains many known equations as particular cases. Setting $M = 0$ in (1), leads to the well-known Falkner-Skan equation (see [11], [12], [16] and the references therein), while the case $M = -m$ reduces (1) to the equation that arises when considering the mixed convection in a fluid saturated porous medium near a semi-infinite vertical flat plate with prescribed temperature studied by many authors like [1], [6], [15], [26] and the references therein. The case $M = m = 0$ is referred to the Blasius equation introduced in [4] and studied by several authors (see for example [2], [3], [46]). Recently, the case $m = -1$ has been studied in [7]. Mention may be made also to the reference [5], where the authors show existence of an infinite number of similarity solutions for the case of a non-Newtonian fluid.

The objective of the present paper is to study the asymptotic behaviours of the solutions of equations (1)-(2). The study of the asymptotic nature of the third order nonlinear differential equation governing wedge flow was initiated by Hartman [18]. Serrin [32] also studied the asymptotic behaviours of third order nonlinear differential equation governing steady two-dimensional laminar flow of an incompressible viscous fluid past a rigid wall. The asymptotic behaviour of ordinary differential equation is also due to Cesari [8]. Later, the study of the asymptotic behaviours of the differential equations of Falkner-Skan type governing the various flow fields was carried out by Chinquing et al. [10], Harri and Pucci [17], Kumar and Singh [25], Parhi and Dass [29], Singh and Kumar [34], Singh and Singh [35]- [39], Singh and verma [40], Singh [41]- [42], Tiryaki and Yaman [45], etc.

Linearization of the third order nonlinear differential equation to second order homogeneous differential equation is due to Kocic [24]. Beside the investigations of nonlinear equations equivalent to linear ones of same order, the nonlinear equations which, by suitable transformations, are reduced to linear ones of different orders are also studied by ([13], [20], [23]). For the present case, linearization of the second order nonlinear differential equation is done by suitable transformation which leads to linear differential equation of same order which is not homogeneous.

2. Governing equations

Let us suppose that an electrically conducting fluid, with electrical conductivity σ, in the presence of a transverse magnetic field $B(x)$ is flowing past a flat plate stretched with a power-law velocity. According to [27] and [33], such phenomenon is described by the following equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e u_{ex} + \nu\frac{\partial^2 u}{\partial x^2} + \frac{\sigma B^2(x)}{\rho}(u_e - u). \tag{4}$$

Here, the induced magnetic field is neglected. In a Cartesian system of co-ordinates (O,x,y), the solution variables $u = u(x,y)$ and $v = v(x,t)$ are the velocity components in the $x-$ and $y-$ directions respectively. Here $u_{ex} = \gamma x^m, \gamma > 0$ denotes the external velocity, $B(x) = B_0 x^{m-1}$ the applied magnetic field, m the power-law velocity exponent, ρ the fluid density and ν the kinematic viscosity.

The boundary conditions for the problem (3)-(4) are

$$u(x,0) = u_w(x) = \alpha x^m, \;\; v_w(x) = \beta x^{\frac{m-1}{2}} \;\text{ and }\; u(x,\infty) = u_{ex}, \tag{5}$$

where $u_w(x)$ and $v_w(x)$ are the stretching and the suction (injection) velocity respectively and α, β are constants. Let us recall that $\alpha > 0$ is referred to suction, $\alpha < 0$ for the injection and $\alpha = 0$ for the impermeable plate.

A little inspection shows that the equations (3) and (4) accompanied by conditions (5) admit a similarity solution. Therefore, we introduce the dimensional stream-function ψ in the usual way to get the following equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = u_e u_{ex} + \nu\frac{\partial^3\psi}{\partial y^3} + \frac{\sigma B^2(x)}{\rho}(u_e - u). \tag{6}$$

The boundary conditions become

$$\frac{\partial\psi}{\partial y}(x,0) = \alpha x^m, \;\; \frac{\partial\psi}{\partial x}(x,0) = -\beta x^{\frac{m-1}{2}}, \;\; \frac{\partial\psi}{\partial y}(x,\infty) = \gamma x^m. \tag{7}$$

Defining the similarity variables as follows

$$\psi(x,y) = x^{\frac{m+1}{2}} f(t)\sqrt{\nu\gamma} \;\text{ and }\; t = x^{\frac{m-1}{2}} y\sqrt{\frac{\nu}{\gamma}},$$

and substituting in equations (6) and (7) we get the boundary value problem (1)-(2) where $a = \frac{2\beta}{(m+1)\sqrt{\nu\gamma}}$, $b = \frac{\alpha}{\gamma}$ and $M = \frac{\sigma B_0^2}{\gamma\rho} > 0$ is the Hartman number.

3. Asymptotic behaviour

The asymptotic behaviours, as $t \to \infty$, of solutions of (1)-(2) under the side condition

$$0 < f' < 1 \;\text{ on }\; [0,\infty) \tag{8}$$

will be discussed. The results will be based on the asymptotic integrations of second order, linear differential equations. The asymptotic integrations will be based on the following theorems.

THEOREM 3.1. *Let* $m \geqslant 0$, $-\infty < a < \infty$, $0 \leqslant b < 1$. *Then there exists one and only one solution* $f(t)$ *of* (1), (2), (8). *This solution also satisfies satisfies* $f''(t) > 0$ *on* $0 \leqslant t < \infty$.

Proof. The proofs for the existence and uniqueness of solution are similar to the proof of Theorem 6.1 (Hartman [19]), p.521). □

THEOREM 3.2. *Let* $-1 \leqslant m < 0$, $M + 2m < 0$ *and* $0 \leqslant b < 1$. *Then there exists a number* $A = A(m,b)$ *and a continuous function* $\gamma'(a)$ *defined for* $a \geqslant A$ *with the properties that* $\gamma'(A) = 0$ *and that* $f(t)$ *is a solution of* (1)-(2), (8) *hold if and only if* $a \geqslant A$ *and* $0 \leqslant f''(0) \leqslant \gamma'(a)$*; in this case,* $f''(t) > 0$ *for* $0 \leqslant t < \infty$.

Proof. The proof of this theorem is similar to the proof of Theorem 7.1 (Hartman [19], p.525). If $f(t)$ is the solution of (1), let us put

$$h(t) = 1 - f'(t). \tag{9}$$

Then $h(t)$ satisfies the differential equation

$$h'' + \left(\frac{m+1}{2}\right) f h' - [m(1+f') + M]h = 0. \tag{10}$$

Differentiating (10) gives

$$h''' + \left(\frac{m+1}{2}\right) f h'' - \left[\left(\frac{3m-1}{2}\right) f' + M\right] h' = 0. \tag{11}$$

Since $h' = -f''$.

In order to eliminate the middle term in (10), let us put

$$h = x \exp\left\{ - \left(\frac{m+1}{4}\right) \int_0^t f d\tau \right\} \tag{12}$$

so that x satisfies

$$x'' - q(t)x = 0, \tag{13}$$

where

$$\begin{aligned} q(t) &= (m+M) + \left(\frac{5m+1}{4}\right) f' + \frac{1}{16}(m+1)^2 f^2 \\ &= \frac{1}{16}(m+1)^2 f^2 \left[1 + \frac{16(m+M)}{(m+1)^2 f^2} + \frac{4(5m+1)}{(m+1)^2} \frac{f'}{f^2}\right]. \end{aligned} \tag{14}$$

Thus

$$q'(t) = \left(\frac{5m+1}{4}\right) f'' + \frac{1}{8}(m+1)^2 f f'$$

and by (1),

$$\begin{aligned} q''(t) = -\frac{(5m+1)(m+M)}{4} &+ \frac{(5m+1)M}{4} f' \\ &- \frac{m(m+1)}{2} f f'' + \left(\frac{11m^2+4m+1}{8}\right) f'^2. \end{aligned}$$

Since $0 < f' < 1$, $f'' > 0$ and $f' \sim 1$, $f \sim t$ as $t \to \infty$, there is a constant k such that for large t,

$$\frac{q'^2}{q^{5/2}} \leqslant k\left[\frac{f''^2}{t^5} + \frac{1}{t^3}\right] \quad \text{and} \quad \frac{|\, q''\,|}{q^{3/2}} \leqslant k\left[\frac{f''}{t^2} + \frac{1}{t^3}\right].$$

In addition, $\int^{\infty} f'' dt$ is absolutely convergent (since $f' \to 1$ as $t \to \infty$), so that

$$\int^{\infty} \frac{q'^2}{q^{5/2}} dt < \infty \quad \text{and} \quad \int^{\infty} \frac{|q''|}{q^{3/2}} dt < \infty \tag{15}$$

provided that

$$\int^{\infty} \frac{f''^2}{t^5} dt < \infty. \tag{16}$$

It is easy to check the validity of (16), for an integration by parts (integrating f'' and differentiating f''/t^5) gives

$$\int \frac{f''^2}{t^5} dt = \frac{f' f''}{t^5} + \int \frac{f'}{t^5} \Big[\Big(\frac{m+1}{2} \Big) f f'' + m(1 - f'^2) + M(1 - f') + \frac{5f''}{t} \Big] dt$$

by (1). The last integral is absolutely convergent and $\liminf f''(t) = 0$ as $t \to \infty$. Thus (16) holds. Consequently, (15) holds, and thus (13) has a principal solution $x(t)$ satisfying, as $t \to \infty$,

$$x \sim K q^{-1/4} \exp\Big(- \int_0^t q^{1/2}(s) ds \Big), \tag{17}$$

where $K \neq 0$ is a constant, while linearly independent solutions satisfy

$$x \sim K q^{-1/4} \exp\Big(\int_0^t q^{1/2}(s) ds \Big) \tag{18}$$

cf. Exercise XI 9.6 (Hartman [19], p.382). From (14) and $f \sim t$,

$$q^{1/2}(t) = \frac{(m+1)}{4} f + \frac{(5m+1)}{(2m+2)} \frac{f'}{f} + \Big(\frac{2M+2m}{m+1} \Big) \frac{1}{f} + O(1/t^3),$$
$$q^{1/4}(t) \sim \Big(\frac{1}{2} t \Big)^{1/2},$$

hence

$$\int^t q^{1/2}(s) ds = \frac{(m+1)}{4} \int^t f dt + \frac{(5m+1)}{(2m+2)} \log f + \frac{(2M+2m)}{(m+1)} \int^t \frac{dt}{f} + K^0 + O(1),$$

where K^0 is a constant. Thus, (17) and (18) become

$$x \sim K t^{-\frac{1}{2} - \frac{5m+1}{2m+2}} \exp\Big(- \int^t \Big[\Big(\frac{m+1}{4} \Big) f + \Big(\frac{2M+2m}{m+1} \Big) \frac{1}{f} \Big] dt \Big) \tag{19}$$

$$x \sim K t^{-\frac{1}{2} - \frac{5m+1}{2m+2}} \exp\Big(\int^t \Big\{ \Big(\frac{m+1}{4} \Big) f + \Big(\frac{2M+2m}{m+1} \Big) \frac{1}{f} \Big\} dt \Big). \tag{20}$$

In view of (12), the (10) has a principal solution satisfying

$$h \sim K t^{-\frac{3m+1}{m+1}} \exp\Big(- \int^t \Big\{ \Big(\frac{m+1}{4} \Big) f + \Big(\frac{2M+2m}{m+1} \Big) \frac{1}{f} \Big\} dt \Big) \tag{21}$$

and that the linearly independent solutions satisfy

$$h \sim Kt^{\frac{2m}{m+1}} \exp\left(\int^t \left\{\left(\frac{2M+2m}{m+1}\right)\frac{1}{f}\right\}dt\right), \tag{22}$$

where $K \neq 0$. By treating (11) as a second order equation for h' in the same way that (10) was handled, it is seen that (11) has the principal solution satisfying

$$h' = K't^{-\frac{4m}{m+1}} \exp\left(-\int^t \left\{\left(\frac{m+1}{2}\right)f + \left(\frac{2M}{m+1}\right)\frac{1}{f}\right\}dt\right), \quad K' \neq 0, \tag{23}$$

and that the linearly independent solutions satisfy

$$h' = K't^{\frac{3m-1}{m+1}} \exp\left(\int^t \left\{\left(\frac{2M}{m+1}\right)\frac{1}{f}\right\}dt\right), \quad K' \neq 0, \tag{24}$$

as $t \to \infty$.

If (11) satisfies (23), then since $f \sim t$, it follows that $\int^\infty h t dt < \infty$; thus

$$f = t + K_1 + O(1) \quad \text{and} \quad \int^t f dt = \frac{t^2}{2} + K_1 t + K_2 + O(1) \text{ as } t \to \infty.$$

Substituting this into (21) and (23), gives

$$1 - f' \sim K_0 t^{-\frac{2M+5m+1}{m+1}} \exp\left\{-\left(\frac{m+1}{2}\right)\left(\frac{t^2}{2} + K_1 t\right)\right\}, \quad f'' \sim t(1-f') \tag{25}$$

as $t \to \infty$, where $K_0 > 0$, K_1 are the constants.

If (9) satisfies (22), then $f \sim t$ implies that $h = 1 - f' \sim ct^{(2M+4m)/(m+1)}$ as $t \to \infty$. Hence,

$$f = t + O\left(t^{\frac{2M+3m-1}{m+1}+\varepsilon}\right) \text{ as } t \to \infty \text{ for all } \varepsilon > 0.$$

If this is substituted into (22), (24) and if it is assumed that $-1 \leqslant m < 0$, $M + 2m < 0$, (and $(2M+4m)/(m+1) + \varepsilon < 0$), then

$$1 - f' \sim K_0 t^{\frac{2M+4m}{m+1}} \quad \text{and} \quad f'' \sim \left(\frac{2M+4m}{m+1}\right) K_0 t^{\frac{2M+3m-1}{m+1}} \tag{26}$$

as $t \to \infty$, where $K_0 > 0$ is a constant.

4. Results

The results pertaining to the asymptotic behaviour can be expressed in terms of the following theorems.

THEOREM 4.1. *Let $m \geqslant 0$ and $f(t)$ be a solution of* (1)-(2) *and* (8). *Then there exist constants $K_0 > 0$, K_1 such that* (25) *holds as $t \to \infty$.*

Proof. For a given $f(t)$, it has to be decided whether $h = 1 - f'$ satisfies (21), (23) or (22), (24). If $m \geqslant 0$, (22) cannot hold, for otherwise $h = 1 - f' \to 0$, $t \to \infty$ fails to hold. Thus, (21), (23) are valid and, as was seen, this gives (25). Hence (25) holds as $t \to \infty$. $\square$

THEOREM 4.2. *Let* $-1 \leqslant m < 0$, $M + 2m < 0$, $0 \leqslant b < 1$ *and* $a \geqslant A(m,b)$, *where* $A(m,b)$, $\gamma'(a)$ *are given by Theorem 3.2. Let* $f(t)$ *be a solution of* (1)-(2). *Then there exist constants* $K_0 > 0$, K_1 *such that* (25) *holds if and only if* $f''(0) = \gamma'(a)$; *for other solutions* $f(t)$ *of* (1), (2), (8), *with* $a > A(m,b)$ *and* $0 \leqslant f''(0) < \gamma'(a)$, *the asymptotic relations* (26) *hold with a suitable constant* $K_0 > 0$.

Proof. The proof is based on the change of variables introduced by Grohne and Iglisch [14]. If $f = f(t)$ is a solution of (1) on some t-interval satisfying $f'(t) > 0$, so that $f(t)$ is an increasing function. Let f be the new independent variable and $z = f'^2$ the new dependent variable.
Thus,

$$\frac{d}{dt} = f'\frac{d}{df} = z^{1/2}\frac{d}{df} \quad \text{or} \quad \frac{d}{df} = z^{-1/2}\frac{d}{dt}$$

and if a dot denotes differentiation with respect to f,

$$f' = z^{1/2} \geqslant 0, \;\; f'' = \frac{1}{2}\dot{z}, \;\; f''' = \frac{1}{2}z^{1/2}\ddot{z}. \tag{27}$$

The equation (1) is transformed into

$$z^{1/2}\ddot{z} + (\frac{m+1}{2})f\dot{z} + 2m(1-z) + 2M(1-z^{1/2}) = 0, \tag{28}$$

where $\dot{z} = \frac{dz}{df}$, and the boundary conditions (2) into

$$z(a) = b^2, \;\; z(\infty) = 1 \tag{29}$$

and the side condition (8) into

$$0 < z(f) < 1, \;\; \text{for } \; a < f < \infty. \tag{30}$$

Based on the results of Iglisch and Kemnitz [21], it can be shown that if $z(f)$ is a solution of (28) determined by the initial conditions

$$z(a) = b^2, \;\; \dot{z}(a) = 2\gamma' \;\; \text{where } 0 \leqslant b < 1, \;\; \gamma' > 0, \tag{31}$$

then:

(a) $z(f) > 0$, $\dot{z}(f) > 0$ for small $f - a$;

(b) as f increases, $\dot{z}(f)$ remains positive as long as (30) holds ; and

(c) if $z(f)$ exists for $f \geqslant a$ and satisfies (30) for $f > a$, then $z(f) \to 1$ as $f \to \infty$, so that (29) holds and the corresponding solution $f(t)$ of (1) satisfies (2), (8).

For a given solution $z(f)$ of (28), let us introduce the functions

$$w = 1 - z(f), \quad r = \frac{\dot{w}}{w}, \tag{32}$$

then (28) becomes a second order equation for w:

$$(1-w)^{1/2}\ddot{w} + (\frac{m+1}{2})f\dot{w} - 2\lambda^* w = 0, \tag{33}$$

where $\lambda^* = m[1 + \frac{M}{1+f'}]$ and r satisfies the corresponding Riccati equation

$$\dot{r} = -r^2 + \frac{2\lambda^* - (\frac{m+1}{2})fr}{(1-w)^{1/2}}. \tag{34}$$

If $z(f) \to 1$ as $f \to \infty$ so that $w \to 0$ as $f \to \infty$, then (33) is a perturbation of the Weber linear differential equation

$$\ddot{V} + (\frac{m+1}{2})f\dot{V} - 2\lambda^* V = 0, \tag{35}$$

where $\dot{V} = \frac{dV}{df}$. If $V \neq 0$ is a solution of (35), then

$$s = \frac{\dot{V}}{V} \tag{36}$$

satisfies the Riccati equation

$$\dot{s} = -s^2 + \left(2\lambda^* - \frac{m+1}{2}fs\right). \tag{37}$$

As is well known, the (37) has a principal solution $V(f)$ which is positive for large f and satisfies

$$s \sim -f \ \text{ as } f \to \infty. \tag{38}$$

After the variation of constants $V = x\exp(-\frac{m+1}{4}f^2)$, this fact is implied by the result mentioned above concerning (13), (15) and (17). Let $a_0 > 0$ be so large that

$$0 < V(f) < 1 \quad \text{and} \quad 2\lambda^* - (\frac{m+1}{2})fs > 0 \ \text{ for } \ f \geqslant a_0 > 0. \tag{39}$$

Let $z(f)$ be a solution of (28) on some f-interval $[f_0, f_0+\varepsilon]$, $f_0 \geqslant a_0$, such that $0 \leqslant z(f) < 1$ and $w(f)$, $r(f)$ are the functions (32). Let us suppose that

$$r(f) > s(f) \tag{40}$$

at $f = f_0$. Then simple results on differential inequalities imply that (41) holds for all $f > f_0$ as long as $z(f)$ exists and $z(f) < 1$. On the other hand, if $r(f) \geqslant s(f)$ on some interval on $f \geqslant f_0$, then the last part of (32), (36) and a quadrature show that $w(f) \geqslant k'V(f) > 0$ on that interval for some $k' > 0$; i.e. $z(f) < 1$. Consequently, if

$z(f)$ is such that $0 \leqslant z(f_0) < 1$ and (40) holds at $f = f_0$, then z(f) exists for all $f \geqslant f_0$, $0 < z(f) < 1$ for $f > f_0$ and following (31), $z(f) \to 1$ as $f \to \infty$.

The case when: $f''(0) = \gamma'(a)$. After these preliminaries, it will be shown that the solution of (1) determined by

$$f(0) = a, \ \ f'(0) = b, \ \ f''(0) = \gamma'(a), \tag{41}$$

which satisfies (2), (8), also satisfies (25).

Let us denote the solution of (1) by $f^*(t)$. Let $z^*(f)$ be the corresponding solution of (28), $w^* = 1 - z^*(f)$ and $r^*(f) = \frac{\dot{w^*}}{w^*}$. It will be first verified that if $V = V(f)$ is a fixed solution of Weber's equation (35) satisfying (38), (39), then

$$r^*(f) \leqslant s(f) \ \text{ for } \ f \geqslant a_0. \tag{42}$$

Let us assume that $r^*(f_0) > s(f_0)$ for some $f_0 \geqslant a_0$. Then the continuity considerations show that if $z(f)$ is the solution of (28), (31) with $\gamma'(\geqslant 0)$ near to $\gamma'(a)$, then $z(f)$ exists on an interval containing $[a, f_0]$ and that the function $r(f)$ belonging to $z(f)$ satisfies (40) at $f = f_0$. In this case, the remarks above imply that $z(f)$ exists for $f \geqslant a$ and the corresponding solution $f(t)$ of (1) satisfies (2), (8). But this contradicts the maximal property of $\gamma'(a)$ in Theorem 3.2 and so (42) holds good.

A quadrature of (42), where $r^* = \frac{\dot{w^*}}{w^*}$ and $s = \frac{\dot{V}}{V}$ gives $w^*(f) \leqslant k''(f)$ for some $k'' > 0$ and $f \geqslant a_0$; i.e. $1 - z^*(f) \leqslant k''V(f)$. In view of (38),

$$V(f) = O\big(e^{-(\frac{1}{2}-\varepsilon)t^2}\big) \ \text{ for large } f \text{ and if } \varepsilon \text{ is fixed, } 0 < \varepsilon < 1/2.$$

Since $f^*(t) \sim t$ as $t \to \infty$ and $z^*(f) = f^{*2}(t)$ at the t-value where $f^*(t) = f$, it follows that

$$1 - f^{*\prime 2}(t) = O\Big(e^{-(\frac{1}{2}-\varepsilon)t^2}\Big) \ \text{ as } t \to \infty \text{ if } 0 < \varepsilon < 1/2.$$

This implies that

$$1 - f^*(t) = O\Big(e^{-(\frac{1}{2}-\varepsilon)t^2}\Big) \ \text{ as } t \to \infty.$$

Hence $f = f^*(t)$ can not satisfy (26) for any $K_0 > 0$ and so (25) holds for $f = f^*(t)$ for suitable constants $K_0 > 0$ and K_1.

The case when: $a > A(m,b)$ and $0 \leqslant f''(0) < \gamma'(a)$. The assertion (26) can be proved in this case by showing that (1), (2), (8) cannot have two distinct solutions satisfying (25).

Let $f(t)$ be the solution of (1), (2), (8) and $z(f)$ be the corresponding solution of (28)-(30). Since $z(f) > 0$ for $a \leqslant f < \infty$, the function $z = z(f)$ is increasing and has an inverse $f = U(z)$ on $b^2 < z < 1$. Let us put $G(z) = \dot{z}(U(z))$. Then $\frac{dU}{dz} = \frac{1}{G}$, so that

$$\frac{d(-U)}{dz} = -\frac{1}{G} \tag{43}$$

and $\frac{dG}{dz} = \ddot{z}\frac{dU}{dz}$, so that by (28)

$$\frac{dG}{dz} = -\Big(\frac{m+1}{2}\Big)\frac{U}{z^{1/2}} - \frac{2(1-z^{1/2})}{z^{1/2}G}[m(1+z^{1/2})+M]. \tag{44}$$

Following Iglisch and Kemnitz [21], if $f_1(t)$ and $f_2(t)$ are the two solutions of (1), (2), (8) such that

$$0 \leqslant f_2''(0) < f_1''(0) \leqslant \gamma'(a)$$

and if $z_1(f)$, $z_2(f)$ are the corresponding solutions of (28), $U_j(z)$ the inverse of $f = z_j(f)$ and $G_j(U_j(z))$ then for $-1 \leqslant m < 0$ and $M + 2m < 0$,

$$U_2(Z) - U_1(Z) \quad \text{is positive and increasing for } b^2 < z < 1, \tag{45}$$

$$G_2(Z) - G_1(Z) > 0 \ \text{ for } \ b^2 < z < 1. \tag{46}$$

Actually, $U_1(b^2) = U_2(b^2) = a$ and $G_1(b^2) > G_2(b^2)$, so that (45), (46) follow from the Theorem of Kamke [22] since the right side of (43) is an increasing function of $G > 0$ (and non-decreasing function of $-U$), while right side of (44) is an increasing function of $-U$.

Let us suppose, if possible, that both solutions $f = f_1(t)$, $f_2(t)$ of (1), (2), (8) satisfy (25). Then, for $j = 1,2$, $f_j''(t) \to 0$ and $f_j''(t) \sim t(1 - f_j'(t))$ as $t \to \infty$. Since $f_j(t) \sim t$, it follows from (27) that

$$\dot{z}_j(f) \sim 2f(1 - z_j^{1/2}) \ \text{ as } f \to \infty.$$

Or since

$$1 - \dot{z}^{1/2} = \frac{(1 - z_j)}{1 + z_j^{1/2}} \sim \Big\{\frac{1}{2}(1 - z_j)\Big\} \ \text{ as } f \to \infty,$$

one has $\dot{z}_j \sim f(1 - z_j)$ as $f \to \infty$. Thus

$$G_j \sim U_j(1 - z) \ \text{ as } z \to 1 \tag{47}$$

and since $f_j''(t) \to 0$ as $t \to \infty$,

$$G_j(z) \to 0 \ \text{ as } z \to 1. \tag{48}$$

By (44), as $z \to 1$,

$$\frac{dG_j}{dz} = -\Big(\frac{m+1}{2}\Big)\frac{U_j(z)}{z^{1/2}} + O\Big(\frac{1}{U_j}\Big).$$

Consequently, as $z \to 1$,

$$\frac{d(G_1 - G_2)}{dz} = (\frac{m+1}{2})\frac{(U_2 - U_1)}{z^{1/2}} + O\Big(\frac{1}{U_1}\Big) + O\Big(\frac{1}{U_2}\Big). \tag{49}$$

In view of (45), there exists a constant k''' such that $U_2(z) - U_1(z) \geqslant k''' > 0$ for z near 1. Since $U_j(t) \to \infty$ as $z \to 1$, implies that $\frac{d(G_1 - G_2)}{dz} \geqslant (\frac{m+1}{2})\frac{k'''}{2}$ for z near 1. By (46), $0 < \lim(G_1 - G_2) \leqslant \infty$ as $z \to 1$. This contradicts (48). Hence the assumption that (1), (2), (8) have two solutions $f_1(t)$, $f_2(t)$ satisfying (25) is untenable. This completes the proof of Theorem. □

5. Concluding remarks

In this paper the asymptotic behaviours of the solutions for steady laminar incompressible boundary layer equations governing the magneto-hydrodynamic flow near the forward stagnation point of two-dimensional and axisymmetric bodies have been discussed. One of the most important problems in the study of the differential equations and their applications to the boundary layer theory is that of describing the nature of the solutions for the large positive values of the independent variables. The solutions of a system will show asymptotic behaviour if it approaches zero as the independent variable tends to infinity or is very small for all the independent variables, or is bounded as the independent variable tends to infinity.

If f(t) is the solution of (1), (2), (8) in Theorem 3.1, then there exist constants $K_0 > 0$, K_1 such that (25) holds as $t \to \infty$ for $m \geqslant 0$ i.e. the solution (25) shows asymptotic behaviour. If $f(t)$ is the solution of (1), (2), (8) in Theorem 3.2 with $a > A(m,b)$, $0 \leqslant f''(0) < \gamma'(a)$, then there exist constant $K_0 > 0$ such that (26) holds as $t \to \infty$; i.e. it shows asymptotic behaviour.

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B. B. Singh and I. M. Chandarki
Department of Mathematics
Dr.Babasaheb Ambedkar Technological University
Lonere-402103, Dist.-Raigad(M.S.)
India
e-mail: brijbhansingh@yahoo.com, imran_2204@yahoo.co.in