EXISTENCE OF TWO SOLUTIONS FOR A NAVIER BOUNDARY VALUE PROBLEM INVOLVING THE p–BIHARMONIC

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Abstract. In this paper, we study the Navier boundary value problem involving the p-biharmonic system with sign-changing weight function. By using the Nehari manifold and variational methods, the existence of two nontrivial solutions is obtained when the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$.

1. Introduction

In this paper, we consider the Navier boundary value problem involving the p-biharmonic system

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \frac{1}{\sigma}\frac{\partial F(x,u,v)}{\partial u} + \lambda g(x)|u|^{q-2}u & \text{in } \Omega, \\ \Delta(|\Delta v|^{p-2}\Delta v) = \frac{1}{\sigma}\frac{\partial F(x,u,v)}{\partial v} + \mu h(x)|v|^{q-2}v & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Δ is the Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ is positively homogeneous of degree σ, that is, $F(x,tu,tv) = t^\sigma F(x,u,v)$ $(t>0)$ holds for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R}^2$. We assume that $\sigma \in (p, \frac{pN}{N-2p})$ if $p < \frac{N}{2}$ or $\sigma \in (0,+\infty)$ if $p = \frac{N}{2}$, $1 < q < p$, the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and the weight functions g, h are satisfying the following conditions:

(A) $g \in C(\Omega)$ with $\| g \|_\infty = 1, g^\pm = \max\{\pm g, 0\} \not\equiv 0$;

(B) $h \in C(\Omega)$ with $\| h \|_\infty = 1, h^\pm = \max\{\pm h, 0\} \not\equiv 0$.

In recent years, there are many papers concerned with the existence and multiplicity of positive solutions for p-biharmonic elliptic problems. Results relating to these

problems can be found in [1, 5, 7, 10-14], and the references therein. Brown and Wu [3] considered the following equation

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} f(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} f(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda g(x)|u|^{q-2}u, \;\; \frac{\partial v}{\partial n} = \mu h(x)|v|^{q-2}v & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

When $F(x,u,v) = f(x)|u|^{\alpha}|v|^{\beta}$, $\alpha > 1, \beta > 1$ satisfing $p < \alpha+\beta < p^*$, the authors have found that if the parameters λ, μ satisfy $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, then problem (1.2) has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^{\pm} \geqslant 0, v_0^{\pm} \geqslant 0$ in Ω and $u_0^{\pm} \neq 0, v_0^{\pm} \neq 0$. Furthermore, if $f \geqslant 0$, then $u_0^{\pm} > 0, v_0^{\pm} > 0$ in Ω.

In this paper, we give a very simple variational method which is similar to the "fibering method" of Pohozaev's (see [4] or [8]) to prove the existence of at least two nontrivial solutions of problem (1.1). In fact, we use the decomposition of the Nehari manifold as (λ, μ) vary to prove the following result.

Before stating our results, we need the following assumptions:

(f_1) $F : \overline{\Omega} \times R \times R \to R$ is a C^1 function such that $F(x,tu,tv) = t^{\sigma}F(x,u,v)$ $(t > 0)$ and $F(x,-u,-v) = F(x,u,v), \forall x \in \overline{\Omega}, (u,v) \in \mathbb{R}^2$;

(f_2) $F(x,u,0) = F(x,0,v) = \frac{\partial F}{\partial u}(x,u,0) = \frac{\partial F}{\partial v}(x,0,v) = 0$, where $u,v \in \mathbb{R}$;

(f_3) $F^{\pm}(x,u,v) = \max\{\pm F(x,u,v),0\} \not\equiv 0, \; \forall \, u,v \in \mathbb{R}, \; uv \neq 0$.

Now we consider the even functional

$$I(u) := \frac{\int_{\Omega} |\Delta u|^p dx}{\int_{\Omega} |u|^p dx}, \;\; \forall u \in (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \backslash \{0\},$$

and the manifold

$$H := \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}} = 1 \right\}.$$

Evidently, H is a nonempty smooth manifold. By a standard argument (or similar to the proof in [9]), $I|_H$ has a sequence of increasing critical values with the variational characterization

$$\lambda_k := \inf_{M \in \Sigma_k} \sup_{u \in M} I(u),$$

where $\Sigma_k := \{M \subset H :$ there exists a continuous, odd and surjective $h : S^{k-1} \to M\}$ and S^{k-1} denotes the unit sphere in $\mathbb{R}^k$. It is not difficult to check that the critical values and critical points of $I|_H$ respectively correspond to the eigenvalues and eigenfunctions of the following equation

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that

$$\int_\Omega |\Delta u|^p dx \geqslant \lambda_1 \int_\Omega |u|^p dx, \ \forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{1.3}$$

Let $C_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} C(\sigma, p, q, K, \lambda_1)$ be a positive number where

$$C(\sigma, p, q, K, \lambda_1) = \left(\frac{\sigma - q}{p - q} K \lambda_1^{-\sigma}\right)^{\frac{p}{p-\sigma}} \left(\frac{\sigma - p}{\sigma - q} \lambda_1^q\right)^{\frac{p}{p-q}}.$$

Our main results are summarized in the following theorems.

THEOREM 1.1. *If the parameters* λ, μ *satisfy*

$$0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma, p, q, K, \lambda_1),$$

and (f_1)-(f_3) *hold, then problem* (1.1) *has at least one nontrivial solution* (u_0^+, v_0^+).

THEOREM 1.2. *If the parameters* λ, μ *satisfy* $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C_0$, *and* (f_1)-(f_3) *hold, then problem* (1.1) *has at least two nontrivial solutions* (u_0^+, v_0^+) *and* (u_0^-, v_0^-).

REMARK 1.3. There are many functions F satisfying the conditions of Theorem 1.1 and 1.2 for the problem (1.1). For examples: let

$$F_1(x, u, v) = \begin{cases} f_1^2(x)|u|^{\frac{3}{2}}|v|^{\frac{5}{2}} + f_2^2(x)\frac{u^3 v^3}{u^2 + v^2}, & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}$$

where $f_1, f_2 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ with $\max\{\pm f_1, \pm f_2, 0\} \not\equiv 0$;

$$F_2(x, u, v) = \begin{cases} \frac{f_1(x)u^3 v^3}{u^2 + 2v^2} + \frac{f_2(x)u^5 v^3}{u^4 + v^4}, & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}$$

where $f_1, f_2 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ with $\max\{\pm f_1, \pm f_2, 0\} \not\equiv 0$, $f_1 f_2 \geqslant 0$;

$$F_3(x, u, v) = \begin{cases} \frac{f_1(x)u^5 v^3}{2u^2 + v^2} + \frac{f_2(x)u^4 v^6}{u^4 + 3v^4} + \frac{f_3(x)u^7 v^5}{u^6 + v^6}, & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}$$

where $f_1, f_2, f_3 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ with $\max\{\pm f_1, \pm f_2, \pm f_3, 0\} \not\equiv 0$, $f_1 f_i \geqslant 0$, $i = 2, 3$. Obviously, F_1, F_2 and F_3 satisfy (f_1)-(f_3).

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we prove Theorem 1.1 and Theorem 1.2.

2. Some notations and preliminaries

Problem (1.1) is posed in the framework of the Sobolev space

$$E = (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \times (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$$

with the standard norm

$$\|(u,v)\| = \left(\int_\Omega |\Delta u|^p dx + \int_\Omega |\Delta v|^p dx \right)^{\frac{1}{p}}.$$

In addition, we define $\|u\|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$ as the norm of the Sobolev space $L^p(\Omega)$.

Using assumption (f_1), we have the so-called Euler identity

$$(u,v) \cdot \nabla F(x,u,v) = \sigma F(x,u,v) \tag{2.1}$$

and

$$|F(x,u,v)| \leqslant K(|u|^\sigma + |v|^\sigma) \qquad \text{for some constant } K > 0. \tag{2.2}$$

Moreover, a pair of functions $(u,v) \in E$ is said to be a weak solution of problem (1.1) if

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \varphi_1 dx + \int_\Omega |\Delta v|^{p-2} \Delta v \Delta \varphi_2 dx - \frac{1}{\sigma} \int_\Omega \frac{\partial F(x,u,v)}{\partial u} \varphi_1 dx$$

$$- \frac{1}{\sigma} \int_\Omega \frac{\partial F(x,u,v)}{\partial v} \varphi_2 dx - \lambda \int_\Omega g|u|^{q-2} u \varphi_1 dx - \mu \int_\Omega h|v|^{q-2} v \varphi_2 dx = 0$$

for all $(\varphi_1, \varphi_2) \in E$. Thus, by (2.1) the corresponding energy functional of problem (1.1) is defined by

$$J_{\lambda,\mu}(u,v) = \frac{1}{p} \|(u,v)\|^p - \frac{1}{\sigma} \int_\Omega F(x,u,v) dx - \frac{1}{q} K_{\lambda,\mu}(u,v)$$

for $(u,v) \in E$, where

$$K_{\lambda,\mu}(u,v) = \lambda \int_\Omega g|u|^q dx + \mu \int_\Omega h|v|^q dx.$$

In order to verify $J_{\lambda,\mu} \in C^1(E,\mathbb{R})$, we need the following lemmas.

LEMMA 2.1. *Assume that $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ is positively homogeneous of degree σ with $\sigma > 1$, then $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ are positively homogeneous of degree $\sigma - 1$.*

Proof. The proof is almost the same as that in Chu and Tang [6].

In addition, by the Lemma 2.1, we get the existence of positive constant M such that

$$\left| \frac{\partial F}{\partial u}(x,u,v) \right| \leqslant M(|u|^{\sigma-1} + |v|^{\sigma-1}), \tag{2.3}$$

$$\left|\frac{\partial F}{\partial v}(x,u,v)\right| \leqslant M(|u|^{\sigma-1}+|v|^{\sigma-1}), \; \forall x \in \overline{\Omega}, \; u,v \in \mathbb{R}. \tag{2.4}$$

Similar to $[15, \text{Theorem A.2}]$, we consider the continuity of the superposition operator

$$A : L^p(\Omega) \times L^p(\Omega) \to L^q(\Omega) : (u,v) \mapsto f(x,u,v).$$

LEMMA 2.2. *Assume that* $|\Omega| < \infty, 1 \leqslant p$, $r < \infty, f \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *and*

$$|f(x,u,v)| \leqslant c(1+|u|^{\frac{p}{r}}+|v|^{\frac{p}{r}}).$$

Then, for every $(u,v) \in L^p(\Omega) \times L^p(\Omega), f(\cdot, u,v) \in L^r(\Omega)$ *and the operator*

$$A : L^p(\Omega) \times L^p(\Omega) \to L^r(\Omega) : (u,v) \mapsto f(x,u,v) \;\; \text{is continuous.}$$

It is well known that these following embedding mappings

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^r(\Omega), \; r < \frac{Np}{N-2p}, & \text{when } p < \frac{N}{2}, \\ L^r(\Omega), \; r < +\infty, & \text{when } p = \frac{N}{2}, \\ C(\overline{\Omega}), & \text{when } p > \frac{N}{2}, \end{cases}$$

are compact. Thus, it is not difficult to verify the following result.

LEMMA 2.3. (see Proposition 1 in [9]) *Suppose that* $\nabla F \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *verifies condition* (2.3) *and* (2.4). *Then the functional* $J_{\lambda,\mu}$ *belongs to* $C^1(E,\mathbb{R})$, *and*

$$\langle J'_{\lambda,\mu}(u,v), (u,v) \rangle = \|(u,v)\|^p - \int_\Omega F(x,u,v)dx - K_{\lambda,\mu}(u,v).$$

As the energy functional $J_{\lambda,\mu}$ is not bounded below on E, it is useful to consider the functional on the Nehari manifold

$$N_{\lambda,\mu} = \big\{(u,v) \in E \backslash \{(0,0)\} | \langle J'_{\lambda,\mu}(u,v), (u,v) \rangle = 0 \big\}.$$

Thus, $(u,v) \in N_{\lambda,\mu}$ if and only if

$$\langle J'_{\lambda,\mu}(u,v), (u,v) \rangle = \|(u,v)\|^p - \int_\Omega F(x,u,v)dx - K_{\lambda,\mu}(u,v) = 0. \tag{2.5}$$

Note that $N_{\lambda,\mu}$ contains every nonzero solution of problem (1.1). Moreover, we have the following result.

LEMMA 2.4. *The energy functional* $J_{\lambda,\mu}$ *is coercive and bounded below on* $N_{\lambda,\mu}$.

Proof. If $(u,v) \in N_{\lambda,\mu}$, then by the Sobolev imbedding theorem

$$J_{\lambda,\mu}(u,v) = \frac{\sigma-p}{p\sigma}\|(u,v)\|^p - \frac{\sigma-q}{q\sigma}K_{\lambda,\mu}(u,v)$$

$$\geqslant \frac{\sigma - p}{p\sigma}\|(u,v)\|^p - \lambda_1^{-q}\frac{\sigma - q}{q\sigma}\left(|\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\|(u,v)\|^q. \qquad (2.6)$$

Thus, $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

Define

$$\Phi_{\lambda,\mu}(u,v) = \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle.$$

Then for $(u,v) \in N_{\lambda,\mu}$,

$$\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = p\|(u,v)\|_W^p - \sigma \int_\Omega F(x,u,v)dx - qK_{\lambda,\mu}(u,v) \qquad (2.7)$$

$$= (p-\sigma)\int_\Omega F(x,u,v)dx - (q-p)K_{\lambda,\mu}(u,v) \qquad (2.8)$$

$$= (p-q)\|(u,v)\|^p - (\sigma - q)\int_\Omega F(x,u,v)dx \qquad (2.9)$$

$$= (p-\sigma)\|(u,v)\|^p - (q-\sigma)K_{\lambda,\mu}(u,v). \qquad (2.10)$$

Now, we split $N_{\lambda,\mu}$ into three parts:

$$N_{\lambda,\mu}^+ = \{(u,v) \in N_{\lambda,\mu}|\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0\},$$
$$N_{\lambda,\mu}^0 = \{(u,v) \in N_{\lambda,\mu}|\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\},$$
$$N_{\lambda,\mu}^- = \{(u,v) \in N_{\lambda,\mu}|\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle < 0\}.$$

Then, we have the following result.

LEMMA 2.5. *Assume that (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ and that $(u_0,v_0) \notin N_{\lambda,\mu}^0$. Then $J'_{\lambda,\mu}(u_0,v_0) = 0$ in E^{-1} (the dual space of the Sobolev space E).*

Proof. The proof is almost the same as that in Wu [16]. □

LEMMA 2.6. *We have:*
(i) *if $(u,v) \in N_{\lambda,\mu}^+$, then $K_{\lambda,\mu}(u,v) > 0$;*
(ii) *if $(u,v) \in N_{\lambda,\mu}^0$, then $K_{\lambda,\mu}(u,v) > 0$ and $\int_\Omega F(x,u,v)dx > 0$;*
(iii) *if $(u,v) \in N_{\lambda,\mu}^-$, then $\int_\Omega F(x,u,v)dx > 0$.*

Proof. The proof is immediate from (2.5),(2.8),(2.9) and (2.10). □

Moreover, we have the following result.

LEMMA 2.7. *If $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma,p,q,K,\lambda_1)$, then $N_{\lambda,\mu}^0 = \emptyset$.*

Proof. Suppose otherwise, that is there exists $(\lambda,\mu) \in \mathbb{R}^2\setminus\{(0,0)\}$ with

$$0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma,p,q,K,\lambda_1)$$

such that $N_{\lambda,\mu}^0 \neq \emptyset$. Then for $(u,v) \in N_{\lambda,\mu}^0$, by (2.9), (2.10) we have

$$
\begin{aligned}
0 &= \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = (p-q)\|(u,v)\|^p - (\sigma-q)\int_\Omega F(x,u,v)dx \\
&= (p-\sigma)\|(u,v)\|^p - (q-\sigma)K_{\lambda,\mu}(u,v).
\end{aligned}
$$

From the Hölder inequality, (2.2) and (1.3), it follows that

$$
\int_\Omega F(x,u,v)dx \leqslant \int_\Omega |F(x,u,v)|dx \leqslant K\int_\Omega (|u|^\sigma + |v|^\sigma)dx \leqslant K\lambda_1^{-\sigma}\|(u,v)\|^\sigma.
$$

Thus,

$$
\|(u,v)\| \geqslant \left(\frac{\sigma-q}{p-q}K\lambda_1^{-\sigma}\right)^{\frac{1}{p-\sigma}}
$$

and

$$
\|(u,v)\| \leqslant \left(\frac{\sigma-q}{\sigma-p}\right)^{\frac{1}{p-q}}\lambda_1^{-\frac{q}{p-q}}(|\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}})^{\frac{1}{p}}.
$$

This implies that

$$
|\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} \geqslant C(\sigma,p,q,K,\lambda_1),
$$

which is a contradiction. Thus, we can conclude that if

$$
0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma,p,q,K,\lambda_1),
$$

we have $N_{\lambda,\mu}^0 = \emptyset$. $\square$

By Lemma 2.7, we write $N_{\lambda,\mu} = N_{\lambda,\mu}^+ \cup N_{\lambda,\mu}^-$ and define

$$
\theta_{\lambda,\mu} = \inf_{(u,v)\in N_{\lambda,\mu}} J_{\lambda,\mu}(u,v); \quad \theta_{\lambda,\mu}^+ = \inf_{(u,v)\in N_{\lambda,\mu}^+} J_{\lambda,\mu}(u,v);
$$

$$
\theta_{\lambda,\mu}^- = \inf_{(u,v)\in N_{\lambda,\mu}^-} J_{\lambda,\mu}(u,v).
$$

Then we have the following result.

LEMMA 2.8. (i) *If*

$$
0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma,p,q,K,\lambda_1),
$$

then we have $\theta_{\lambda,\mu} \leqslant \theta_{\lambda,\mu}^+ < 0$;
(ii) *If*

$$
0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C_0,
$$

then $\theta_{\lambda,\mu}^- > d_0$ *for some* $d_0 = d_0(\sigma,p,q,K,\lambda_1,\lambda,\mu) > 0$.

Proof. (i) Let $(u,v) \in N_{\lambda,\mu}^+$. By (2.9)

$$\frac{p-q}{\sigma-q}\|(u,v)\|^p > \int_\Omega F(x,u,v)dx$$

and so

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &= \left(\frac{1}{p}-\frac{1}{q}\right)\|(u,v)\|^p + \left(\frac{1}{q}-\frac{1}{\sigma}\right)\int_\Omega F(x,u,v)dx \\
&< \left[\left(\frac{1}{p}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{\sigma}\right)\frac{p-q}{\sigma-q}\right]\|(u,v)\|^p \\
&= -\frac{(p-q)(\sigma-p)}{pq\sigma}\|(u,v)\|^p < 0.
\end{aligned}$$

Thus, from the definition of $\theta_{\lambda,\mu}$ and $\theta_{\lambda,\mu}^+$, we can deduce that $\theta_{\lambda,\mu} \leqslant \theta_{\lambda,\mu}^+ < 0$.

(ii) Let $(u,v) \in N_{\lambda,\mu}^-$. It follows from (2.9)

$$\frac{p-q}{\sigma-q}\|(u,v)\|^p < \int_\Omega F(x,u,v)dx.$$

Moreover, by (1.3) we obtain

$$\int_\Omega F(x,u,v)dx \leqslant K\lambda_1^{-\sigma}\|(u,v)\|^\sigma.$$

This implies that

$$\|(u,v)\| > \left(\frac{(p-q)\lambda_1^\sigma}{(\sigma-q)K}\right)^{\frac{1}{\sigma-p}} \quad \text{for all} \quad (u,v) \in N_{\lambda,\mu}^-. \tag{2.11}$$

By (2.6) in the proof of Lemma 2.4

$$\begin{aligned}
&J_{\lambda,\mu}(u,v) \\
&\geqslant \|(u,v)\|^q\left[\frac{\sigma-p}{p\sigma}\|(u,v)\|^{p-q} - \lambda_1^{-q}\frac{\sigma-q}{q\sigma}\left(|\lambda|^{\frac{p}{p-q}}+|\mu|^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right] \\
&> \left(\frac{(p-q)\lambda_1^\sigma}{(\sigma-q)K}\right)^{\frac{q}{\sigma-p}}\left[\frac{\sigma-p}{p\sigma}\left(\frac{(p-q)\lambda_1^\sigma}{(\sigma-q)K}\right)^{\frac{p-q}{\sigma-p}} - \frac{\sigma-q}{q\sigma\lambda_1^q}\left(|\lambda|^{\frac{p}{p-q}}+|\mu|^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right].
\end{aligned}$$

Thus, if

$$0 < |\lambda|^{\frac{p}{p-q}}+|\mu|^{\frac{p}{p-q}} < C_0,$$

then

$$J_{\lambda,\mu}(u,v) > d_0 \text{ for all } (u,v) \in N_{\lambda,\mu}^-,$$

for some $d_0 = d_0(\sigma,p,q,K,\lambda_1,\lambda,\mu) > 0$. This completes the proof. $\square$

For each $(u,v) \in E$ with $\int_\Omega F(x,u,v)dx > 0$, set

$$t_{\max} = \Big(\frac{(p-q)\|(u,v)\|^p}{(\sigma-q)\int_\Omega F(x,u,v)dx} \Big)^{\frac{1}{\sigma-p}} > 0.$$

Then the following lemma hold.

LEMMA 2.9. *For each $(u,v) \in E$ with $\int_\Omega F(x,u,v)dx > 0$, we have:*

(i) *if $K_{\lambda,\mu}(u,v) \leqslant 0$, then there is unique $t^- > t_{\max}$ such that $(t^-u, t^-v) \in N^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geqslant 0} J_{\lambda,\mu}(tu, tv);$$

(ii) *if $K_{\lambda,\mu}(u,v) > 0$, then there are unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+u, t^+v) \in N^+_{\lambda,\mu}, (t^-u, t^-v) \in N^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \leqslant t \leqslant t_{\max}} J_{\lambda,\mu}(tu, tv); \quad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geqslant 0} J_{\lambda,\mu}(tu, tv).$$

Proof. Fix $(u,v) \in E$ with $\int_\Omega F(x,u,v)dx > 0$. Let

$$m(t) = t^{p-q}\|(u,v)\|^p - t^{\sigma-q}\int_\Omega F(x,u,v)dx \quad \text{for } t \geqslant 0. \tag{2.12}$$

Clearly, $m(0) = 0, m(t) \to -\infty$ as $t \to \infty$. Since

$$m'(t) = (p-q)t^{p-q-1}\|(u,v)\|^p - (\sigma-q)t^{\sigma-q-1}\int_\Omega F(x,u,v)dx,$$

we have $m'(t) = 0$ at $t = t_{\max}$, $m'(t) > 0$ for $t \in [0, t_{\max})$ and $m'(t) < 0$ for $t \in (t_{\max}, \infty)$. Then $m(t)$ achieves its maximum at $t_{\max}$, is increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$m(t_{\max}) = \|(u,v)\|^q \Big[\Big(\frac{p-q}{\sigma-q}\Big)^{\frac{p-q}{\sigma-p}} - \Big(\frac{p-q}{\sigma-q}\Big)^{\frac{\sigma-q}{\sigma-p}} \Big] \Big(\frac{\|(u,v)\|^\sigma}{\int_\Omega F(x,u,v)dx} \Big)^{\frac{p-q}{\sigma-p}}$$

$$\geqslant \|(u,v)\|^q \Big(\frac{\sigma-p}{\sigma-q} \Big) \Big(\frac{\sigma-q}{p-q} K\lambda_1^{-\sigma} \Big)^{\frac{p-q}{p-\sigma}}. \tag{2.13}$$

(i) $K_{\lambda,\mu}(u,v) \leqslant 0$. There is a unique $t^- > t_{\max}$ such that $m(t^-) = K_{\lambda,\mu}(u,v)$ and $m'(t^-) < 0$. Now,

$$(p-q)(t^-)^p\|(u,v)\|^p - (\sigma-q)(t^-)^\sigma \int_\Omega F(x,u,v)dx = (t^-)^{1+q}m'(t^-) < 0,$$

and

$$\langle J'_{\lambda,\mu}(t^-u, t^-v), (t^-u, t^-v) \rangle = (t^-)^q[m(t^-) - K_{\lambda,\mu}(u,v)] = 0.$$

Thus, $(t^-u, t^-v) \in N^-_{\lambda,\mu}$. Since for $t > t_{\max}$, we have

$$(p-q)\|(tu,tv)\|^p - (\sigma - q)\int_\Omega F(x,tu,tv)dx < 0, \quad \frac{d^2}{dt^2}J_{\lambda,\mu}(tu,tv) < 0$$

and

$$\frac{d}{dt}J_{\lambda,\mu}(tu,tv) = t^{p-1}\|(u,v)\|^p - t^{\sigma-1}\int_\Omega F(x,u,v)dx - t^{q-1}K_{\lambda,\mu}(u,v)$$
$$= 0 \text{ for } t = t^-.$$

Thus, $J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geqslant 0} J_{\lambda,\mu}(tu,tv)$.

(ii) $K_{\lambda,\mu}(u,v) > 0$. By (2.13) and

$$m(0) = 0 < K_{\lambda,\mu}(u,v)$$
$$\leqslant \lambda_1^{-q}(|\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}})^{\frac{p-q}{p}}\|(u,v)\|^q$$
$$< \|(u,v)\|^q\left(\frac{\sigma-p}{\sigma-q}\right)\left(\frac{\sigma-q}{p-q}K\lambda_1^{-\sigma}\right)^{\frac{p-q}{p-\sigma}}$$
$$< m(t_{\max}),$$

for

$$0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma,p,q,K,\lambda_1),$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$m(t^+) = K_{\lambda,\mu}(u,v) = m(t^-)$$

and

$$m'(t^+) > 0 > m'(t^-).$$

We have $(t^+u, t^+v) \in N^+_{\lambda,\mu}, (t^-u, t^-v) \in N^-_{\lambda,\mu}$, and

$$J_{\lambda,\mu}(t^-u, t^-v) \geqslant J_{\lambda,\mu}(tu,tv) \geqslant J_{\lambda,\mu}(t^+u, t^+v) \text{ for each } t \in [t^+, t^-]$$

and $J_{\lambda,\mu}(t^+u, t^+v) \leqslant J_{\lambda,\mu}(tu,tv)$ for each $t \in [0, t^+]$. Thus,

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \leqslant t \leqslant t_{\max}} J_{\lambda,\mu}(tu,tv) \quad \text{and} \quad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geqslant 0} J_{\lambda,\mu}(tu,tv).$$

This completes the proof. $\square$

For each $(u,v) \in E$ with $K_{\lambda,\mu}(u,v) > 0$, set

$$\bar{t}_{\max} = \left(\frac{(\sigma-q)K_{\lambda,\mu}(u,v)}{(\sigma-p)\|(u,v)\|^p}\right)^{\frac{1}{p-q}} > 0. \tag{2.14}$$

Then we have the following lemma.

LEMMA 2.10. *For each* $(u,v) \in E$ *with* $K_{\lambda,\mu}(u,v) > 0$, *we have:*

(i) *if* $\int_\Omega F(x,u,v)dx \leqslant 0$, *then there is unique* $0 < t^+ < \bar{t}_{\max}$ *such that* $(t^+u, t^+v) \in N_{\lambda,\mu}^+$ *and*

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{t \geqslant 0} J_{\lambda,\mu}(tu, tv);$$

(ii) *if* $\int_\Omega F(x,u,v)dx > 0$, *then there are unique* $0 < t^+ < \bar{t}_{\max} < t^-$ *such that* $(t^+u, t^+v) \in N_{\lambda,\mu}^+, (t^-u, t^-v) \in N_{\lambda,\mu}^-$ *and*

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \leqslant t \leqslant \bar{t}_{\max}} J_{\lambda,\mu}(tu, tv); \quad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geqslant 0} J_{\lambda,\mu}(tu, tv).$$

Proof. Fix $(u,v) \in E$ with $K_{\lambda,\mu}(u,v) > 0$. Let

$$\bar{m}(t) = t^{p-\sigma}\|(u,v)\|^p - t^{q-\sigma}K_{\lambda,\mu}(u,v) \qquad \text{for } t > 0. \tag{2.15}$$

Clearly, $\bar{m}(t) \to -\infty$ as $t \to 0^+$, $\bar{m}(t) \to 0$ as $t \to \infty$. Since

$$\bar{m}'(t) = (p-\sigma)t^{p-\sigma-1}\|(u,v)\|^p - (q-\sigma)t^{q-\sigma-1}K_{\lambda,\mu}(u,v),$$

we have $\bar{m}'(t) = 0$ at $t = \bar{t}_{\max}$, $\bar{m}'(t) > 0$ for $t \in (0, \bar{t}_{\max})$ and $\bar{m}'(t) < 0$ for $t \in (\bar{t}_{\max}, \infty)$. Then $\bar{m}(t)$ achieves its maximum at $\bar{t}_{\max}$, is increasing for $t \in (0, \bar{t}_{\max})$ and decreasing for $t \in (\bar{t}_{\max}, \infty)$. Similar to the argument in Lemma 2.9, we can obtain the results of Lemma 2.10. $\square$

3. Proofs of Theorems 1.1 and 1.2

Before giving the proofs of Theorem 1.1 and 1.2, we need the following lemma.

LEMMA 3.1. (i) *If*

$$0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma, p, q, K, \lambda_1),$$

then there exists a $(PS)_{\theta_{\lambda,\mu}}$-*sequence* $\{(u_n, v_n)\} \subset N_{\lambda,\mu}$ *in* E *for* $J_{\lambda,\mu}$;
(ii) *If*

$$0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C_0,$$

then there exists a $(PS)_{\theta_{\lambda,\mu}^-}$-*sequence* $\{(u_n, v_n)\} \subset N_{\lambda,\mu}^-$ *in* E *for* $J_{\lambda,\mu}$.

Proof. The proof is almost the same as that in Wu [16].

First, we establish the existence of a local minimum for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^+$.

THEOREM 3.2. *If* $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma, p, q, K, \lambda_1)$ *and* (f_1)-(f_3) *hold, then* $J_{\lambda,\mu}$ *has a minimizer* (u_0^+, v_0^+) *in* $N_{\lambda,\mu}^+$ *and it satisfies:*

(i) $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+$;

(ii) (u_0^+, v_0^+) *is a nontrivial solution of problem* (1.1).

Proof. By Lemma 2.4 and Lemma 3.1(i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o(1) \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) = o(1) \quad \text{in } E^{-1}. \tag{3.1}$$

Then by Lemma 2.4, there exists a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+) \in E$ such that

$$\begin{cases} u_n \rightharpoonup u_0^+ \text{ weakly in } E, \\ u_n \to u_0^+ \text{ strongly in } L^q(\Omega) \text{ and in } L^\sigma(\Omega), \\ v_n \rightharpoonup v_0^+ \text{ weakly in } E, \\ v_n \to v_0^+ \text{ strongly in } L^q(\Omega) \text{ and in } L^\sigma(\Omega). \end{cases} \tag{3.2}$$

This implies that $K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+)$ as $n \to \infty$.

Next, we will show that

$$\int_\Omega F(x, u_n, v_n) dx \to \int_\Omega F(x, u_0^+, v_0^+) dx \quad \text{as } n \to \infty.$$

By Lemma 2.2, we have

$$\frac{\partial F(x, u_n, v_n)}{\partial u} \in L^\gamma(\Omega) \quad \text{and} \quad \frac{\partial F(x, u_n, v_n)}{\partial u} \to \frac{\partial F(x, u_0^+, v_0^+)}{\partial u} \quad \text{in } L^\gamma(\Omega),$$

where $\gamma = \frac{\sigma}{\sigma-1}$. It follows from the Hölder inequality,

$$\int_\Omega \left| u_n \frac{\partial F(x, u_n, v_n)}{\partial u} - u_0^+ \frac{\partial F(x, u_0^+, v_0^+)}{\partial u} \right| dx$$

$$\leqslant \int_\Omega |u_n - u_0^+| \left| \frac{\partial F(x, u_n, v_n)}{\partial u} \right| dx$$

$$+ \int_\Omega |u_0^+| \left| \frac{\partial F(x, u_n, v_n)}{\partial u} - \frac{\partial F(x, u_0^+, v_0^+)}{\partial u} \right| dx$$

$$\leqslant \|u_n - u_0^+\|_\sigma \left\| \frac{\partial F(x, u_n, v_n)}{\partial u} \right\|_\gamma$$

$$+ \|u_0^+\|_\sigma \left\| \frac{\partial F(x, u_n, v_n)}{\partial u} - \frac{\partial F(x, u_0^+, v_0^+)}{\partial u} \right\|_\gamma$$

$$\to 0 \quad \text{as } n \to \infty.$$

Hence

$$\int_\Omega \left| u_n \frac{\partial F(x, u_n, v_n)}{\partial u} - u_0^+ \frac{\partial F(x, u_0^+, v_0^+)}{\partial u} \right| dx \to 0 \quad \text{as } n \to \infty.$$

In a similar way, we have that

$$\int_\Omega \left| v_n \frac{\partial F(x, u_n, v_n)}{\partial v} - v_0^+ \frac{\partial F(x, u_0^+, v_0^+)}{\partial v} \right| dx \to 0 \quad \text{as } n \to \infty.$$

So by (2.1), we obtain

$$
\left| \int_\Omega F(x,u_n,v_n)dx - \int_\Omega F(x,u_0^+,v_0^+)dx \right|
$$

$$
\leqslant \int_\Omega \left| F(x,u_n,v_n) - F(x,u_0^+,v_0^+) \right| dx
$$

$$
= \frac{1}{\sigma} \int_\Omega \left| \left(u_n \frac{\partial F(x,u_n,v_n)}{\partial u} + v_n \frac{\partial F(x,u_n,v_n)}{\partial v} \right) \right.
$$

$$
\left. - \left(u_0^+ \frac{\partial F(x,u_0^+,v_0^+)}{\partial u} + v_0^+ \frac{\partial F(x,u_0^+,v_0^+)}{\partial v} \right) \right| dx
$$

$$
\leqslant \frac{1}{\sigma} \int_\Omega \left| u_n \frac{\partial F(x,u_n,v_n)}{\partial u} - u_0^+ \frac{\partial F(x,u_0^+,v_0^+)}{\partial u} \right| dx
$$

$$
+ \frac{1}{\sigma} \int_\Omega \left| v_n \frac{\partial F(x,u_n,v_n)}{\partial v} - v_0^+ \frac{\partial F(x,u_0^+,v_0^+)}{\partial v} \right| dx
$$

$$
\to 0 \ \text{ as } n \to \infty.
$$

Thus, $\int_\Omega F(x,u_n,v_n)dx \to \int_\Omega F(x,u_0^+,v_0^+)dx$ as $n \to \infty$.

By (3.1) and (3.2), it is easy to prove that (u_0^+,v_0^+) is a weak solution of (1.1). Since

$$
J_{\lambda,\mu}(u_n,v_n) = \frac{\sigma-p}{p\sigma}\|(u_n,v_n)\|^p - \frac{\sigma-q}{q\sigma}K_{\lambda,\mu}(u_n,v_n) \geqslant -\frac{\sigma-q}{q\sigma}K_{\lambda,\mu}(u_n,v_n)
$$

and by Lemma 2.8 (i), $J_{\lambda,\mu}(u_n,v_n) \to \theta_{\lambda,\mu} < 0$ as $n \to \infty$, letting $n \to \infty$, we see that $K_{\lambda,\mu}(u_0^+,v_0^+) > 0$. Now it follows that $u_n \to u_0^+$ strongly in E, $v_n \to v_0^+$ strongly in E and $J_{\lambda,\mu}(u_0^+,v_0^+) = \theta_{\lambda,\mu}$.

By $(u_0^+,v_0^+) \in N_{\lambda,\mu}$ and applying Fatou's lemma, we get

$$
\theta_{\lambda,\mu} \leqslant J_{\lambda,\mu}(u_0^+,v_0^+) = \frac{1}{p}\|(u_0^+,v_0^+)\|^p - \frac{1}{\sigma}\int_\Omega F(x,u_0^+,v_0^+) - \frac{1}{q}K_{\lambda,\mu}(u_0^+,v_0^+)
$$

$$
\leqslant \liminf_{n\to\infty} \left(\frac{1}{p}\|(u_n,v_n)\|^p - \frac{1}{\sigma}\int_\Omega F(x,u_n,v_n) - \frac{1}{q}K_{\lambda,\mu}(u_n,v_n) \right)
$$

$$
\leqslant \liminf_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = \theta_{\lambda,\mu}.
$$

This follows that

$$
J_{\lambda,\mu}(u_0^+,v_0^+) = \theta_{\lambda,\mu} \quad \text{and} \quad \lim_{n\to\infty}\|(u_n,v_n)\|^p = \|(u_0^+,v_0^+)\|^p.
$$

Let $(\widetilde{u}_n,\widetilde{v}_n) = (u_n,v_n) - (u_0^+,v_0^+)$, then Brézis-Lieb lemma [12] implies that

$$
\|(\widetilde{u}_n,\widetilde{v}_n)\|^p = \|(u_n,v_n)\|^p - \|(u_0^+,v_0^+)\|^p.
$$

Therefore, $u_n \to u_0^+$ strongly in E, $v_n \to v_0^+$ strongly in E.

Moreover, we have $(u_0^+,v_0^+) \in N_{\lambda,\mu}^+$. In fact, if $(u_0^+,v_0^+) \in N_{\lambda,\mu}^-$, by Lemma 2.6 (iii) and Lemma 2.9, there are unique t_0^+ and t_0^- such that $(t_0^+u_0^+,t_0^+v_0^+) \in N_{\lambda,\mu}^+$ and

$(t_0^- u_0^+, t_0^- v_0^+) \in N_{\lambda,\mu}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \leqslant t_0^-$ such that $J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) < J_{\lambda,\mu}(\bar{t} u_0^+, \bar{t} v_0^+)$. By Lemma 2.9,

$$J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) < J_{\lambda,\mu}(\bar{t} u_0^+, \bar{t} v_0^+) \leqslant J_{\lambda,\mu}(t_0^- u_0^+, t_0^- v_0^+) = J_{\lambda,\mu}(u_0^+, v_0^+),$$

which is a contradiction.

So by Lemma 2.5 we may assume that (u_0^+, v_0^+) is a nontrivial solution of problem (1.1).

Next, we establish the existence of a local minimum for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$.

THEOREM 3.3. *If* $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C_0$, (f_1) *and* (f_2) *hold, then* $J_{\lambda,\mu}$ *has a minimizer* (u_0^-, v_0^-) *in* $N_{\lambda,\mu}^-$ *and it satisfies:*

(i) $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta_{\lambda,\mu}^-$;

(ii) (u_0^-, v_0^-) *is a nontrivial solution of problem* (1.1).

Proof. By Lemma 3.1(ii), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$ such that

$$J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}^- + o(1) \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) = o(1) \quad \text{in } E^{-1}. \tag{3.3}$$

Then by Lemma 2.4, there exists a subsequence $\{(u_n, v_n)\}$ and $(u_0^-, v_0^-) \in E$ such that

$$\begin{cases} u_n \rightharpoonup u_0^+ \text{ weakly in } E, \\ u_n \to u_0^+ \text{ strongly in } L^q(\Omega) \text{ and in } L^\sigma(\Omega), \\ v_n \rightharpoonup v_0^+ \text{ weakly in } E, \\ v_n \to v_0^+ \text{ strongly in } L^q(\Omega) \text{ and in } L^\sigma(\Omega). \end{cases} \tag{3.4}$$

This implies that

$$K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^-, v_0^-) \quad \text{as } n \to \infty,$$

$$\int_\Omega F(x, u_n, v_n)dx \to \int_\Omega F(x, u_0^-, v_0^-)dx \quad \text{as } n \to \infty.$$

Moreover, by (2.9) we obtain

$$\int_\Omega F(x, u_n, v_n)dx > \frac{p-q}{\sigma-q}\|(u_n, v_n)\|^p. \tag{3.5}$$

By (2.11) and (3.5) there exists a positive number $\overline{C}$ such that

$$\int_\Omega F(x, u_n, v_n)dx > \overline{C}.$$

This implies that

$$\int_\Omega F(x,u_0^-,v_0^-)dx \geqslant \overline{C}. \tag{3.6}$$

By (3.3) and (3.4) we obtain that (u_0^-,v_0^-) is a weak solution of (1.1).

Now we prove that $u_n \to u_0^-$ strongly in E and $v_n \to v_0^-$ strongly in E. Supposing otherwise, then either

$$\|u_0^-\| < \liminf_{n\to\infty}\|u_n\| \text{ or } \|v_0^-\| < \liminf_{n\to\infty}\|v_n\|.$$

By Lemma 2.9, there is a unique t_0^- such that $(t_0^- u_0^-, t_0^- v_0^-) \in N_{\lambda,\mu}^-$, since $(u_n,v_n) \in N_{\lambda,\mu}^-, J_{\lambda,\mu}(u_n,v_n) \geqslant J_{\lambda,\mu}(tu_n,tv_n)$ for all $t \geqslant 0$, we have

$$J_{\lambda,\mu}(t_0^- u_0^-, t_0^- v_0^-) < \lim_{n\to\infty} J_{\lambda,\mu}(t_0^- u_n, t_0^- v_n) \leqslant \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = \theta_{\lambda,\mu}^-$$

and this is contradiction. Hence $u_n \to u_0^-$ strongly in E and $v_n \to v_0^-$ strongly in E.

This implies that

$$J_{\lambda,\mu}(u_n,v_n) \to J_{\lambda,\mu}(u_0^-,v_0^-) = \theta_{\lambda,\mu}^- \text{ as } n \to \infty.$$

By Lemma 2.5 and (3.6) we may assume that (u_0^-,v_0^-) is a nontrivial solution of problem (1.1). $\square$

Now, we complete the proof of Theorem 1.1 and Theorem 1.2: by Theorem 3.1, we obtain that for all $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C(\sigma,p,q,K,\lambda_1)$, problem (1.1) has a nontrivial solution $(u_0^+,v_0^+) \in N_{\lambda,\mu}^+$. On the other hand, from Theorems 3.2, we get the second solution $(u_0^-,v_0^-) \in N_{\lambda,\mu}^-$ for all $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C_0 < C(\sigma,p,q,K,\lambda_1)$. Since $N_{\lambda,\mu}^+ \cap N_{\lambda,\mu}^- = \emptyset$, this implies that (u_0^+,v_0^+) and (u_0^-,v_0^-) are distinct.

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