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SMALL DATA SCATTERING FOR A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study the scattering theory for a system of nonlinear Schrödinger equations in space dimension $n \geqslant 3$. In the case $n \geqslant 4$, existence of the scattering operator is proved in small data setting in the Sobolev space $H^{n/2-2}$. In the case $n = 3$, a similar result is proved in the weighted L^2 space $\langle x\rangle^{-1/2}L^2 = \mathscr{F}\left(H^{-1/2}\right)$ under the mass resonance condition.

1. Introduction

We consider the following system of nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \frac{1}{2m}\Delta u = \lambda v\overline{u}, \\ i\partial_t v + \frac{1}{2M}\Delta v = \mu u^2, \end{cases} \tag{1}$$

where u and v are complex-valued functions of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n, m$ and M are positive constants, and λ and μ are complex constants. The interaction terms in the system (1) is quadratic in (u,v). Regarding quadratic nonlinearity in space $\mathbb{R}^n$, we have the associated critical space dimensions. By the space-time scaling argument on (1), the critical function space is $H^{n/2-2}$, where $H^s = (1-\Delta)^{-s/2}L^2$. In particular, L^2 and H^1 are critical for $n = 4$ and $n = 6$, respectively, from the scaling point of view. As far as the asymptotic profiles of solutions are concerned, quadratic nonlinearity is regarded as the borderline between short-range and long-range interactions for $n = 2$. If the argument depends exclusively on the space-time integrability properties, Strauss exponent [18] $\gamma(n) = (n+2+\sqrt{n^2+12n+4})/(2n)$ is a natural critical number and quadratic nonlinearity is critical for $n = 3$ since $\gamma(3) = 2$. In [8], [9], a detailed asymptotic analysis on the long-time behavior of small amplitude solutions has been made for $n = 2$.

The purpose in this paper is to study the scattering theory for (1) in a small data setting for $n \geqslant 3$. In the case $n \geqslant 4$, existence of the scattering operator is proved in the neighborhood of the origin in $H^{n/2-2}\times H^{n/2-2}$ (Theorem 2.1). The method of proof depends on the endpoint Strichartz estimate of Keel and Tao [12] (see also [13]) and on a special bilinear estimate (Proposition 3.1). In the case $n = 3$, existence of scattering

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operator is proved in a neighborhood of the origin in $\mathscr{F}\left(H^{-1/2}\right) \times \mathscr{F}\left(H^{-1/2}\right)$ (Theorem 2.3), where $\mathscr{F}$ is the Fourier transform and $\mathscr{F}\left(H^{-s}\right) = \langle x\rangle^{-s} L^2$ is the weighted L^2 space with weight $\langle x\rangle^s = \left(1+|x|^2\right)^{s/2}$ of order $s \geqslant 0$, under the mass resonance condition

$$M = 2m. \tag{2}$$

The method of proof depends on the gauge invariance of the interactions in (1) under the mass resonance condition (2) and the Strichartz estimates based on the Lorentz spaces in time [16], [17].

The system (1) is regarded as a non relativistic limit of the system of nonlinear Klein-Gordon equations with (2)

$$\begin{cases} \frac{1}{2c^2 m}\partial_t^2 u - \frac{1}{2m}\Delta u + \frac{mc^2}{2}u = -\lambda v\overline{u}, \\ \frac{1}{2c^2 M}\partial_t^2 v - \frac{1}{2M}\Delta v + \frac{Mc^2}{2}v = -\mu u^2, \end{cases} \tag{3}$$

where c is the speed of light, since the wave functions $u_c = e^{\mathrm{i}tmc^2}u$, $v_c = e^{\mathrm{i}tMc^2}v$ satisfy

$$\begin{cases} \frac{1}{2c^2 m}\partial_t^2 u_c - \mathrm{i}\partial_t u_c - \frac{1}{2m}\Delta u_c = -e^{itc^2(2m-M)}\lambda v_c\overline{u_c}, \\ \frac{1}{2c^2 M}\partial_t^2 v_c - \mathrm{i}\partial_t v_c - \frac{1}{2M}\Delta v_c = -e^{itc^2(M-2m)}\mu u_c^2, \end{cases} \tag{4}$$

where the phase oscillations on the right hand sides vanish if and only if (2) holds, and under the resonance condition (2) the limiting system of (4) formally yields (1) as $c \longrightarrow \infty$.

The system (3) is closely related to systems studied in [1], [6], [7] and nonrelativistic limit for the nonlinear Klein-Gordon equations has been studied in [13], [14].

We conclude this section by giving some of the notation used in this paper. For any p with $1 \leqslant p \leqslant \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and for any p with $1 < p < \infty$, $H_p^s = (1-\Delta)^{-s/2}L^p$ denotes the Sobolev space defined in terms of Bessel potentials. We shall also use the homogeneous Sobolev spaces $\dot{H}_p^s = (-\Delta)^{-s/2}L^p$. For simplicity we put $H^s = H_2^s$ and $\dot{H}^s = \dot{H}_2^s$. For any p with $1 \leqslant p \leqslant \infty$, the exponent dual to p is denoted by p'. For any interval $I \subset \mathbb{R}$ and any Banach space X, we denote by $C(I;X)$ the space of strongly continuous functions from I to X and by $L^p(I;X)$ [resp. $L^{p,q}(I;X)$] the space of strongly measurable functions u from I to X such that $\|u(\cdot);X\| \in L^p(I)$ [resp. $\|u(\cdot);X\| \in L^{p,q}(I)$], where $L^{p,q}$ denotes the Lorentz space. We refer to [2], [19] for general information on function spaces. We define the Fourier transform in $\mathbb{R}^n$ by

$$(\mathscr{F}u)(\xi) = \widehat{u}(\xi) = (2\pi)^{-n/2}\int \exp(-\mathrm{i}x\cdot\xi)\,u(x)\,dx.$$

We denote $a \vee b$ the maximum of $a, b \in \mathbb{R}$

2. Main theorems

To state the main theorems, we introduce the following notation. The Cauchy problem for (1) with data $(u(0), v(0)) = (\phi, \psi)$ at $t = 0$ shall be treated in the form of the following integral equations:

$$\begin{cases} u(t) = U_m(t)\phi - \mathrm{i}\int_0^t U_m(t-t')\lambda v\overline{u}(t')\,dt', \\ v(t) = U_M(t)\psi - \mathrm{i}\int_0^t U_M(t-t')\mu u^2(t')\,dt', \end{cases} \tag{5}$$

where $U_m(t) = \exp\left(\mathrm{i}\frac{t}{2m}\Delta\right)$ and $U_M(t) = \exp\left(\mathrm{i}\frac{t}{2M}\Delta\right)$ are free propagators with masses m and M, respectively. For $n \geqslant 4$, we introduce the following Banach space

$$X = (C \cap L^\infty)\left(\mathbb{R}; H^{n/2-2}\right) \cap L^2\left(\mathbb{R}; H^{n/2-2}_{2^*}\right)$$

with norm

$$\left\|u; X\right\| = \left\|u; L^\infty\left(H^{n/2-2}\right)\right\| \vee \left\|u; L^2\left(H^{n/2-2}_{2^*}\right)\right\|,$$

where $2^* = 2n/(n-2)$ is the critical Sobolev exponent.

For $\varepsilon > 0$, we define

$$B_\varepsilon = \left\{(\phi, \psi) \in H^{n/2-2} \times H^{n/2-2}; \left\|\phi; \dot{H}^{n/2-2}\right\| \vee \left\|\psi; \dot{H}^{n/2-2}\right\| \leqslant \varepsilon\right\}.$$

THEOREM 2.1. *Let $n \geqslant 4$. Then there exist ε_0 and C_0 such that $0 < \varepsilon_0 \leqslant 1 \leqslant C_0$ with the following properties.*

(1) *For any ε with $0 < \varepsilon \leqslant \varepsilon_0$ and any $(\phi, \psi) \in B_\varepsilon$ the system (5) has a unique pair of solutions $(u, v) \in X \times X$. Moreover, there exist unique $(\phi_\pm, \psi_\pm) \in B_{C_0\varepsilon}$ such that*

$$\left\|u(t) - U_m(t)\phi_\pm; H^{n/2-2}\right\| \to 0,$$
$$\left\|v(t) - U_M(t)\psi_\pm; H^{n/2-2}\right\| \to 0,$$

as $t \to \pm\infty$.

$(2)_+$ *For any ε with $0 < \varepsilon \leqslant \varepsilon_0$ and any $(\phi_+, \psi_+) \in B_\varepsilon$ the system (5) has a unique pair of solutions $(u, v) \in X \times X$ such that $(u(0), v(0)) \in B_{C_0\varepsilon}$,*

$$\left\|u(t) - U_m(t)\phi_+; H^{n/2-2}\right\| \to 0,$$
$$\left\|v(t) - U_M(t)\psi_+; H^{n/2-2}\right\| \to 0,$$

as $t \to +\infty$.

$(2)_-$ *For any ε with $0 < \varepsilon \leqslant \varepsilon_0$ and any $(\phi_-, \psi_-) \in B_\varepsilon$ the system (5) has a unique pair of solutions $(u, v) \in X \times X$ such that $(u(0), v(0)) \in B_{C_0\varepsilon}$,*

$$\left\|u(t) - U_m(t)\phi_-; H^{n/2-2}\right\| \to 0,$$
$$\left\|v(t) - U_M(t)\psi_-; H^{n/2-2}\right\| \to 0,$$

as $t \to -\infty$.

COROLLARY 2.2. *The wave operators $W_\pm : (\phi_\pm, \psi_\pm) \longmapsto (u(0), v(0))$ are defined as mappings from B_ε to $B_{C_0\varepsilon}$ for any ε with $0 < \varepsilon \leqslant \varepsilon_0$. The scattering operator $S : (\phi_-, \psi_-) \longmapsto (\phi_+, \psi_+)$ is defined as a mapping from $B_{C_0^{-1}\varepsilon}$ to $B_{C_0\varepsilon}$ for any ε with $0 < \varepsilon \leqslant \varepsilon_0$.*

REMARK 2.1. As in the definition of B_ε , smallness of the data is necessary for the scaling invariant part. The smallness assumption L^2 part is necessary only for $n=4$, since $n/2-2=0$. See [3], [4], [11], [15] for the related topics.

For $n=3$, we introduce the following Banach space

$$Y=(C\cap L^\infty)\left(\mathbb{R};L^2\right)\cap L^{4,2}\left(\mathbb{R};L^3\right)$$

with norm

$$\|u;Y\|=\left\|u;L_t^\infty\left(L^2\right)\right\|\vee\left\|u;L_t^{4,2}\left(L^3\right)\right\|,$$

where $L^{4,2}$ is the Lorentz space with second exponent 2, so that $L^{4,1}\subset L^{4,2}\subset L^{4,4}=L^4\subset L^{4,\infty}$. For the free propagator

$$U_m(t)=\exp\left(\mathrm{i}\frac{t}{2m}\Delta\right)=\mathscr{F}^{-1}\exp\left(-\mathrm{i}\frac{t}{2m}|\xi|^2\right)\mathscr{F},$$

we introduce the standard generator of Galilei transformations as

$$J_m(t)=U_m(t)xU_m(-t)=x+\mathrm{i}\frac{t}{m}\nabla,$$

which are also represented as

$$J_m(t)=M_m(t)\mathrm{i}\frac{t}{m}\nabla M_m(-t),M_m(t)=\exp\left(\mathrm{i}\frac{m}{2t}|x|^2\right)$$

for $t\neq 0$. Fractional power of J_m are defined as

$$|J_m|^a(t)=U_m(t)\,|x|^a\,U_m(-t),a>0,$$

which are also represented as (see [10])

$$|J_m|^a(t)=M_m(t)\left(-\frac{t^2}{m^2}\Delta\right)^{a/2}M_m(-t)$$

for $t\neq 0$, since $U_m(t)$ is represented as

$$U_m(t)=M_m(t)D_m(t)\mathscr{F}M_m(t)$$

with

$$\left(D_m(t)\psi\right)(x)=(\mathrm{i}t/m)^{-n/2}\,\psi\left(mx/t\right).$$

We solve (5) in the Banach space

$$Y_m=\left\{u\in Y;|J_m|^{1/2}u\in Y\right\}$$

with norm

$$\|u;Y_m\|=\|u;Y\|\vee\left\||J_m|^{1/2}u;Y\right\|.$$

For $\varepsilon>0$, we define

$$\begin{aligned}\widehat{B}_\varepsilon&=\left\{(\phi,\psi)\in\mathscr{F}\left(H^{-1/2}\right)\times\mathscr{F}\left(H^{-1/2}\right);\left\|\widehat{\phi};\dot{H}^{1/2}\right\|\vee\left\|\widehat{\psi};\dot{H}^{1/2}\right\|\leqslant\varepsilon\right\}\\&=\left\{(\phi,\psi)\in L^2\times L^2;|x|^{1/2}\phi,|x|^{1/2}\psi\in L^2,\left\||x|^{1/2}\phi;L^2\right\|\vee\left\||x|^{1/2}\psi;L^2\right\|\leqslant\varepsilon\right\}.\end{aligned}$$

THEOREM 2.3. *Let* $n = 3$. *Let* m *and* M *satisfy* $M = 2m$. *Then there exist* ε_0 *and* C_0 *such that* $0 < \varepsilon_0 \leqslant 1 \leqslant C_0$ *with the following properties.*

(1) *For any* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$ *and any* $(\phi, \psi) \in \widehat{B}_\varepsilon$ *the system* (5) *has a unique pair of solutions* $(u, v) \in Y_m \times Y_{2m}$. *Moreover, there exist unique* $(\phi_\pm, \psi_\pm) \in \widehat{B}_{C_0\varepsilon}$ *such that*

$$\left\| U_m(-t)u(t) - \phi_\pm; \mathscr{F}\left(H^{1/2}\right) \right\| \to 0,$$
$$\left\| U_{2m}(-t)v(t) - \psi_\pm; \mathscr{F}\left(H^{1/2}\right) \right\| \to 0,$$

as $t \to \pm\infty$.

$(2)_+$ *For any* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$ *and any* $(\phi_+, \psi_+) \in \widehat{B}_\varepsilon$ *the system* (5) *has a unique pair of solutions* $(u, v) \in Y_m \times Y_{2m}$ *such that* $(u(0), v(0)) \in \widehat{B}_{C_0\varepsilon}$,

$$\left\| U_m(-t)u(t) - \phi_+; \mathscr{F}\left(H^{1/2}\right) \right\| \to 0,$$
$$\left\| U_{2m}(-t)v(t) - \psi_+; \mathscr{F}\left(H^{1/2}\right) \right\| \to 0,$$

as $t \to +\infty$.

$(2)_-$ *For any* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$ *and any* $(\phi_-, \psi_-) \in \widehat{B}_\varepsilon$ *the system* (5) *has a unique pair of solutions* $(u, v) \in Y_m \times Y_{2m}$ *such that* $(u(0), v(0)) \in \widehat{B}_{C_0\varepsilon}$,

$$\left\| U_m(-t)u(t) - \phi_-; \mathscr{F}\left(H^{1/2}\right) \right\| \to 0,$$
$$\left\| U_{2m}(-t)v(t) - \psi_-; \mathscr{F}\left(H^{1/2}\right) \right\| \to 0,$$

as $t \to -\infty$.

COROLLARY 2.4. *The wave operators* $W_\pm : (\phi_\pm, \psi_\pm) \longmapsto (u(0), v(0))$ *are defined as mappings from* $\widehat{B}_\varepsilon$ *to* $\widehat{B}_{C_0\varepsilon}$ *for any* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$. *The scattering operator* $S : (\phi_-, \psi_-) \longmapsto (\phi_+, \psi_+)$ *is defined as a mapping from* $\widehat{B}_{C_0^{-1}\varepsilon}$ *to* $\widehat{B}_{C_0\varepsilon}$ *for any* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$.

3. Proof of Theorem 2.1

In this section we prove Theorem 2.1. For that purpose we use the following bilinear estimates.

PROPOSITION 3.1. *Let* $n \geqslant 4$. *Then there exists a constant* C *depending only on* n *such that the following estimates hold:*

$$\left\| uv; L^2 \right\| \leqslant C \left\| u; \dot{H}^{n/2-2}_{2^*} \right\| \left\| v; L^{2^*} \right\|$$

for any $u \in \dot{H}_{2^*}^{n/2-2}$, $v \in L^{2^*}$ *and*

$$\left\| uv; \dot{H}^{n/2-2} \right\| \leqslant C \left\| u; \dot{H}_{2^*}^{n/2-2} \right\| \left\| v; \dot{H}_{2^*}^{n/2-2} \right\|$$

for any $u, v \in \dot{H}_{2^*}^{n/2-2}$.

Proof. The first inequality follows from the Hölder inequality with $1/2 = 1/2^* + 1/n$ and the Sobolev embedding $\dot{H}_{2^*}^{n/2-2} \subset L^n$. When n is even, the second inequality follows from

$$\begin{aligned}
\left\| uv; \dot{H}^{n/2-2} \right\| &\leqslant C \sum_{|\alpha|=n/2-2} \left\| \partial^\alpha (uv); L^2 \right\| \\
&\leqslant C \sum_{|\alpha|=n/2-2} \sum_{\beta+\gamma=\alpha} \left\| \partial^\beta u \partial^\gamma v; L^2 \right\| \\
&\leqslant C \sum_{|\alpha|=n/2-2} \sum_{\beta+\gamma=\alpha} \left\| \partial^\beta u; L^{n/(|\beta|+1)} \right\| \left\| \partial^\gamma v; L^{n/(|\gamma|+1)} \right\| \\
&\leqslant C \left\| u; \dot{H}_{2^*}^{n/2-2} \right\| \left\| v; \dot{H}_{2^*}^{n/2-2} \right\|,
\end{aligned}$$

where the last inequality follows from the Sobolev embedding

$$\dot{H}_{2^*}^{n/2-2} \subset \dot{H}_{n/(s+1)}^{s}$$

for any $s \in \mathbb{R}$ with $0 \leqslant s \leqslant n/2-2$. When n is odd, we estimate

$$\begin{aligned}
\left\| uv; \dot{H}^{n/2-2} \right\| &\leqslant C \sum_{|\alpha|=(n-5)/2} \left\| \partial^\alpha (uv); \dot{H}^{1/2} \right\| \\
&\leqslant C \sum_{|\alpha|=(n-5)/2} \sum_{\beta+\gamma=\alpha} \left\| \partial^\beta u \partial^\gamma v; \dot{H}^{1/2} \right\| \\
&\leqslant C \sum_{|\alpha|=(n-5)/2} \sum_{\beta+\gamma=\alpha} \left\| \partial^\beta u; \dot{H}_{n/(|\beta|+3/2)}^{1/2} \right\| \left\| \partial^\gamma v; L^{n/(|\gamma|+1)} \right\| \\
&\quad + C \sum_{|\alpha|=(n-5)/2} \sum_{\beta+\gamma=\alpha} \left\| \partial^\beta u; L^{n/(|\beta|+1)} \right\| \left\| \partial^\gamma v; \dot{H}_{n/(|\gamma|+3/2)}^{1/2} \right\| \\
&\leqslant C \left\| u; \dot{H}_{2^*}^{n/2-2} \right\| \left\| v; \dot{H}_{2^*}^{n/2-2} \right\|,
\end{aligned}$$

where we have used fractional Leibniz estimate of Kato and Ponce [11] and the Sobolev embedding of the same type as above. □

3.1. Proof of Theorem 2.1

We introduce two auxiliary function spaces:

$$X_0 = (C \cap L^\infty)\left(\mathbb{R}; L^2\right) \cap L^2\left(\mathbb{R}; L^{2^*}\right),$$

$$\dot{X} = (C \cap L^\infty)\left(\mathbb{R}; \dot{H}^{n/2-2}\right) \cap L^2\left(\mathbb{R}; \dot{H}_{2^*}^{n/2-2}\right)$$

with norms

$$\left\|u; X_0\right\| = \left\|u; L_t^\infty\left(L^2\right)\right\| \vee \left\|u; L_t^2\left(L^{2^*}\right)\right\|,$$

$$\left\|u; \dot{X}\right\| = \left\|u; L_t^\infty\left(\dot{H}^{n/2-2}\right)\right\| \vee \left\|u; L_t^2\left(\dot{H}_{2^*}^{n/2-2}\right)\right\|.$$

For $\rho, \delta > 0$ we define

$$X(\rho, \delta) = \left\{(u, v) \in X \times X; \|u; X_0\| \vee \|v; X_0\| \leqslant \rho, \left\|u; \dot{X}\right\| \vee \left\|v; \dot{X}\right\| \leqslant \delta\right\}$$

with metric

$$d\left((u,v),(u',v')\right) = \left\|u - u'; X_0\right\| \vee \left\|u - u'; \dot{X}\right\| \vee \left\|v - v'; X_0\right\| \vee \left\|v - v'; \dot{X}\right\|$$

which is equivalent to the metric induced by the norm in X. Let $\varepsilon > 0$. For any $(\phi, \psi) \in B_\varepsilon$ and any $(u, v) \in X(\rho, \delta)$ we define

$$(\Phi(u,v))(t) = U_m(t)\phi - \mathrm{i}\int_0^t U_m\left(t - t'\right)\lambda v\bar{u}\left(t'\right)dt',$$

$$(\Psi(u,v))(t) = U_M(t)\psi - \mathrm{i}\int_0^t U_M\left(t - t'\right)\mu u^2\left(t'\right)dt'.$$

We prove that there exist δ, ε_0 such that for any ε with $0 < \varepsilon \leqslant \varepsilon_0$ and any $(\phi, \psi) \in B_\varepsilon$ the mapping $(u, v) \longmapsto (\Phi(u, v), \Psi(u, v))$ is a contraction on $X(\rho, \delta)$ for some $\rho > 0$. By the Strichartz estimate [12] and Proposition 3.1, we estimate

$$\begin{aligned}\left\|\Phi(u,v); X_0\right\| &\leqslant C\left\|\phi; L^2\right\| + C\left\|v\bar{u}; L_t^1\left(L^2\right)\right\| \\ &\leqslant C\left\|\phi; L^2\right\| + C\left\|v; L_t^2\left(\dot{H}_{2^*}^{n/2-2}\right)\right\|\left\|u; L_t^2\left(L^{2^*}\right)\right\|,\end{aligned}$$

$$\begin{aligned}\left\|\Psi(u,v); X_0\right\| &\leqslant C\left\|\psi; L^2\right\| + C\left\|u^2; L_t^1\left(L^2\right)\right\| \\ &\leqslant C\left\|\psi; L^2\right\| + C\left\|u; L_t^2\left(\dot{H}_{2^*}^{n/2-2}\right)\right\|\left\|u; L_t^2\left(L^{2^*}\right)\right\|,\end{aligned}$$

$$\left\|\Phi(u,v); \dot{X}\right\| \leqslant C\left\|\phi; \dot{H}^{n/2-2}\right\| + C\left\|v\bar{u}; L_t^1\left(\dot{H}^{n/2-2}\right)\right\|$$

$$\leqslant C\left\|\phi;\dot{H}^{n/2-2}\right\|+C\left\|v;L^2\left(\dot{H}_{2^*}^{n/2-2}\right)\right\|\left\|u;L^2\left(\dot{H}_{2^*}^{n/2-2}\right)\right\|,$$

$$\begin{aligned}\left\|\Psi(u,v);\dot{X}\right\| &\leqslant C\left\|\psi;\dot{H}^{n/2-2}\right\|+C\left\|u^2;L_t^1\left(\dot{H}^{n/2-2}\right)\right\| \\ &\leqslant C\left\|\psi;\dot{H}^{n/2-2}\right\|+C\left\|u;L^2\left(\dot{H}_{2^*}^{n/2-2}\right)\right\|^2.\end{aligned}$$

Therefore, we have for $(u,v)\in X(\rho,\delta)$,

$$\|\Phi(u,v);X_0\|\vee\|\Psi(u,v);X_0\|\leqslant C\left(\|\phi;L^2\|\vee\|\psi;L^2\|\right)+C\rho\delta,$$

$$\left\|\Phi(u,v);\dot{X}\right\|\vee\left\|\Psi(u,v);\dot{X}\right\|\leqslant C\varepsilon+C\delta^2,$$

and similarly, for $(u,v),(u',v')\in X(\rho,\delta)$,

$$\begin{aligned}&\left\|\Phi(u,v)-\Phi(u',v');X_0\right\|\vee\left\|\Psi(u,v)-\Psi(u',v');X_0\right\| \\ &\qquad\leqslant C\delta\left(\|u-u';X_0\|\vee\|v-v';X_0\|\right),\end{aligned}$$

$$\begin{aligned}&\left\|\Phi(u,v)-\Phi(u',v');\dot{X}\right\|\vee\left\|\Psi(u,v)-\Psi(u',v');\dot{X}\right\| \\ &\qquad\leqslant C\delta\left(\left\|u-u';\dot{X}\right\|\vee\left\|v-v';\dot{X}\right\|\right),\end{aligned}$$

we choose $\rho,\delta,$ and ε_0 as

$$C\left(\|\phi;L^2\|\vee\|\psi;L^2\|\right)\leqslant\rho/2, C\delta\leqslant 1/2, C\varepsilon_0=\delta/2.$$

Then $(u,v)\longmapsto(\Phi(u,v),\Psi(u,v))$ is a contraction on $X(\rho,\delta)$ so that the corresponding fixed point provides the required pair of solutions (u,v) of (5). The uniqueness of solutions in $X\times X$ follows by the standard argument by taking into account that

$$\left\|u;L^2\left(I;\dot{H}_{2^*}^{n/2-2}\right)\right\|\vee\left\|v;L^2\left(I;\dot{H}_{2^*}^{n/2-2}\right)\right\|$$

can be arbitrarily small by making the length $|I|$ of the interval $I\subset\mathbb{R}$ sufficiently small. The existence of scattering states $(\phi_\pm,\psi_\pm)$ follows for instance by estimating

$$\begin{aligned}&\left\|U_m(-t)u(t)-U_m(-s)u(s);H^{n/2-2}\right\| \\ &\quad=\left\|\int_s^t U_m(-t')\lambda v\overline{u}(t')dt';H^{n/2-2}\right\| \\ &\quad\leqslant C\left\|v;L^2\left(s,t;H^{n/2-2}\right)\right\|\left\|u;L^2\left(s,t;H^{n/2-2}\right)\right\|\to 0\end{aligned}$$

as $t > s \to +\infty$. Moreover, $(\phi_\pm, \psi_\pm)$ belong to $B_{C_0\varepsilon}$ since

$$\begin{aligned}\left\|\phi_+; \dot{H}^{n/2-2}\right\| &= \left\|U_m(t)\phi_+; \dot{H}^{n/2-2}\right\| \\ &\leqslant \left\|u(t); \dot{H}^{n/2-2}\right\| + \left\|u(t) - U_m(t)\phi_+; \dot{H}^{n/2-2}\right\| \\ &\leqslant \left\|u(t); L_t^\infty\left(\dot{H}^{n/2-2}\right)\right\| + \left\|u(t) - U_m(t)\phi_+; \dot{H}^{n/2-2}\right\| \\ &\leqslant C\varepsilon_0 + \left\|u(t) - U_m(t)\phi_+; \dot{H}^{n/2-2}\right\|\end{aligned}$$

for instance. This proves part (1). Finally we consider part $(2)_\pm$. For any $(\phi_\pm, \psi_\pm) \in B_\varepsilon$ and any $(u, v) \in X(\rho, \delta)$ we define

$$(\Phi_\pm(u,v))(t) = U_m(t)\,\phi_\pm - \mathrm{i}\int_{\pm\infty}^{t} U_m(t-t')\,\lambda v\overline{u}(t')\,dt',$$

$$(\Psi_\pm(u,v))(t) = U_M(t)\,\psi_\pm - \mathrm{i}\int_{\pm\infty}^{t} U_M(t-t')\,\mu u^2(t')\,dt'.$$

Then the contraction argument proceeds in almost the same way.

4. Proof of Theorem 2.3

In this section we prove Theorem 2.3. For that purpose we use the following bilinear estimates.

PROPOSITION 4.1. *Let $n = 3$. Then there exists a constant C depending only on n such that the following estimates hold:*

$$\left\|uv; L^{3/2}\right\| \leqslant C|t|^{-1/2}\left\||J_m|^{1/2}u; L^2\right\|\left\|v; L^3\right\|,$$

$$\begin{aligned}\left\||J_m|^{1/2}(v\overline{u}); L^{3/2}\right\| \leqslant C|t|^{-1/2}\Big(&\left\||J_m|^{1/2}u; L^2\right\|\left\||J_{2m}|^{1/2}v; L^3\right\| \\ &+ \left\||J_m|^{1/2}u; L^3\right\|\left\||J_{2m}|^{1/2}v; L^2\right\|\Big),\end{aligned}$$

$$\begin{aligned}\left\||J_{2m}|^{1/2}(vu); L^{3/2}\right\| \leqslant C|t|^{-1/2}\Big(&\left\||J_m|^{1/2}u; L^2\right\|\left\||J_m|^{1/2}v; L^3\right\| \\ &+ \left\||J_m|^{1/2}u; L^3\right\|\left\||J_m|^{1/2}v; L^2\right\|\Big),\end{aligned}$$

for any $t \neq 0$ and any $m > 0$.

Proof. The first inequality follows from the Hölder inequality and the Sobolev embedding $\dot{H}^{1/2} \subset L^3$. The second inequality follows from fractional Leibniz estimate of Kato and Ponce [11] and the Sobolev embedding $\dot{H}^{1/2} \subset L^3$, since

$$\begin{aligned}
\left\| |J_m|^{1/2}(v\overline{u}); L^{3/2} \right\| &= \left\| \Big(-\frac{t^2}{m^2}\Delta \Big)^{1/4} \big(M_{2m}^{-1} v \cdot \overline{M_m^{-1} u} \big); L^{3/2} \right\| \\
&\leqslant C \left\| \big(-\frac{t^2}{m^2}\Delta \big)^{1/4} M_{2m}^{-1} v; L^3 \right\| \left\| \overline{M_m^{-1} u}; L^3 \right\| \\
&\qquad + C \left\| M_{2m}^{-1} v; L^3 \right\| \left\| \Big(-\frac{t^2}{m^2}\Delta \Big)^{1/4} \overline{M_m^{-1} u}; L^3 \right\| \\
&\leqslant C \left\| |J_{2m}|^{1/2} v; L^3 \right\| \left\| M_m^{-1} u; L^3 \right\| \\
&\qquad + C \left\| M_{2m}^{-1} v; L^3 \right\| \left\| |J_m|^{1/2} u; L^3 \right\|.
\end{aligned}$$

The third inequality follows in the same way, since

$$\left\| |J_{2m}|^{1/2} (vu); L^{3/2} \right\| = \left\| \left(-\frac{t^2}{4m^2}\Delta \right)^{1/4} \left(M_m^{-1} v \cdot M_m^{-1} u \right); L^{3/2} \right\|.$$

4.1. Proof of Theorem 2.3

For $\rho, \delta > 0$ we define

$$\begin{aligned}
Y(\rho,\delta) = \Big\{ &(u,v) \in Y_m \times Y_{2m}; \|u;Y\| \vee \|v;Y\| \leqslant \rho, \\
&\left\| |J_m|^{1/2} u; Y \right\| \vee \left\| |J_{2m}|^{1/2} v; Y \right\| \leqslant \delta \Big\}
\end{aligned}$$

with metric

$$d\big((u,v),(u',v')\big) = \left\| u-u'; Y_m \right\| \vee \left\| v-v'; Y_{2m} \right\|.$$

Let $\varepsilon > 0$. For any $(\phi,\psi) \in \widehat{B}_\varepsilon$ and any $(u,v) \in Y(\rho,\delta)$ we define

$$\big(\Phi(u,v)\big)(t) = U_m(t)\phi - \mathrm{i}\int_0^t U_m(t-t')\lambda v\overline{u}(t')dt',$$

$$\big(\Psi(u,v)\big)(t) = U_{2m}(t)\psi - \mathrm{i}\int_0^t U_{2m}(t-t')\mu u^2(t')dt'.$$

By the Strichartz estimate in Lorentz spaces [16], [17], Proposition 4.1, and the Hölder inequality in Lorentz spaces, we obtain

$$\begin{aligned}
\left\| \Phi(u,v); Y \right\| &\leqslant C \left\| \phi; L^2 \right\| + C \left\| v\overline{u}; L_t^{4/3,2} \big(L^{3/2} \big) \right\| \\
&\leqslant C \left\| \phi; L^2 \right\| + C \left\| |t|^{-1/2} \left\| |J_{2m}|^{1/2} v; L^2 \right\| \left\| u; L^3 \right\|; L_t^{4/3,2} \right\| \\
&\leqslant C \left\| \phi; L^2 \right\| + C \left\| |t|^{-1/2}; L_t^{2,\infty} \right\| \left\| |J_{2m}|^{1/2} v; L_t^\infty \big(L^2 \big) \right\| \left\| u; L_t^{4,2} \big(L^3 \big) \right\|
\end{aligned}$$

$$\leqslant C\left\|\phi;L^2\right\|+C\left\||J_{2m}|^{1/2}v;L_t^{\infty}\left(L^2\right)\right\|\left\|u;L_t^{4,2}\left(L^3\right)\right\|,$$

$$\begin{aligned}\left\|\Psi(u,v);Y\right\| &\leqslant C\left\|\psi;L^2\right\|+C\left\|u^2;L_t^{4/3,2}\left(L^{3/2}\right)\right\|\\ &\leqslant C\left\|\psi;L^2\right\|+C\left\||J_m|^{1/2}u;L_t^{\infty}\left(L^2\right)\right\|\left\|u;L_t^{4,2}\left(L^3\right)\right\|,\end{aligned}$$

$$\begin{aligned}&\left\||J_m|^{1/2}\Phi(u,v);Y\right\|\\ &\quad\leqslant C\left\||x|^{1/2}\phi;L^2\right\|+C\left\||J_m|^{1/2}(v\overline{u});L_t^{4/3,2}\left(L^{3/2}\right)\right\|\\ &\quad\leqslant C\left\|\phi;\mathscr{F}\left(\dot{H}^{1/2}\right)\right\|+C\left\||J_m|^{1/2}u;L_t^{\infty}\left(L^2\right)\right\|\left\||J_{2m}|^{1/2}v;L_t^{4,2}\left(L^3\right)\right\|\\ &\qquad+C\left\||J_m|^{1/2}u;L_t^{4,2}\left(L^3\right)\right\|\left\||J_{2m}|^{1/2}v;L_t^{\infty}\left(L^2\right)\right\|,\end{aligned}$$

$$\begin{aligned}\left\||J_{2m}|^{1/2}\Psi(u,v);Y\right\| &\leqslant C\left\||x|^{1/2}\psi;L^2\right\|+C\left\||J_{2m}|^{1/2}u^2;L_t^{4/3,2}\left(L^{3/2}\right)\right\|\\ &\leqslant C\left\|\psi;\mathscr{F}\left(\dot{H}^{1/2}\right)\right\|\\ &\qquad+C\left\||J_m|^{1/2}u;L_t^{\infty}\left(L^2\right)\right\|\left\||J_m|^{1/2}u;L_t^{4,2}\left(L^3\right)\right\|.\end{aligned}$$

Therefore, we have for $(u,v)\in Y(\rho,\delta)$,

$$\|\Phi(u,v);Y\|\vee\|\Psi(u,v);Y\|\leqslant C\left(\left\|\phi;L^2\right\|\vee\left\|\psi;L^2\right\|\right)+C\rho\delta,$$

$$\left\||J_m|^{1/2}\Phi(u,v);Y\right\|\vee\left\||J_{2m}|^{1/2}\Psi(u,v);Y\right\|\leqslant C\varepsilon+C\delta^2.$$

Similarly, for $(u,v),(u',v')\in Y(\rho,\delta)$,

$$\begin{aligned}&\left\|\Phi(u,v)-\Phi(u',v');Y_m\right\|\vee\left\|\Psi(u,v)-\Psi(u',v');Y_{2m}\right\|\\ &\qquad\leqslant C\delta\left(\|u-u';Y_m\|\vee\|v-v';Y_{2m}\|\right).\end{aligned}$$

We choose $\rho,\delta,$ and ε_0 as

$$C\left(\|\phi;L^2\|\vee\|\psi;L^2\|\right)\leqslant\rho/2,\ C\delta\leqslant 1/2,\ C\varepsilon_0=\delta/2.$$

Then $(u,v)\longmapsto\left(\Phi(u,v),\Psi(u,v)\right)$ is a contraction on $Y(\rho,\delta)$ and the proof proceeds in almost the same way as in the proof of Theorem 2.1.

REFERENCES

[1] A. BACHELOT, *Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon,* Ann. Inst. Henri Poincaré, **48** (1988), 387–422.
[2] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces*, Springer, 1976.
[3] T. CAZENAVE, *Semilinear Schrödinger Equations,* Courant Lecture Notes in Mathematics vol. **10**, American Mathematical Society (2003).

[4] T. CAZENAVE AND F. B. WEISSLER, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Analysis TMA, **14** (1990), 807–836.

[5] M. COLIN AND T. COLIN, *On a quasilinear Zakharov system describing laser-plasma interactions*, Diff. Integral Equations, **17** (2004), 297–330.

[6] V. GEORGIEV, *Global solution of the system of wave and Klein-Gordon equations*, Math. Z., **203** (1990), 683–698.

[7] N. HAYASHI, M. IKEDA AND P. I. NAUMKIN, Wave operator for the system of the Dirac-Klein-Gordon equations, to appear in Math. Method. Appl. Sci.

[8] N. HAYASHI, C. LI AND P. I .NAUMKIN, *On a system of nonlinear Schrödinger equations in 2d*, preprint, Differental Integral Equations, **24** (2011), 417–434.

[9] N. HAYASHI, C. LI AND P. I. NAUMKIN, *Modified wave operator for a system of nonlinear Schrödinger equations in 2d,* preprint.

[10] N. HAYASHI AND T. OZAWA, *Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for some Schrödinger equations,* Ann. Inst. H. Poincarè Phys., **48** (1988), 17–37.

[11] T. KATO, *On nonlinear Schrödinger equations II, H^s-solutions and nonconditional well-posedness*, J. Anal. Math., 67 (1995), 281–306.

[12] M. KEEL AND T. TAO, *Endpoint Strichartz estimates*, Amer. J. Math., **120** (1998), 955–980

[13] S. MACHIHARA, K. NAKANISHI AND T. OZAWA, *Nonrelativistic limit in the energy space for non-linear Klein-Gordon equations*, Math. Ann., **332** (2002), 603–621.

[14] S. MACHIHARA, K. NAKANISHI AND T. OZAWA, *Small global solutions and the nonrelativistic limitfor the nonlinear Dirac equation*, Rev. Math. Iberoamericana, **19** (2003), 179–194.

[15] M. NAKAMURA AND T. OZAWA, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces,* Rev. Math. Phys., **9** (1997), 397–410.

[16] K. NAKANISHI, *Asymptotically-free solutions for the short-range nonlinear Schrödinger equation,* SIAM J. Math. Anal., **32** (2001), 1265–1271.

[17] K. NAKANISHI AND T.OZAWA, *Remarks on scattering for nonlinear Schrödinger equations,* NoDEA Nonlinear Differential Equations Appl., **9** (2002), 45–68.

[18] W. STRAUSS, *Nonlinear Wave Equations*, CBMS Regional Conference Series in Math., **73**, AMS, 1989.

[19] H. TRIEBEL, *Theory of Function Spaces*, Monographs in Math., **78**, Birkhäuser, 1983.

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