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WELL–POSEDNESS OF A DISSIPATIVE SYSTEM MODELING ELECTROHYDRODYNAMICS IN LEBESGUE SPACES

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Abstract. In this paper, we study a dissipative system of partial differential equations modeling the flow of electrohydrodynamics. This system consists of the Navier-Stokes equations with a source term coupled with the Nernst-Planck-Poisson equations for electronic charges. We establish local well-posedness of the initial value problem of this system in the critical and subcritical vector Lebesgue spaces. Moreover, we also prove that if the initial data is sufficiently small in critical Lebesgue spaces, then the solution is a global.

1. Introduction

In this paper, we study the initial value problem for a system of dissipative nonlinear partial differential equations modeling the motion of an isothermal, incompressible and viscous Newtonian fluid of uniform and homogeneous composition of a high number of positively and negatively charged particles. The problem reads as follows (cf. [26]):

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \Delta\phi\nabla\phi \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.2}$$

$$\partial_t v + \mathbf{u}\cdot\nabla v = \nabla\cdot(\nabla v - v\nabla\phi) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.3}$$

$$\partial_t w + \mathbf{u}\cdot\nabla w = \nabla\cdot(\nabla w + w\nabla\phi) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.4}$$

$$\Delta\phi = v - w \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.5}$$

$$(\mathbf{u}, v, w)|_{t=0} = (\mathbf{u}_0, v_0, w_0) \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

Here $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^N$ denotes the velocity field of the fluid, $P = P(x,t) \in \mathbb{R}$ is the pressure inside the fluid, $\phi = \phi(x,t) \in \mathbb{R}$ is the electrostatic potential caused by the charged particles, $v = v(x,t) \in \mathbb{R}$ and $w = w(x,t) \in \mathbb{R}$ respectively represent the charge densities of the negatively and positively charged particles, and $\mathbf{u}_0$, v_0 and w_0 are initial data of $\mathbf{u}$, v and w, respectively. Equations (1.1) and (1.2) are the momentum conservation and the mass conservation equations of the flow, and the right-hand side

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term in (1.1) is the Lorentz force caused by the charges. Equations (1.3) and (1.4) model the balance between diffusion and convective transport of the charges by the flow and the electric fields. The equation (1.5) is the Poisson equation for the electrostatic potential ϕ. We refer the reader to see [5], [8], [9], [28], [30] and the references therein for more details of the physical background of this problem and some different models of similar equations. Note that for simplicity we have assumed that the fluid density, the viscosity coefficient, the charge mobility and the dielectric constant are all equal to unit.

In the case that the flow is charge-free, i.e., $v = w = \phi = 0$, the system (1.1)-(1.6) reduces into the well-known Navier-Stokes equations:

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.7}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.8}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.9}$$

This problem has drawn great attention of researchers for many years, and a huge number of works can be found from the literature, cf., e.g., [4], [6], [10], [11], [17]-[21], [25], [31] and the references therein. If, on the other hand, the velocity field $\mathbf{u}$ is identically vanishing, then (1.1)-(1.6) reduces into the following problem:

$$\partial_t v = \nabla \cdot (\nabla v - v \nabla \phi) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.10}$$

$$\partial_t w = \nabla \cdot (\nabla w + w \nabla \phi) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.11}$$

$$\Delta \phi = v - w \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.12}$$

$$(v, w)|_{t=0} = (v_0, w_0) \quad \text{in} \quad \mathbb{R}^N. \tag{1.13}$$

This problem is the so-called Nernst-Planck-Poisson system which was formulated by W. Nernst and M. Planck at the end of the nineteenth century as a basic model for the diffusion of ions in an electrolytes (cf. [7]). In some literatures it is also called Debye-Hückel system (cf. [3]). It has drawn much attention of analysts during the past twenty years (cf. [1], [2], [3], [12], [15] and [16]). As for the problem (1.1)-(1.6), the results are much less. In [13], Jerome established the first existence result for this system in Kato's semigroup framework. More precisely, he proved that (1.1)-(1.6) has a unique local smooth solution for smooth initial data where he verified the local existence in Kato's semigroup framework. In [29], by using the energy inequalities and the Schauder's fixed point theorem, Schmuck obtained existence of global weak solutions to the system (1.1)-(1.6) in a bounded domain Ω with homogeneous Neumann boundary conditions under the assumption that $\mathbf{u}_0 \in [L^2(\Omega)]^N$ and v_0, $w_0 \in L^\infty(\Omega)$ for $N = 2, 3$. In [14], the existence of a global weak solution was proved to hold for the initial/boundary-value problem of the system (1.1)-(1.6). In [27], Ryham studied existence, uniqueness and regularity of weak solutions of (1.1)-(1.6) in a bounded domain with no-flux boundary conditions for general L^2 initial data in $N = 2$ and for small initial data in $N = 3$. For computational simulations of the problem (1.1)-(1.6), see [22]-[24].

In this paper we study well-posedness of the problem (1.1)-(1.6) in the vector Lebesgue spaces $[L^q(\mathbb{R}^N)]^N \times L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for suitable q and p. By a standard scaling argument we can easily see that $[L^N(\mathbb{R}^N)]^N \times L^{\frac{N}{2}}(\mathbb{R}^N) \times L^{\frac{N}{2}}(\mathbb{R}^N)$ is the critical vector Lebesgue space for the system (1.1)-(1.5). We can thus expect that the problem (1.1)-(1.6) is locally well-posed in this space for arbitrarily large initial data, and globally well-posed for sufficiently small ones. Our first goal of this paper is to prove that this is indeed the case. The subcritical spaces are $[L^q(\mathbb{R}^N)]^N \times L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ with $q > N$ and $\frac{N}{2} < p < N$. The second goal of this paper is to prove that at least for (p,q) satisfying the conditions $N < q < \infty$, $\frac{N}{2} < p < N$ and $\frac{2}{p} < \frac{3}{N} + \frac{1}{q}$, the problem (1.1)-(1.6) is locally well-posed in these subcritical spaces. The restriction $\frac{N}{2} < p < N$ is due to some similar reasons as illustrated in [15] and [16], and the condition $\frac{2}{p} < \frac{3}{N} + \frac{1}{q}$ is caused by the coupling between the components $\mathbf{u}$ and (v,w). To state our main results we first introduce some notations in the following paragraph.

For $1 < p < \infty$, we let $\mathbf{L}^p(\mathbb{R}^N) = [L^p(\mathbb{R}^N)]^N$, and denote by $\mathbf{L}^p_\omega(\mathbb{R}^N)$ the subspace of $\mathbf{L}^p(\mathbb{R}^N)$ consisting of all divergence-free vector fields, i.e.,

$$\mathbf{L}^p_\omega(\mathbb{R}^N) = \{\mathbf{u} \in \mathbf{L}^p(\mathbb{R}^N) : \ \mathrm{div}\, \mathbf{u} = 0\}.$$

For a given Banach space X, we use the notation $\|\cdot\|_X$ to denote the norm of X. However, the norms of the Lebesque space $L^p(\mathbb{R}^N)$ and its N-vector counterpart will be simply written as $\|\cdot\|_{L^p}$. Given $0 < T \leqslant \infty$ and a Banach space X, we denote by $BC([0,T),X)$ the Banach space of all bounded and continuous mappings from $[0,T)$ to X, with norm

$$\|\phi\|_{BC([0,T),X)} = \sup_{t\in[0,T)} \|\phi(t)\|_X \quad \text{for } \phi \in BC([0,T),X).$$

For T, X as before and $\sigma > 0$, we use the notation $BC_\sigma([0,T),X)$ to denote the Banach space of all continuous mappings from $(0,T)$ to X such that $t^\sigma \phi(t) \in BC([0,T),X)$, with norm

$$\|\phi\|_{BC_\sigma([0,T),X)} = \sup_{t\in[0,T)} t^\sigma \|\phi(t)\|_X.$$

The notation $B\dot{C}_\sigma([0,T),X)$ denotes the following subspace of $BC_\sigma([0,T),X)$:

$$B\dot{C}_\sigma([0,T),X) = \big\{\phi : \ \phi \in BC_\sigma([0,T),X), \quad \lim_{t\to 0^+} t^\sigma \|\phi\|_X = 0\big\}.$$

Finally, we denote $\mathbb{P} = I + \nabla(-\Delta)^{-1}\mathrm{div}$, i.e., $\mathbb{P}$ is the $N\times N$ matrix pseudo-differential operator in $\mathbb{R}^N$ with the symbol $(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2})_{i,j=1}^N$, where I represents the unit operator and δ_{ij} is the Kronecker symbol. Note that denoting by R the Riesz transform in $\mathbb{R}^N$, i.e., $R = (R_1, R_2, \cdots, R_N)$ and R_j is the pseudo-differential operator in $\mathbb{R}^N$ with symbol $\frac{\xi_j}{|\xi|}$, $j = 1,2,\cdots,N$, then $\mathbb{P} = I - R\otimes R$, where $\otimes$ denotes the tensor product between N-vectors. Hence by the well-known theory of Calderon and Zygmund on singular integral operators, we see that for any $1 < p < \infty$, $\mathbb{P}$ is a bounded linear mapping from $\mathbf{L}^p(\mathbb{R}^N)$ to itself. Note also that for any $\mathbf{u} \in [S(\mathbb{R}^N)]^N$, div $(\mathbb{P}\mathbf{u}) = 0$ and $\mathbb{P}\mathbf{u} = \mathbf{u}$ if div $\mathbf{u} = 0$. Hence, since $[S(\mathbb{R}^N)]^N$ is dense in $\mathbf{L}^p(\mathbb{R}^N)$, we see that when restricted on $\mathbf{L}^p(\mathbb{R}^N)$, $\mathbb{P}$ is the projection onto the subspace $\mathbf{L}^p_\omega(\mathbb{R}^N)$, so that $\mathbf{L}^p_\omega(\mathbb{R}^N) = \mathbb{P}\mathbf{L}^p(\mathbb{R}^N)$.

The main results of this paper are the following Theorems 1.1-1.4:

THEOREM 1.1. *Let $N \geqslant 2$. Assume that $\mathbf{u}_0 \in \mathbf{L}^N_\omega(\mathbb{R}^N)$ and $v_0,\ w_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Then there exists $T > 0$ and a unique solution $(\mathbf{u}, v, w)$ of* (1.1)-(1.6) *such that*

$$\mathbf{u} \in BC([0,T), \mathbf{L}^N_\omega(\mathbb{R}^N)) \cap B\dot{C}_\sigma([0,T), \mathbf{L}^q_\omega(\mathbb{R}^N)),$$
$$v, w \in BC([0,T), L^{\frac{N}{2}}(\mathbb{R}^N)) \cap B\dot{C}_\eta([0,T), L^p(\mathbb{R}^N)),$$

where $N < q \leqslant 2N$, $\frac{N}{2} < p < N$, $\frac{2}{p} < \frac{3}{N} + \frac{1}{q}$, $\sigma = \frac{1}{2}(1 - \frac{N}{q})$ and $\eta = \frac{1}{2}(2 - \frac{N}{p})$. In addition, there exists $\varepsilon > 0$ such that if

$$\|(\mathbf{u}_0, v_0, w_0)\|_{L^N \times [L^{\frac{N}{2}}]^2} \leqslant \varepsilon,$$

then the above assertion holds for $T = \infty$, i.e., the solution $(\mathbf{u}, v, w)$ of (1.1)-(1.6) *is global.*

THEOREM 1.2. *Let $(\mathbf{u}, v, w)$ be the solution of* (1.1)-(1.6) *given by Theorem* 1.1. *Then there exists $0 < T_1 \leqslant T$ such that for any $N < q \leqslant 2N$, $N < r < \infty$, $\frac{2}{r} = 1 - \frac{N}{q}$, $\frac{N}{2} < p < N$, $\frac{N}{2} < s < \infty$, $\frac{2}{s} = 2 - \frac{N}{p}$ and $\frac{2}{p} < \frac{3}{N} + \frac{1}{q}$, we have*

$$\mathbf{u} \in L^r((0,T_1), \mathbf{L}^q_\omega(\mathbb{R}^N)) \ \text{ and } \ v, w \in L^s((0,T_1), L^p(\mathbb{R}^N)).$$

In addition, there is $\varepsilon > 0$ such that if $\|(\mathbf{u}_0, v_0, w_0)\|_{L^N \times [L^{\frac{N}{2}}]^2} \leqslant \varepsilon$, then we may take $T_1 = \infty$.

THEOREM 1.3. *Let $N \geqslant 2$, $N < q_0 < \infty$ and $\frac{N}{2} < p_0 < N$ satisfying $\frac{2}{p_0} < \frac{3}{N} + \frac{1}{q_0}$. Assume that $\mathbf{u}_0 \in \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)$ and $v_0, w_0 \in L^{p_0}(\mathbb{R}^N)$. Then there exists $T > 0$ and a unique solution $(\mathbf{u}, v, w)$ of* (1.1)-(1.6) *such that*

$$\mathbf{u} \in BC([0,T), \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)) \quad \text{and} \quad v, w \in BC([0,T), L^{p_0}(\mathbb{R}^N)).$$

THEOREM 1.4. *Let $(\mathbf{u}, v, w)$ be the solution of* (1.1)-(1.6) *given by Theorem* 1.3. *Then there exists $0 < T_1 \leqslant T$ such that for any $N < q_0 < q < \infty$ and $\frac{N}{2} < p_0 < p < N$ satisfying $\frac{2}{p_0} < \frac{3}{N} + \frac{1}{q_0}$ and $\frac{2}{p} < \frac{3}{N} + \frac{1}{q}$, we have*

$$\mathbf{u} \in L^r((0,T_1), \mathbf{L}^q_\omega(\mathbb{R}^N)) \ \text{ and } \ v, w \in L^s((0,T_1), L^p(\mathbb{R}^N)),$$

where $q_0 < r < \infty$ and $p_0 < s < \infty$ satisfying $\frac{1}{r} = \frac{N}{2}(\frac{1}{q_0} - \frac{1}{q})$ and $\frac{1}{s} = \frac{N}{2}(\frac{1}{p_0} - \frac{1}{p})$.

REMARK 1.1. Note that in the above theorems we did not consider P and ϕ. The reason is that when $\mathbf{u}$, v and w are determined, then P and ϕ can be easily obtained from (1.2) and (1.5).

REMARK 1.2. Theorem 1.1 implies that for small initial data, the solution is not only global but also decays to zero as $t \to \infty$. In fact, this theorem shows that as $t \to \infty$,

$$\|\mathbf{u}(t)\|_{L^q} \sim Ct^{-\frac{1}{2}(1-\frac{N}{q})} \ \ (N < q \leqslant 2N)$$

and

$$(\|v(t)\|_{L^p}, \|w(t)\|_{L^p}) \sim Ct^{-\frac{1}{2}(2-\frac{N}{p})} \ (\tfrac{N}{2} < p < N).$$

The rest of this paper is organized as follows. In Section 2, we reformulate the system (1.1)-(1.6) into an equivalent mild integral equations and state some preliminary results. Section 3 is devoted to giving the proofs of Theorems 1.1 and 1.2. The proofs are based on the analytic C_0-semigroup theory and the L^p-L^q estimates for the heat semigroup. In Section 4 we give the proofs of Theorems 1.3 and 1.4.

2. Preliminaries

As a standard practice, we can reformulate the problem (1.1)-(1.6) into an equivalent system of integral equations. For this purpose, we first solve the equation (1.5) to get ϕ as a functional of $v-w$:

$$\phi = (-\Delta)^{-1}(w-v) = K*(w-v), \tag{2.1}$$

where $K = K(x)$ is defined for all $x \in \mathbb{R}^N \backslash \{0\}$ by:

$$K(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } N = 2, \\ \frac{1}{4}\pi^{-\frac{N}{2}}\Gamma\left(\frac{N}{2}-1\right)|x|^{-(N-2)} & \text{if } N \geqslant 3. \end{cases}$$

Next, we use the standard argument in the theory of Navier-Stokes equations to eliminate the pressure P, namely, we apply the operator $\mathbb{P}$ to both sides of (1.1). Then (1.1)-(1.2) reduce into the following equation:

$$\partial_t \mathbf{u} - \Delta\mathbf{u} + \mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbb{P}\Delta\phi\nabla\phi. \tag{2.2}$$

Hence, the problem (1.1)-(1.6) can be reduced into the following system of integral equations by the well-known Duhamel formula

$$\mathbf{u} = e^{t\Delta}\mathbf{u}_0 + \int_0^t e^{(t-\tau)\Delta}G_1(\mathbf{u}(\tau), v(\tau), w(\tau))d\tau, \tag{2.3}$$

$$v = e^{t\Delta}v_0 + \int_0^t e^{(t-\tau)\Delta}G_2(\mathbf{u}(\tau), v(\tau), w(\tau))d\tau, \tag{2.4}$$

$$w = e^{t\Delta}w_0 + \int_0^t e^{(t-\tau)\Delta}G_3(\mathbf{u}(\tau), v(\tau), w(\tau))d\tau, \tag{2.5}$$

where

$$\begin{cases} G_1(\mathbf{u}, v, w) = -\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \mathbb{P}(v-w)\nabla((-\Delta)^{-1}(w-v)), \\ G_2(\mathbf{u}, v, w) = -\nabla\cdot(\mathbf{u}v) - \nabla\cdot(v\nabla(-\Delta)^{-1}(w-v)), \\ G_3(\mathbf{u}, v, w) = -\nabla\cdot(\mathbf{u}w) + \nabla\cdot(w\nabla(-\Delta)^{-1}(w-v)). \end{cases}$$

Later on we shall work on this system of integral equations.

Now we collect some basic results concerning the operator $e^{t\Delta}$, which are the main tools for the proofs of Theorems 1.1-1.4.

LEMMA 2.1. ([18]) *For any* $1 \leqslant p < \infty$, $\{e^{t\Delta}\}_{t\geqslant 0}$ *is a contractive and analytic* C_0*-semigroups in both* $\mathbf{L}^p(\mathbb{R}^N)$ *and* $\mathbf{L}^p_\omega(\mathbb{R}^N)$.

LEMMA 2.2. ([15], [17]) *Let* $k \in \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geqslant 0\}$, $\alpha \in \mathbb{Z}_+^N$ *and* $1 \leqslant p \leqslant q \leqslant \infty$. *Then for any* $\varphi \in L^p(\mathbb{R}^N)$ *we have*

$$\|\partial_t^k \partial_x^\alpha (e^{t\Delta}\varphi)\|_{L^q} \leqslant C t^{-\sigma} \|\varphi\|_{L^p}, \quad \forall t > 0, \tag{2.6}$$

where $\sigma = k + \frac{|\alpha|}{2} + \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$. *Moreover, if either* $k + \frac{|\alpha|}{2} \neq 0$ *or* $k + \frac{|\alpha|}{2} = 0$ *and* $1 \leqslant p < q \leqslant \infty$, *then*

$$\lim_{t \to 0^+} t^\sigma \|\partial_t^k \partial_x^\alpha (e^{t\Delta}\varphi)\|_{L^q} = 0. \tag{2.7}$$

LEMMA 2.3. ([17]) *Let* $G(\varphi,\psi) = \int_0^t e^{(t-\tau)\Delta}\varphi\psi d\tau$. *Then for any* $\alpha, \beta, \gamma \geqslant 0$ *such that* $\gamma \leqslant \alpha + \beta < \min\{N, 2+\gamma\}$, *we have the following estimate:*

$$\|G(\varphi,\psi)\|_{L^{\frac{N}{\gamma}}} \leqslant C \int_0^t (t-\tau)^{-\frac{1}{2}(\alpha+\beta-\gamma)} \|\varphi\|_{L^{\frac{N}{\alpha}}} \|\psi\|_{L^{\frac{N}{\beta}}} d\tau. \tag{2.8}$$

Furthermore, if $\gamma \leqslant \alpha + \beta < \min\{N, 1+\gamma\}$, *then we also have the following estimate:*

$$\|\nabla G(\varphi,\psi)\|_{L^{\frac{N}{\gamma}}} \leqslant C \int_0^t (t-\tau)^{-\frac{1}{2}(1+\alpha+\beta-\gamma)} \|\varphi\|_{L^{\frac{N}{\alpha}}} \|\psi\|_{L^{\frac{N}{\beta}}} d\tau. \tag{2.9}$$

LEMMA 2.4. ([11]) *Let* $1 < p < q \leqslant r < \infty$. *Then for any* $\varphi \in L^p(\mathbb{R}^N)$ *we have*

$$\int_0^t \|e^{\tau\Delta}\varphi\|_{L^q}^r d\tau \leqslant C \|\varphi\|_{L^p}^r \ \textit{for}\ t > 0, \tag{2.10}$$

where $\frac{1}{r} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$, *and the constant* C *depends only on* p, q *and* N.

LEMMA 2.5. *Let* p, s *and* σ *be positive constants such that* $\frac{N}{2} < p < N$ *and* $s > \frac{N}{2}$. *Then for any* $0 < T \leqslant \infty$ *we have the following assertion: if*

$$v, w \in \begin{cases} BC([0,T), L^p(\mathbb{R}^N)) & \textit{or} \\ B\dot{C}_\sigma([0,T), L^p(\mathbb{R}^N)) & \textit{or} \\ L^s((0,T), L^p(\mathbb{R}^N)) & \textit{respectively,} \end{cases}$$

then

$$\nabla((-\Delta)^{-1}(w-v)) \in \begin{cases} BC([0,T), \mathbf{L}^{\frac{Np}{N-p}}(\mathbb{R}^N)) & \textit{or} \\ B\dot{C}_\sigma([0,T), \mathbf{L}^{\frac{Np}{N-p}}(\mathbb{R}^N)) & \textit{or} \\ L^s((0,T), \mathbf{L}^{\frac{Np}{N-p}}(\mathbb{R}^N)) & \textit{respectively,} \end{cases}$$

and the following estimates respectively hold:

$$\sup_{t\in[0,T)} \|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np}{N-p}}} \leqslant C \sup_{t\in[0,T)} \|(v,w)\|_{L^p}, \tag{2.11}$$

$$\sup_{t\in[0,T)} t^\sigma \|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np}{N-p}}} \leqslant C \sup_{t\in[0,T)} t^\sigma \|(v,w)\|_{L^p}, \tag{2.12}$$

$$\|\nabla((-\Delta)^{-1}(w-v))\|_{L^s((0,T),L^{\frac{Np}{N-p}})} \leqslant C \|(v,w)\|_{L^s((0,T),L^p)}. \tag{2.13}$$

Proof. By (2.1) we have

$$\nabla((-\Delta)^{-1}(w-v)) = \nabla K * (w-v).$$

Since $|\nabla K(x)| \leqslant C|x|^{-(N-1)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$, by using the Hardy-Littlewood-Sobolev inequality we see that for any $t \in [0,T)$, $v(\cdot,t)$ and $w(\cdot,t)$ belong to $L^p(\mathbb{R}^N)$ implies that

$$\nabla((-\Delta)^{-1}(w(\cdot,t)-v(\cdot,t))) \in \mathbf{L}^{\frac{Np}{N-p}}(\mathbb{R}^N)$$

and

$$\|\nabla((-\Delta)^{-1}(w(\cdot,t)-v(\cdot,t)))\|_{L^{\frac{Np}{N-p}}} \leqslant C\|(v(\cdot,t),w(\cdot,t))\|_{L^p}.$$

This yields the desired assertions easily. □

3. Proofs of Theorems 1.1 and 1.2

In this section we give the proofs of Theorems 1.1-1.2. Thus, throughout this section we assume that

$$\mathbf{u}_0 \in \mathbf{L}^N_\omega(\mathbb{R}^N) \quad \text{and} \quad v_0, w_0 \in L^{\frac{N}{2}}(\mathbb{R}^N).$$

THE PROOF OF THEOREM 1.1 Under the assumptions of Theorem 1.1, we introduce two spaces X_T and Y_T as follows:

$$\begin{aligned}
X_T &= B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \times B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N)) \times B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N)),\\
Y_T &= \big\{(\mathbf{u},v,w): \ \mathbf{u} \in BC([0,T),\mathbf{L}^N_\omega(\mathbb{R}^N)) \cap B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \text{ and}\\
&\qquad v,w \in BC([0,T),L^{\frac{N}{2}}(\mathbb{R}^N)) \cap B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N))\big\},
\end{aligned}$$

where $\sigma = \frac{1}{2}(1-\frac{N}{q})$ and $\eta = \frac{1}{2}(2-\frac{N}{p})$. The norms in X_T and Y_T are respectively defined by

$$\begin{aligned}
\|(\mathbf{u},v,w)\|_{X_T} &= \sup_{t\in[0,T)} t^\sigma\|\mathbf{u}(t)\|_{L^q} + \sup_{t\in[0,T)} t^\eta\|(v(t),w(t))\|_{L^p},\\
\|(\mathbf{u},v,w)\|_{Y_T} &= \sup_{t\in[0,T)} \|\mathbf{u}(t)\|_{L^N} + \sup_{t\in[0,T)} t^\sigma\|\mathbf{u}(t)\|_{L^q}\\
&\quad + \sup_{t\in[0,T)} \|(v(t),w(t))\|_{L^{\frac{N}{2}}} + \sup_{t\in[0,T)} t^\eta\|(v(t),w(t))\|_{L^p}.
\end{aligned}$$

It can be easily checked that $(X_T,\|\cdot\|_{X_T})$ and $(Y_T,\|\cdot\|_{Y_T})$ are both Banach spaces, and $(Y_T,\|\cdot\|_{Y_T})$ is an embedded Banach subspace of $(X_T,\|\cdot\|_{X_T})$. We now introduce a mapping $\mathfrak{F}$ defined in the space X_T as follows: Given $(\mathbf{u},v,w) \in X_T$, we let $\mathfrak{F}(\mathbf{u},v,w) = (\hat{\mathbf{u}},\hat{v},\hat{w})$, where

$$\hat{\mathbf{u}}(t) = e^{t\Delta}\mathbf{u}_0 + \int_0^t e^{(t-\tau)\Delta}G_1(\mathbf{u}(\tau),v(\tau),w(\tau))d\tau, \tag{3.1}$$

$$\hat{v}(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-\tau)\Delta}G_2(\mathbf{u}(\tau),v(\tau),w(\tau))d\tau, \tag{3.2}$$

$$\hat{w}(t) = e^{t\Delta}w_0 + \int_0^t e^{(t-\tau)\Delta}G_3(\mathbf{u}(\tau),v(\tau),w(\tau))d\tau. \tag{3.3}$$

In what follows we prove that $\mathfrak{F}$ is well-defined and maps X_T into Y_T, and if T is sufficiently small then it is a contraction mapping from a closed ball of X_T into itself. Besides, there exists $\varepsilon > 0$ such that if $\|(\mathbf{u}_0,v_0,w_0)\|_{L^N\times[L^{\frac{N}{2}}]^2} \leqslant \varepsilon$, then we can take $T = \infty$.

LEMMA 3.1. *For any $0 < T \leqslant \infty$, $\mathfrak{F}$ is well-defined and maps X_T into Y_T.*

Proof. We first consider $\hat{\mathbf{u}}$. By (3.1) we have $\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 + \hat{\mathbf{u}}_3$, where

$$\begin{cases} \hat{\mathbf{u}}_1(t) = e^{t\Delta}\mathbf{u}_0, \\ \hat{\mathbf{u}}_2(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}[-\nabla\cdot(\mathbf{u}\otimes\mathbf{u})](\tau)d\tau, \\ \hat{\mathbf{u}}_3(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(v-w)\nabla((-\Delta)^{-1}(w-v))](\tau)d\tau. \end{cases} \tag{3.4}$$

For $\hat{\mathbf{u}}_1$, since $\mathbf{u}_0 \in \mathbf{L}^N_\omega(\mathbb{R}^N)$, by Lemmas 2.1 and 2.2 we immediately see that

$$\hat{\mathbf{u}}_1 \in BC([0,T),\mathbf{L}^N_\omega(\mathbb{R}^N)) \cap B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \tag{3.5}$$

and

$$\|\hat{\mathbf{u}}_1(t)\|_{L^N} + t^\sigma\|\hat{\mathbf{u}}_1\|_{L^q} \leqslant C\|\mathbf{u}_0\|_{L^N}. \tag{3.6}$$

For $\hat{\mathbf{u}}_2$, by applying (2.9) with $\alpha = \beta = \frac{N}{q}$ and $\gamma = 1$, we get

$$\begin{aligned} \|\hat{\mathbf{u}}_2(t)\|_{L^N} &= \Big\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[-\nabla\cdot(\mathbf{u}\otimes\mathbf{u})](\tau)d\tau\Big\|_{L^N} \\ &\leqslant \int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q}+\frac{N}{q}-1)}\|\mathbf{u}\|^2_{L^q}d\tau \\ &\leqslant C\Big(\sup_{t\in[0,T)} t^\sigma\|\mathbf{u}\|_{L^q}\Big)^2\int_0^t (t-\tau)^{-\frac{N}{q}}\tau^{-2\sigma}d\tau \leqslant C\Big(\sup_{t\in[0,T)} t^\sigma\|\mathbf{u}\|_{L^q}\Big)^2. \end{aligned}$$

Here we used the assumption $N < q \leqslant 2N$ which ensures that $\gamma \leqslant \alpha+\beta$ and $\frac{N}{q} < 1$. By a standard argument (cf. [17] and [11]), the above estimate implies that $\hat{\mathbf{u}}_2 \in BC([0,T),\mathbf{L}^N_\omega(\mathbb{R}^N))$. Besides, by applying (2.9) with $\alpha = \beta = \gamma = \frac{N}{q}$,

$$\begin{aligned} \|\hat{\mathbf{u}}_2(t)\|_{L^q} &= \Big\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[-\nabla\cdot(\mathbf{u}\otimes\mathbf{u})](\tau)d\tau\Big\|_{L^q} \\ &\leqslant \int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\|\mathbf{u}\|^2_{L^q}d\tau \\ &\leqslant C\Big(\sup_{t\in[0,T)} t^\sigma\|\mathbf{u}\|_{L^q}\Big)^2\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\tau^{-2\sigma}d\tau \end{aligned}$$

$$\leqslant Ct^{-\sigma}\Big(\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)^2.$$

This estimate yields that $\hat{\mathbf{u}}_2 \in B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N))$. Hence,

$$\hat{\mathbf{u}}_2 \in BC([0,T),\mathbf{L}^N_\omega(\mathbb{R}^N)) \cap B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \tag{3.7}$$

and

$$\|\hat{\mathbf{u}}_2(t)\|_{L^N} + t^\sigma\|\hat{\mathbf{u}}_2(t)\|_{L^q} \leqslant C(\sup_{t\in[0,T)} t^\sigma\|\mathbf{u}\|_{L^q})^2. \tag{3.8}$$

For $\hat{\mathbf{u}}_3$, by applying (2.8) with $\alpha=\frac{N}{p}$, $\beta=\frac{N-p}{p}$, $\gamma=1$ and (2.12),

$$\begin{aligned}
\|\hat{\mathbf{u}}_3(t)\|_{L^N} &= \Big\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(v-w)\nabla((-\Delta)^{-1}(w-v))](\tau)d\tau\Big\|_{L^N}\\
&\leqslant C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{2}{p}-\frac{2}{N})}\|(v-w)\|_{L^p}\|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np}{N-p}}}d\tau\\
&\leqslant C\Big(\sup_{t\in[0,T)} t^\eta\|(v,w)\|_{L^p}\Big)^2\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{2}{p}-\frac{2}{N})}\tau^{-2\eta}d\tau\\
&\leqslant C\Big(\sup_{t\in[0,T)} t^\eta\|(v,w)\|_{L^p}\Big)^2.
\end{aligned}$$

This estimate yields that $\hat{\mathbf{u}}_3 \in BC([0,T),\mathbf{L}^N_\omega(\mathbb{R}^N))$. Furthermore, by applying (2.8) with $\alpha=\frac{N}{p}$, $\beta=\frac{N-p}{p}$, $\gamma=\frac{N}{q}$ and (2.12),

$$\begin{aligned}
\|\hat{\mathbf{u}}_3(t)\|_{L^q} &= \Big\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(v-w)\nabla((-\Delta)^{-1}(w-v))](\tau)d\tau\Big\|_{L^q}\\
&\leqslant C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{N}-\frac{1}{q})}\|(v-w)\|_{L^p}\|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np}{N-p}}}d\tau\\
&\leqslant C\Big(\sup_{t\in[0,T)} t^\eta\|(v,w)\|_{L^p}\Big)^2\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{N}-\frac{1}{q})}\tau^{-2\eta}d\tau\\
&\leqslant Ct^{-\sigma}\Big(\sup_{t\in[0,T)} t^\eta\|(v,w)\|_{L^p}\Big)^2.
\end{aligned}$$

Here we used the assumptions $\frac{N}{2}<p<N$ and $\frac{2}{p}<\frac{3}{N}+\frac{1}{q}$ to ensure that the above integral

$$\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{N}-\frac{1}{q})}\tau^{-2\eta}d\tau$$

is convergent and independent of t, thus the above estimate implies that

$$\hat{\mathbf{u}}_3 \in B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)).$$

Hence, we have proved that

$$\hat{\mathbf{u}}_3 \in BC([0,T),\mathbf{L}^N_\omega(\mathbb{R}^N)) \cap B\dot{C}_\sigma([0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \tag{3.9}$$

and

$$\|\hat{\mathbf{u}}_3(t)\|_{L^N}+t^{\sigma}\|\hat{\mathbf{u}}_3(t)\|_{L^q}\leqslant C\Big(\sup_{t\in[0,T)}t^{\eta}\|(v,w)\|_{L^p}\Big)^2. \tag{3.10}$$

Combining (3.5)-(3.10), we obtain

$$\hat{\mathbf{u}}\in BC([0,T),\mathbf{L}^N_{\omega}(\mathbb{R}^N))\cap B\dot{C}_{\sigma}([0,T),\mathbf{L}^q_{\omega}(\mathbb{R}^N)) \tag{3.11}$$

and

$$\begin{aligned}&\sup_{t\in[0,T)}\|\hat{\mathbf{u}}(t)\|_{L^N}+\sup_{t\in[0,T)}t^{\sigma}\|\hat{\mathbf{u}}(t)\|_{L^q}\\&\qquad\leqslant C\Big[\|\mathbf{u}_0\|_{L^N}+\Big(\sup_{t\in[0,T)}t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)^2+\Big(\sup_{t\in[0,T)}t^{\eta}\|(v,w)\|_{L^p}\Big)^2\Big].\end{aligned} \tag{3.12}$$

Next we consider $\hat{v}$. By (3.2) we have $\hat{v}=\hat{v}_1+\hat{v}_2+\hat{v}_3$, where

$$\begin{cases}\hat{v}_1(t)=e^{t\Delta}v_0,\\ \hat{v}_2(t)=\displaystyle\int_0^t e^{(t-\tau)\Delta}[-\nabla\cdot(\mathbf{u}v)](\tau)d\tau,\\ \hat{v}_3(t)=\displaystyle\int_0^t e^{(t-\tau)\Delta}[-\nabla\cdot(v\nabla((-\Delta)^{-1}(w-v)))](\tau)d\tau.\end{cases} \tag{3.13}$$

Since $v_0\in L^{\frac{N}{2}}(\mathbb{R}^N)$, similarly as for $\hat{\mathbf{u}}_1$ it can be easily seen that

$$\hat{v}_1\in BC([0,T),L^{\frac{N}{2}}(\mathbb{R}^N))\cap B\dot{C}_{\eta}([0,T),L^p(\mathbb{R}^N)) \tag{3.14}$$

and

$$\|\hat{v}_1(t)\|_{L^{\frac{N}{2}}}+t^{\eta}\|\hat{v}_1(t)\|_{L^p}\leqslant C\|v_0\|_{L^{\frac{N}{2}}}. \tag{3.15}$$

For $\hat{v}_2$, by using (2.9) with $\alpha=\frac{N}{q}$, $\beta=\frac{N}{p}$ and $\gamma=2$, we get

$$\begin{aligned}\|\hat{v}_2(t)\|_{L^{\frac{N}{2}}}&=\Big\|\int_0^t e^{(t-\tau)\Delta}[-\nabla\cdot(\mathbf{u}v)](\tau)d\tau\Big\|_{L^{\frac{N}{2}}}\\&\leqslant C\int_0^t(t-\tau)^{-\frac12-\frac{N}{2}(\frac1p+\frac1q-\frac2N)}\|\mathbf{u}\|_{L^q}\|v\|_{L^p}d\tau\\&\leqslant C\Big(\sup_{t\in[0,T)}t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)}t^{\eta}\|v\|_{L^p}\Big)\int_0^t(t-\tau)^{\frac12-\frac{N}{2}(\frac1p+\frac1q)}\tau^{-\sigma-\eta}d\tau\\&\leqslant C\Big(\sup_{t\in[0,T)}t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)}t^{\eta}\|v\|_{L^p}\Big).\end{aligned}$$

Thus, we obtain $\hat{v}_2\in BC([0,T),L^{\frac{N}{2}}(\mathbb{R}^N))$, and

$$\|\hat{v}_2(t)\|_{L^{\frac{N}{2}}}\leqslant C\Big(\sup_{t\in[0,T)}t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)}t^{\eta}\|v\|_{L^p}\Big).$$

Besides, by applying (2.9) with $\alpha = \frac{N}{q}$ and $\beta = \gamma = \frac{N}{p}$,

$$\begin{aligned}
\|\hat{v}_2(t)\|_{L^p} &= \Big\| \int_0^t e^{(t-\tau)\Delta}[-\nabla\cdot(\mathbf{u}v)](\tau)d\tau \Big\|_{L^p} \\
&\leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\|\mathbf{u}\|_{L^q}\|v\|_{L^p}d\tau \\
&\leqslant C\Big(\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)} t^{\eta}\|v\|_{L^p}\Big)\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\tau^{-\sigma-\eta}d\tau \\
&\leqslant Ct^{-\eta}\Big(\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)} t^{\eta}\|v\|_{L^p}\Big).
\end{aligned}$$

Thus $\hat{v}_2 \in B\dot{C}_\eta([0,T), L^p(\mathbb{R}^N))$, and

$$t^{\eta}\|\hat{v}_2(t)\|_{L^p} \leqslant C\Big(\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)} t^{\eta}\|v\|_{L^p}\Big).$$

Hence, we see that

$$\hat{v}_2 \in BC([0,T), L^{\frac{N}{2}}(\mathbb{R}^N)) \cap B\dot{C}_\eta([0,T), L^p(\mathbb{R}^N)) \tag{3.16}$$

and

$$\|\hat{v}_2(t)\|_{L^{\frac{N}{2}}} + t^{\eta}\|\hat{v}_2(t)\|_{L^p} \leqslant C\Big(\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\Big)\Big(\sup_{t\in[0,T)} t^{\eta}\|v\|_{L^p}\Big). \tag{3.17}$$

For $\hat{v}_3$, by applying (2.9) with $\alpha = \frac{N}{p}$, $\beta = \frac{N-p}{p}$, $\gamma = 2$ and (2.12), we obtain

$$\begin{aligned}
\|\hat{v}_3(t)\|_{L^{\frac{N}{2}}} &= \Big\| \int_0^t [e^{(t-\tau)\Delta}\nabla\cdot(v\nabla((-\Delta)^{-1}(w-v)))(\tau)]d\tau \Big\|_{L^{\frac{N}{2}}} \\
&\leqslant C\int_0^t (t-\tau)^{-(\frac{N}{p}-1)}\|v\|_{L^p}\|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np}{N-p}}}d\tau \\
&\leqslant C\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2\int_0^t (t-\tau)^{-(\frac{N}{p}-1)}\tau^{-2\eta}d\tau \\
&\leqslant C\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2.
\end{aligned}$$

Thus $\hat{v}_3 \in BC([0,T), L^{\frac{N}{2}}(\mathbb{R}^N))$, and

$$\|\hat{v}_3(t)\|_{L^{\frac{N}{2}}} \leqslant C\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2.$$

Besides, by applying (2.9) with $\alpha = \gamma = \frac{N}{p}$, $\beta = \frac{N-p}{p}$ and (2.12),

$$\|\hat{v}_3(t)\|_{L^p} = \Big\| \int_0^t [e^{(t-\tau)\Delta}\nabla\cdot(v\nabla((-\Delta)^{-1}(w-v)))(\tau)]d\tau \Big\|_{L^p}$$

$$\leqslant C\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2\int_0^t (t-\tau)^{-\frac{N}{2p}}\tau^{-2\eta}d\tau$$
$$\leqslant Ct^{-\eta}\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2.$$

Thus, $\hat{v}_3\in B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N))$, and

$$t^{\eta}\|\hat{v}_3(t)\|_{L^p}\leqslant C\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2.$$

Hence, we obtain that

$$\hat{v}_3\in BC([0,T),L^{\frac{N}{2}}(\mathbb{R}^N))\cap B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N)) \tag{3.18}$$

and

$$\|\hat{v}_3(t)\|_{L^{\frac{N}{2}}}+t^{\eta}\|\hat{v}_3(t)\|_{L^p}\leqslant C\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2. \tag{3.19}$$

Combining (3.14)-(3.19), we have proved that

$$\hat{v}\in BC([0,T),L^{\frac{N}{2}}(\mathbb{R}^N))\cap B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N)), \tag{3.20}$$

and

$$\begin{aligned}\sup_{t\in[0,T)}\|\hat{v}\|_{L^{\frac{N}{2}}}+\sup_{t\in[0,T)} t^{\eta}\|\hat{v}\|_{L^p}&\\ \leqslant C\Big[\|v_0\|_{L^{\frac{N}{2}}}&+\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\sup_{t\in[0,T)} t^{\eta}\|v\|_{L^p}\\ &+\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2\Big].\end{aligned} \tag{3.21}$$

Similarly we can prove that

$$\hat{w}\in BC([0,T),L^{\frac{N}{2}}(\mathbb{R}^N))\cap B\dot{C}_\eta([0,T),L^p(\mathbb{R}^N)) \tag{3.22}$$

and

$$\begin{aligned}\sup_{t\in[0,T)}\|\hat{w}\|_{L^{\frac{N}{2}}}+\sup_{t\in[0,T)} t^{\eta}\|\hat{w}\|_{L^p}&\\ \leqslant C\Big[\|w_0\|_{L^{\frac{N}{2}}}&+\sup_{t\in[0,T)} t^{\sigma}\|\mathbf{u}\|_{L^q}\sup_{t\in[0,T)} t^{\eta}\|w\|_{L^p}\\ &+\Big(\sup_{t\in[0,T)} t^{\eta}\|(v,w)\|_{L^p}\Big)^2\Big].\end{aligned} \tag{3.23}$$

Putting the estimates (3.11), (3.12), (3.20), (3.21), (3.22) and (3.23) together, we complete the proof of Lemma 3.1. □

By Lemma 3.1, there exists a constant $C_0 > 0$ independent of T such that for $(\mathbf{u}, v, w) \in X_T$ and $(\hat{\mathbf{u}}, \hat{v}, \hat{w}) = \mathfrak{F}(\mathbf{u}, v, w)$, we have the following estimate

$$\|(\hat{\mathbf{u}}, \hat{v}, \hat{w})\|_{X_T} \leqslant \|e^{t\Delta}(\mathbf{u}_0, v_0, w_0)\|_{X_T} + C_0\|(\mathbf{u}, v, w)\|_{X_T}^2. \tag{3.24}$$

Since $\mathbf{u}_0 \in \mathbf{L}_\omega^N(\mathbb{R}^N)$ and $v_0, w_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$, by Lemma 2.2 we get

$$\lim_{T\to 0^+} \|e^{t\Delta}(\mathbf{u}_0, v_0, w_0)\|_{X_T} = \lim_{T\to 0^+}\Big(\sup_{t\in[0,T)} t^\sigma \|e^{t\Delta}\mathbf{u}_0\|_{L^q} + \sup_{t\in[0,T)} t^\eta \|e^{t\Delta}(v_0, w_0)\|_{L^p}\Big) = 0.$$

Thus, for any $0 < \delta < (4C_0)^{-1}$ there exists corresponding $T > 0$ (depending on the initial data $(\mathbf{u}_0, v_0, w_0)$) such that

$$\|e^{t\Delta}(\mathbf{u}_0, v_0, w_0)\|_{X_T} \leqslant \delta. \tag{3.25}$$

For such a T, (3.24) implies that

$$\|(\hat{\mathbf{u}}, \hat{v}, \hat{w})\|_{X_T} \leqslant \delta + C_0\|(\mathbf{u}, v, w)\|_{X_T}^2. \tag{3.26}$$

Let $\mathfrak{X}_T$ be a closed ball in X_T with radius 2δ, i.e.,

$$\mathfrak{X}_T = \big\{(\mathbf{u}, v, w) \in X_T : \ \|(\mathbf{u}, v, w)\|_{X_T} \leqslant 2\delta\big\}.$$

For any $(\mathbf{u}, v, w) \in \mathfrak{X}_T$, from (3.26) we have

$$\|(\hat{\mathbf{u}}, \hat{v}, \hat{w})\|_{X_T} \leqslant \delta + C_0(2\delta)^2 = (1 + 4C_0\delta)\delta \leqslant 2\delta.$$

Hence, $\mathfrak{F}$ maps $\mathfrak{X}_T$ into itself.

LEMMA 3.2. *Let δ, T and $\mathfrak{X}_T$ be as above. Then when restricted in $\mathfrak{X}_T$, $\mathfrak{F}$ is a contraction mapping. Moreover, there exists $\varepsilon > 0$ such that if $\|(\mathbf{u}_0, v_0, w_0)\|_{L^N\times[L^{\frac{N}{2}}]^2} \leqslant \varepsilon$, then we may take $T = \infty$.*

Proof. Let $(\mathbf{u}_1, v_1, w_1), (\mathbf{u}_2, v_2, w_2) \in \mathfrak{X}_T$, and let $(\hat{\mathbf{u}}_j, \hat{v}_j, \hat{w}_j) = \mathfrak{F}(\mathbf{u}_j, v_j, w_j)$, $j = 1, 2$. Then by a similar argument as in the proof of Lemma 3.1 we have the following estimate:

$$\begin{aligned}
&\|(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{v}_1 - \hat{v}_2, \hat{w}_1 - \hat{w}_2)\|_{X_T} \\
&\qquad \leqslant C_0\big(\|(\mathbf{u_1}, v_1, w_1)\|_{X_T} + \|(\mathbf{u_2}, v_2, w_2)\|_{X_T}\big)\|(\mathbf{u_1} - \mathbf{u_2}, v_1 - v_2, w_1 - w_2)\|_{X_T} \\
&\qquad \leqslant 4C_0\delta\|(\mathbf{u_1} - \mathbf{u_2}, v_1 - v_2, w_1 - w_2)\|_{X_T}.
\end{aligned}$$

Since $4C_0\delta < 1$, we see that $\mathfrak{F}$ is a contraction mapping.

Next, from the proof of Lemma 3.1 we see that there exists constant $C_0' > 0$ such that

$$\|e^{t\Delta}(\mathbf{u}_0, v_0, w_0)\|_{X_T} \leqslant C_0'\Big(\|\mathbf{u}_0\|_{L^N} + \|(v_0, w_0)\|_{L^{\frac{N}{2}}}\Big).$$

Let δ be as before (i.e. $0<\delta<(4C_0)^{-1}$) and choose $\varepsilon>0$ sufficiently small such that $2C_0'\varepsilon\leqslant\delta$. Then if

$$\|(\mathbf{u}_0,v_0,w_0)\|_{L^N\times[L^{\frac{N}{2}}]^2}\leqslant\varepsilon,$$

we see that (3.25) holds for all $t\geqslant 0$. Hence, the above argument shows that in this case we can take $T=\infty$. This completes the proof of Lemma 3.2. □

Theorem 1.1 follows from Lemmas 3.1-3.2 and the Banach fixed point theorem.

THE PROOF OF THEOREM 1.2 Under the assumptions of Theorem 1.2, we denote by

$$\begin{aligned}&\tilde{X}_T=L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N))\times L^s((0,T),L^p(\mathbb{R}^N))\times L^s((0,T),L^p(\mathbb{R}^N)),\\&\tilde{Y}_T=Y_T\cap\tilde{X}_T,\end{aligned}$$

where Y_T is as before. It is clear that $\tilde{X}_T$ and $\tilde{Y}_T$ are both Banach spaces.

LEMMA 3.3. *For any* $0<T\leqslant\infty$, $\mathfrak{F}$ *maps* $\tilde{Y}_T$ *into itself.*

Proof. Due to Lemma 3.1, we only need to prove that

$$\hat{\mathbf{u}}\in L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N))\quad\text{and}\quad\hat{v},\hat{w}\in L^s((0,T),L^p(\mathbb{R}^N)).$$

Recall that $\hat{\mathbf{u}}=\hat{\mathbf{u}}_1+\hat{\mathbf{u}}_2+\hat{\mathbf{u}}_3$, where $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$ are defined by (3.4). For $\hat{\mathbf{u}}_1$, since $\mathbf{u}_0\in\mathbf{L}^N_\omega(\mathbb{R}^N)$, by Lemma 2.4,

$$\|\hat{\mathbf{u}}_1\|_{L^r((0,T),L^q)}\leqslant C\|\mathbf{u}_0\|_{L^N}.\tag{3.27}$$

For $\hat{\mathbf{u}}_2$, by using (2.9) with $\alpha=\beta=\gamma=\frac{N}{q}$,

$$\|\hat{\mathbf{u}}_2(t)\|_{L^q}\leqslant C\int_0^t(t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\|\mathbf{u}(\tau)\|^2_{L^q}d\tau.$$

Applying the Hardy-Littlewood-Sobolev inequality to the above estimate yields that

$$\|\hat{\mathbf{u}}_2\|_{L^r((0,T),L^q)}\leqslant C\|\mathbf{u}\|^2_{L^r((0,T),L^q)}\leqslant C\|(\mathbf{u},v,w)\|^2_{\tilde{X}_T}.\tag{3.28}$$

For $\hat{\mathbf{u}}_3$, by using (2.8) with $\alpha=\frac{N}{p}$, $\beta=\frac{N-p}{p}$ and $\gamma=\frac{N}{q}$,

$$\|\hat{\mathbf{u}}_3(t)\|_{L^q}\leqslant C\int_0^t(t-\tau)^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{N}-\frac{1}{q})}\|(v(\tau),w(\tau))\|_{L^p}\,\|\nabla((-\Delta)^{-1}(w-v)(\tau))\|_{L^{\frac{Np}{N-p}}}d\tau.$$

Applying the Hardy-Littlewood-Sobolev inequality, the Hölder inequality and (2.13) yield that

$$\|\hat{\mathbf{u}}_3\|_{L^r((0,T),L^q)}\leqslant C\|(v,w)\|^2_{L^s((0,T),L^p)}\leqslant C\|(\mathbf{u},v,w)\|^2_{\tilde{X}_T}.\tag{3.29}$$

From (3.27)-(3.29) we see that

$$\|\hat{\mathbf{u}}\|_{L^r((0,T),L^q)} \leqslant C\big(\|\mathbf{u}_0\|_{L^N} + \|(\mathbf{u},v,w)\|_{\tilde{X}_T}^2\big). \tag{3.30}$$

Next we consider $\hat{v}$. Recall that $\hat{v} = \hat{v}_1 + \hat{v}_2 + \hat{v}_3$, where $\hat{v}_1$, $\hat{v}_2$ and $\hat{v}_3$ are defined by (3.13). For $\hat{v}_1$, by Lemma 2.4, it is easy to see that

$$\|\hat{v}_1\|_{L^s((0,T),L^p)} \leqslant C\|v_0\|_{L^{\frac{N}{2}}} \tag{3.31}$$

due to $v_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$. For $\hat{v}_2$, by using (2.9) with $\alpha = \frac{N}{q}$ and $\beta = \gamma = \frac{N}{p}$,

$$\|\hat{v}_2(t)\|_{L^p} \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\|\mathbf{u}\|_{L^q}\|v\|_{L^p}d\tau.$$

Applying the Hardy-Littlewood inequality and the Hölder inequality, we get

$$\|\hat{v}_2\|_{L^r((0,T),L^{\frac{Nq}{N+q}})} \leqslant C\|\mathbf{u}\|_{L^r((0,T),L^q)}\|v\|_{L^r((0,T),L^{\frac{Nq}{N+q}})} \leqslant C\|(\mathbf{u},v,w)\|_{\tilde{X}_T}^2. \tag{3.32}$$

For $\hat{v}_3$, by using (2.9) with $\alpha = \gamma = \frac{N}{p}$ and $\beta = \frac{N-p}{p}$,

$$\|\hat{v}_3(t)\|_{L^p} \leqslant C\int_0^t (t-\tau)^{-\frac{N}{2p}}\|v\|_{L^p}\|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np}{N-p}}}d\tau.$$

Again, by using the Hardy-Littlewood-Sobolev inequality, the Hölder inequality and (2.13), we obtain

$$\|\hat{v}_3\|_{L^s((0,T),L^p)} \leqslant C\|(v,w)\|_{L^s((0,T),L^p)}^2 \leqslant C\|(\mathbf{u},v,w)\|_{\tilde{X}_T}^2. \tag{3.33}$$

Combining (3.31)-(3.33), we see that

$$\|\hat{v}\|_{L^r((0,T),L^q)} \leqslant C\big(\|v_0\|_{L^{\frac{N}{2}}} + \|(\mathbf{u},v,w)\|_{\tilde{X}_T}^2\big). \tag{3.34}$$

Similarly for $\hat{w}$, we have

$$\|\hat{w}\|_{L^r((0,T),L^q)} \leqslant C\big(\|w_0\|_{L^{\frac{N}{2}}} + \|(\mathbf{u},v,w)\|_{\tilde{X}_T}^2\big). \tag{3.35}$$

From (3.30), (3.34) and (3.35), we see that the desired assertion follows. □

LEMMA 3.4. *For any $\mathbf{u}_0 \in \mathbf{L}^N_\omega(\mathbb{R}^N)$ and $v_0, w_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ there exists corresponding $T > 0$, such that $\mathfrak{F}$ maps a closed ball in $\tilde{Y}_T$ into itself and is a contraction mapping when restricted to this ball. Moreover, there exists $\varepsilon > 0$ such that if $\|(\mathbf{u}_0,v_0,w_0)\|_{L^N\times[L^{\frac{N}{2}}]^2} \leqslant \varepsilon$, then we may take $T = \infty$.*

Proof. From the proofs of Lemmas 3.1 and 3.3 we see that for any $\mathbf{u}_0 \in \mathbf{L}^N_\omega(\mathbb{R}^N)$ and $v_0, w_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ the following inequality holds:

$$\|(\hat{\mathbf{u}},\hat{v},\hat{w})\|_{\tilde{Y}_T} \leqslant \|e^{t\Delta}(\mathbf{u}_0,v_0,w_0)\|_{X_T\cap\tilde{X}_T} + C\|(\mathbf{u},v,w)\|_{\tilde{Y}_T}^2.$$

Note that for any $\mathbf{u}_0 \in \mathbf{L}^N_\omega(\mathbb{R}^N)$ and $v_0, w_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ we have

$$\lim_{T\to 0} \|e^{t\Delta}(\mathbf{u}_0, v_0, w_0)\|_{X_T\cap\tilde{X}_T} = 0$$

and

$$\|e^{t\Delta}(\mathbf{u}_0, v_0, w_0)\|_{X_T\cap\tilde{X}_T} \leqslant C\|(\mathbf{u}_0, v_0, w_0)\|_{L^N\times[L^{\frac{N}{2}}]^2}.$$

Hence, by using a similar argument as in the proof of Lemma 3.2 we obtain the desired assertion. □

By Lemmas 3.3 and 3.4, Theorem 1.2 follows the Banach fixed point theorem.

4. Proofs of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3-1.4. Throughout this section we let q_0 and p_0 be two fixed numbers satisfying the conditions $N < q_0 < \infty$, $\frac{N}{2} < p_0 < N$ and $\frac{2}{p_0} < \frac{3}{N} + \frac{1}{q_0}$, and assume that

$$\mathbf{u}_0 \in \mathbf{L}^{q_0}_\omega(\mathbb{R}^N) \ \text{ and } \ v_0, w_0 \in L^{p_0}(\mathbb{R}^N).$$

PROOF OF THEOREM 1.3. For a constant $T > 0$ to be specified later, without loss of generality, we assume that $T < 1$. We define Z_T by

$$Z_T = BC([0,T), \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)) \times BC([0,T), L^{p_0}(\mathbb{R}^N)) \times BC([0,T), L^{p_0}(\mathbb{R}^N)).$$

The norm in Z_T is defined by

$$\|(\mathbf{u}, v, w)\|_{Z_T} = \sup_{t\in[0,T)} \|\mathbf{u}\|_{L^{q_0}} + \sup_{t\in[0,T)} \|(v,w)\|_{L^{p_0}}.$$

It is obvious that $(Z_T, \|\cdot\|_{Z_T})$ is a Banach space.

LEMMA 4.1. *For any $0 < T < \infty$, $\mathfrak{F}$ is well-defined and maps Z_T into itself.*

Proof. We first consider $\hat{\mathbf{u}}$. Recall that $\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 + \hat{\mathbf{u}}_3$, where $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$ are defined by (3.4). For $\hat{\mathbf{u}}_1$, since $\mathbf{u}_0 \in \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)$, from Lemma 2.1 we see that

$$\hat{\mathbf{u}}_1 \in BC([0,T), \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)) \ \text{ and } \ \sup_{t\in[0,T)} \|\hat{\mathbf{u}}_1\|_{L^{q_0}} \leqslant C\|\mathbf{u}_0\|_{L^{q_0}}. \tag{4.1}$$

For $\hat{\mathbf{u}}_2$, by applying (2.9) with $\alpha = \beta = \gamma = \frac{N}{q_0}$,

$$\|\hat{\mathbf{u}}_2\|_{L^{q_0}} \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q_0})} \|\mathbf{u}\|^2_{L^{q_0}} d\tau \leqslant Ct^{\frac{1}{2}(1-\frac{N}{q_0})} \Big(\sup_{t\in[0,T)} \|\mathbf{u}\|_{L^{q_0}}\Big)^2.$$

Hence

$$\hat{\mathbf{u}}_2 \in BC([0,T), \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)) \ \text{ and } \ \sup_{t\in[0,T)} \|\hat{\mathbf{u}}_2\|_{L^{q_0}} \leqslant CT^{\frac{1}{2}(1-\frac{N}{q_0})} \|(\mathbf{u}, v, w)\|^2_{Z_T}. \tag{4.2}$$

For $\hat{\mathbf{u}}_3$, by applying (2.8) with $\alpha = \frac{N}{p_0}$, $\beta = \frac{N-p_0}{p_0}$, $\gamma = \frac{N}{q_0}$ and (2.11) with $p = p_0$,

$$\begin{aligned}\|\hat{\mathbf{u}}_3\|_{L^{q_0}} &\leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{2N}{p_0}-\frac{N}{q_0}-1)}\|(v-w)\|_{L^{p_0}}\|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np_0}{N-p_0}}}d\tau \\ &\leqslant C\Big(\sup_{t\in[0,T)}\|(v,w)\|_{L^{p_0}}\Big)^2\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{2N}{p_0}-\frac{N}{q_0}-1)}d\tau \\ &\leqslant Ct^{\frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}}\Big(\sup_{t\in[0,T)}\|(v,w)\|_{L^{p_0}}\Big)^2.\end{aligned}$$

Here we used the assumption $\frac{2}{p_0} < \frac{3}{N} + \frac{1}{q_0}$ which ensures that $\frac{1}{2}(\frac{2N}{p_0} - \frac{N}{q_0} - 1) < 1$. Hence

$$\hat{\mathbf{u}}_3 \in BC([0,T),\mathbf{L}^{q_0}_{\omega}(\mathbb{R}^N)) \text{ and } \sup_{t\in[0,T)}\|\hat{\mathbf{u}}_2\|_{L^{q_0}} \leqslant CT^{\frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}}\|(\mathbf{u},v,w)\|^2_{Z_T}. \tag{4.3}$$

Let

$$\theta_1 = \min\Big\{\frac{1}{2}\Big(1-\frac{N}{q_0}\Big), \frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}\Big\}.$$

From (4.1)-(4.3) we get

$$\hat{\mathbf{u}} \in BC\big([0,T),\mathbf{L}^{q_0}_{\omega}(\mathbb{R}^N)\big)$$

and

$$\sup_{t\in[0,T)}\|\hat{\mathbf{u}}\|_{L^{q_0}} \leqslant C\big(\|\mathbf{u}_0\|_{L^{q_0}} + T^{\theta_1}\|(\mathbf{u},v,w)\|^2_{Z_T}\big). \tag{4.4}$$

Next we consider $\hat{v}$. Recall that $\hat{v} = \hat{v}_1 + \hat{v}_2 + \hat{v}_3$, where $\hat{v}_1$, $\hat{v}_2$ and $\hat{v}_3$ are defined by (3.13). For $\hat{v}_1$, since $v_0 \in L^{p_0}(\mathbb{R}^N)$, Lemma 2.1 implies immediately that

$$\hat{v}_1 \in BC([0,T),L^{p_0}(\mathbb{R}^N)) \text{ and } \sup_{t\in[0,T)}\|\hat{v}_1\|_{L^{p_0}} \leqslant C\|v_0\|_{L^{p_0}}. \tag{4.5}$$

For $\hat{v}_2$, by applying (2.9) with $\alpha = \frac{N}{q_0}$ and $\beta = \gamma = \frac{N}{p_0}$,

$$\begin{aligned}\|\hat{v}_2\|_{L^{p_0}} &\leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q_0})}\|\mathbf{u}\|_{L^{q_0}}\|v\|_{L^{p_0}}d\tau \\ &\leqslant Ct^{\frac{1}{2}(1-\frac{N}{q_0})}\Big(\sup_{t\in[0,T)}\|\mathbf{u}\|_{L^{q_0}}\Big)\Big(\sup_{t\in[0,T)}\|v\|_{L^{p_0}}\Big).\end{aligned}$$

Hence

$$\hat{v}_2 \in BC([0,T),L^{p_0}(\mathbb{R}^N)) \text{ and } \sup_{t\in[0,T)}\|\hat{v}_2\|_{L^{p_0}} \leqslant CT^{\frac{1}{2}(1-\frac{N}{q_0})}\|(\mathbf{u},v,w)\|^2_{Z_T}. \tag{4.6}$$

For $\hat{v}_3$, by applying (2.9) with $\alpha = \gamma = \frac{N}{p_0}$, $\beta = \frac{N-p_0}{p_0}$ and (2.11) with $p = p_0$,

$$\|\hat{v}_3\|_{L^{p_0}} \leqslant C\int_0^t (t-\tau)^{-\frac{N}{2p_0}}\|v\|_{L^{p_0}}\|\nabla((-\Delta)^{-1}(w-v))\|_{L^{\frac{Np_0}{N-p_0}}}d\tau$$

$$\leqslant Ct^{1-\frac{N}{2p_0}}\Big(\sup_{t\in[0,T)}\|(v,w)\|_{L^{p_0}}\Big)^2.$$

Hence

$$\hat{v}_3\in BC([0,T),L^{p_0}(\mathbb{R}^N)) \text{ and } \sup_{t\in[0,T)}\|\hat{v}_3\|_{L^{p_0}}\leqslant CT^{1-\frac{N}{2p_0}}\|(\mathbf{u},v,w)\|_{Z_T}^2. \tag{4.7}$$

Let $\theta_2=\min\{\frac{1}{2}(1-\frac{N}{q_0}),1-\frac{N}{2p_0}\}$. Combining (4.5)-(4.7), we get

$$\hat{v}\in BC([0,T),L^{p_0}(\mathbb{R}^N))$$

and

$$\sup_{t\in[0,T)}\|\hat{v}\|_{L^{p_0}}\leqslant C\big(\|v_0\|_{L^{p_0}}+T^{\theta_2}\|(\mathbf{u},v,w)\|_{Z_T}^2\big). \tag{4.8}$$

Similarly we can prove that $\hat{w}\in BC([0,T),L^{p_0}(\mathbb{R}^N))$ and

$$\sup_{t\in[0,T)}\|\hat{w}\|_{L^{p_0}}\leqslant C\big(\|w_0\|_{L^{p_0}}+T^{\theta_2}\|(\mathbf{u},v,w)\|_{Z_T}^2\big). \tag{4.9}$$

Combining (4.4), (4.8) and (4.9), we see that the desired assertion follows. □

From the proof of Lemma 4.1 we see that there exists a constant $C_1>0$ independent of T and $\theta=\min\{\theta_1,\theta_2\}$, such that for any $(\mathbf{u},v,w)\in Z_T$ and $(\hat{\mathbf{u}},\hat{v},\hat{w})=\mathfrak{F}(\mathbf{u},v,w)$ we have

$$\|(\hat{\mathbf{u}},\hat{v},\hat{w})\|_{Z_T}\leqslant C_1\big(\|\mathbf{u}_0\|_{L^{q_0}}+\|(v_0,w_0)\|_{L^{p_0}}+T^{\theta}\|(\mathbf{u},v,w)\|_{Z_T}^2\big). \tag{4.10}$$

Let $R=C_1(\|\mathbf{u}_0\|_{L^{q_0}}+\|(v_0,w_0)\|_{L^{p_0}})$ and B_T be a closed ball in Z_T with radius $2R$, i.e.,

$$B_T=\big\{(\mathbf{u},v,w)\in Z_T:\ \|(\mathbf{u},v,w)\|_{Z_T}\leqslant 2R\big\}.$$

For any $(\mathbf{u},v,w)\in B_T$, from (4.10) we have

$$\|(\hat{\mathbf{u}},\hat{v},\hat{w})\|_{Z_T}\leqslant R+4R^2T^{\theta}.$$

Hence, by choosing T sufficiently small such that $4RT^{\theta}<1$, we see that $\mathfrak{F}$ maps B_T into itself. Furthermore, by using the similar argument as in the proof of Lemma 3.2 we can prove $\mathfrak{F}$ is a contraction mapping in B_T.

LEMMA 4.2. *Let R, T and B_T be as above. Then when restricted in B_T, $\mathfrak{F}$ is a contraction mapping.*

By using Lemmas 4.1, 4.2 and the Banach fixed point theorem, we obtain Theorem 1.3.

THE PROOF OF THEOREM 1.4. Under the assumptions of Theorem 1.4, we define $\tilde{Z}_T$ to be the space

$$\tilde{Z}_T=L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N))\times L^s((0,T),L^p(\mathbb{R}^N))\times L^s((0,T),L^p(\mathbb{R}^N))$$

with norm

$$\|(\mathbf{u},v,w)\|_{\tilde{Z}_T} = \|\mathbf{u}(t)\|_{L^r((0,T),L^q)} + \|(v(t),w(t))\|_{L^s((0,T),L^p)}.$$

It is clear that $(\tilde{Z}_T, \|\cdot\|_{\tilde{Z}_T})$ is Banach space.

LEMMA 4.3. *For any $0<T<\infty$, $\mathfrak{F}$ maps $\tilde{Z}_T$ into itself.*

Proof. Due to Lemma 4.1, we only need to prove that

$$\hat{\mathbf{u}} \in L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \quad \text{and} \quad \hat{v},\hat{w} \in L^s((0,T),L^p(\mathbb{R}^N)).$$

We first consider $\hat{\mathbf{u}}$. Again, recall that $\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 + \hat{\mathbf{u}}_3$ with $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$ given by (3.4). For $\hat{\mathbf{u}}_1$, since $\mathbf{u}_0 \in \mathbf{L}^{q_0}_\omega(\mathbb{R}^N)$, from Lemmas 2.4 we see that

$$\hat{\mathbf{u}}_1 \in L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \text{ and } \|\hat{\mathbf{u}}_1\|_{L^r((0,T),L^q)} \leqslant C\|\mathbf{u}_0\|_{L^{q_0}}. \tag{4.11}$$

For $\hat{\mathbf{u}}_2$, by using (2.9) with $\alpha=\beta=\gamma=\frac{N}{q}$,

$$\|\hat{\mathbf{u}}_2\|_{L^q} \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\|\mathbf{u}(\tau)\|^2_{L^q}d\tau.$$

Note that $\frac{1}{2}(1+\frac{N}{q})<1$ (since $q>N$), applying the Hardy-Littlewood-Sobolev inequality yields that

$$\|\hat{\mathbf{u}}_2\|_{L^r((0,T),L^q)} \leqslant CT^{\frac{1}{2}(1-\frac{N}{q_0})}\|\mathbf{u}\|^2_{L^r((0,T),L^q)}.$$

Hence

$$\hat{\mathbf{u}}_2 \in L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \text{ and } \|\hat{\mathbf{u}}_2\|_{L^r((0,T),L^q)} \leqslant CT^{\frac{1}{2}(1-\frac{N}{q_0})}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}. \tag{4.12}$$

For $\hat{\mathbf{u}}_3$, by using (2.8) with $\alpha=\frac{N}{p}$, $\beta=\frac{N-p}{p}$ and $\gamma=\frac{N}{q}$,

$$\|\hat{\mathbf{u}}_3\|_{L^q} \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(\frac{2N}{p}-\frac{N}{q}-1)}\|(v-w)\|_{L^p}\|\nabla((-\Delta)^{-1}(v-w))\|_{L^{\frac{Np}{N-p}}}d\tau.$$

From the assumption $\frac{2}{p}<\frac{3}{N}+\frac{1}{q}$ we see that $\frac{1}{2}(\frac{2N}{p}-\frac{N}{q}-1)<1$. Hence, by applying the Hardy-Littlewood-Sobolev inequality and (2.13) we get

$$\|\hat{\mathbf{u}}_3\|_{L^r((0,T),L^q)} \leqslant CT^{\frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}}\|(v,w)\|^2_{L^s((0,T),L^p)}.$$

Note that the assumption $\frac{2}{p_0}<\frac{3}{N}+\frac{1}{q_0}$ ensures that $\frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}>0$. Hence, we obtain

$$\hat{\mathbf{u}}_3 \in L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)) \quad \text{and} \quad \|\hat{\mathbf{u}}_3\|_{L^r((0,T),L^q)} \leqslant CT^{\frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}. \tag{4.13}$$

Since $\theta_1=\min\{\frac{1}{2}(1-\frac{N}{q_0}),\frac{3}{2}+\frac{N}{2q_0}-\frac{N}{p_0}\}$, from (4.11)-(4.13) we get

$$\hat{\mathbf{u}} \in L^r((0,T),\mathbf{L}^q_\omega(\mathbb{R}^N)),$$

and

$$\|\hat{\mathbf{u}}\|_{L^r((0,T),L^q)} \leqslant C\big(\|\mathbf{u}_0\|_{L^{q_0}} + T^{\theta_1}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}\big). \tag{4.14}$$

Next we consider $\hat{v}$. Recall that $\hat{v} = \hat{v}_1 + \hat{v}_2 + \hat{v}_3$, where $\hat{v}_1$, $\hat{v}_2$ and $\hat{v}_3$ are defined by (3.13). For $\hat{v}_1$, since $v_0 \in L^{p_0}(\mathbb{R}^N)$, similarly as for $\hat{\mathbf{u}}_1$, we get

$$\hat{v}_1 \in L^s((0,T),L^p(\mathbb{R}^N)) \text{ and } \|\hat{v}_1\|_{L^s((0,T),L^p)} \leqslant C\|v_0\|_{L^{p_0}}. \tag{4.15}$$

For $\hat{v}_2$, by applying (2.9) with $\alpha = \frac{N}{q}$ and $\beta = \gamma = \frac{N}{p}$,

$$\|\hat{v}_2\|_{L^p} \leqslant C\int_0^t (t-\tau)^{-\frac{1}{2}(1+\frac{N}{q})}\|\mathbf{u}\|_{L^q}\|v\|_{L^p}d\tau.$$

Using the Hardy-Littlewood-Sobolev inequality again yields that

$$\|\hat{v}_2\|_{L^s((0,T),L^p)} \leqslant CT^{\frac{1}{2}(1-\frac{N}{q_0})}\|\mathbf{u}\|_{L^r((0,T),L^q)}\|v\|_{L^s((0,T),L^p)}.$$

Hence

$$\hat{v}_2 \in L^s((0,T),L^p(\mathbb{R}^N)) \text{ and } \|\hat{v}_2\|_{L^s((0,T),L^p)} \leqslant CT^{\frac{1}{2}(1-\frac{N}{q_0})}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}. \tag{4.16}$$

For $\hat{v}_3$, by applying (2.9) with $\alpha = \gamma = \frac{N}{p}$ and $\beta = \frac{N-p}{p}$,

$$\|\hat{v}_3\|_{L^p} \leqslant C\int_0^t (t-\tau)^{-\frac{N}{2p}}\|v\|_{L^p}\|\nabla((-\Delta)^{-1}(v-w))\|_{L^{\frac{Np}{N-p}}}d\tau.$$

Note that $\frac{N}{2p} < 1$ (because $p > \frac{N}{2}$). By applying the Hardy-Littlewood-Sobolev inequality and (2.13) we obtain

$$\|\hat{v}_3\|_{L^s((0,T),L^p)} \leqslant CT^{1-\frac{N}{2p_0}}\|(v,w)\|^2_{L^s((0,T),L^p)}.$$

Hence

$$\hat{v}_3 \in L^s((0,T),L^p(\mathbb{R}^N)) \text{ and } \|\hat{v}_3\|_{L^s((0,T),L^p)} \leqslant CT^{1-\frac{N}{2p_0}}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}. \tag{4.17}$$

Since $\theta_2 = \min\{\frac{1}{2}(1-\frac{N}{q_0}), 1-\frac{N}{2p_0}\}$, from (4.15)-(4.17) we get $\hat{v} \in L^s((0,T),L^p(\mathbb{R}^N))$, and

$$\|\hat{v}\|_{L^s((0,T),L^p)} \leqslant C\big(\|v_0\|_{L^{p_0}} + T^{\theta_2}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}\big). \tag{4.18}$$

Similarly we have $\hat{w} \in L^s((0,T),L^p(\mathbb{R}^N))$, and

$$\|\hat{w}\|_{L^s((0,T),L^p)} \leqslant C\big(\|w_0\|_{L^{p_0}} + T^{\theta_2}\|(\mathbf{u},v,w)\|^2_{\tilde{Z}_T}\big). \tag{4.19}$$

From (4.14), (4.18) and (4.19), we see that the desired assertion follows. □

From Lemma 4.3, we see that $\mathfrak{F}$ is well-defined. Now we can use a similar argument as in the proof of Lemma 4.2, which shows that if T is sufficiently small then $\mathfrak{F}$ is a contraction mapping from some closed ball in $\tilde{Z}_T$ into itself. Hence, by using the Banach fixed point theorem, we obtain Theorem 1.4.

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REFERENCES

[1] N. BEN ABDALLAH, F. MÉHATS AND N. VAUCHELET, *A note on the long time behavior for the drift-diffusion-Poisson system*, C. R. Math. Acad. Sci. Paris, **339**, 10 (2004), 683–688.

[2] P. BILER AND J. DOLBEAULT, *Long time behavior of solutions to Nernst-Planck and Debye-Hückel drift-diffusion systems*, Ann. Henri Poincaré, **1** (2000), 461–472.

[3] P. BILER, W. HEBISCH AND T. NADZIEJA, *The Debye system: existence and large time behavior of solutions*, Nonlinear Anal., **23** (1994), 1189–1209.

[4] J. BOURGAIN AND N. PAVLOVIĆ, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, J. Funct. Anal., **255** (2008), 2233–2247.

[5] P. B. BALBUENA AND Y. WANG, *Lithium-ion Batteries, Solid-electrolyte Interphase*, Imperial College Press, 2004.

[6] M. CANNONE, *A generalization of a theorem by Kato on Navier-Stokes equations*, Rev. Mat. Iberoam., **13** (1997), 515–541.

[7] P. DEBYE AND E. HÜCKEL, *Zur Theorie der Elektrolyte, II: Das Grenzgesetz für die elektrische Leitfähigkeit*, Phys. Z., **24** (1923) 305–325.

[8] E. T. ENIKOV AND B. J. NELSON, *Electrotransport and deformation model of ion exchange membrane based actuators*, Smart Structures and Materials, Vol. **3978** (2000), 129–139.

[9] E. T. ENIKOV AND G. S. SEO, *Analysis of water and proton fluxes in ion-exchange polymer-metal composite (IPMC) actuators subjected to large external potentials*, Sensors and Actuators, **122** (2005), 264–272.

[10] H. FUJITA AND T. KATO, *On the Navier-Stokes initial value problem I*, Arch. Rational Mech. Anal., **16** (1964), 269–315.

[11] Y. GIGA, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations, **62** (1986), 186–212.

[12] H. GAJEWSKI AND K. GRÖGER, *On the basic equations for carrier transport in semiconductors*, J. Math. Anal. Appl., **113** (1986), 12–35.

[13] J.W. JEROME, *Analytical approaches to charge transport in a moving medium*, Tran. Theo. Stat. Phys., **31** (2002), 333–366.

[14] J. W. JEROME AND R. SACCO, *Global weak solutions for an incompressible charged fluid with multi-scale couplings: Initial-boundary-value problem*, Nonlinear Anal., **71** (2009), 2487–2497.

[15] G. KARCH, *Scaling in nonlinear parabolic equations*, J. Math. Anal. Appl., **234** (1999), 534–558.

[16] G. KARCH, *Scaling in nonlinear parabolic equations: applications to Debye system*, AIP Conf. Proc., **553**(1) (2001), 243–248.

[17] T. KATO, *Strong L^p solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Math. Z., **187** (1984), 471–480.

[18] T. KATO AND G. PONCE, *Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^s_p(\mathbb{R}^2)$*, Rev. Mat. Iberoam., **2** (1986), 73–88.

[19] T. KATO AND G. PONCE, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **41** (1988), 891–907.

[20] H. KOCH AND D. TATARU, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157** (2001), 22–35.

[21] J. LERAY, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., **63** (1934), 193–248.

[22] M. LONGARETTI, B. CHINI, J.W. JEROME AND R. SACCO, *Electrochemical modeling and characterization of voltage operated channels in nano-bio-electronics*, Sensor Letters, **6** (2008), 49–56.

[23] M. LONGARETTI, B. CHINI, J.W. JEROME AND R. SACCO, *Computational modeling and simulation of complex systems in bio-electronics*, J. Computational Electronics, **7** (2008), 10–13.

[24] M. LONGARETTI, G. MARINO, B. CHINI, J.W. JEROME AND R. SACCO, *Computational models in nano-bio-electronics: simulation of ionic transport in voltage operated channels*, J. Nanoscience and Nanotechnology, **8** (2008), 3686–3694.

[25] F. PLANCHON, *Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$*, Ann. Inst. Henri Poincare, **13** (1996), 319–336.

[26] I. RUBINSTEIN, *Electro-Diffusion of Ions*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1990.

[27] R. J. RYHAM, *Existence, uniqueness, regularity and long-term behavior for dissipative systems modeling electrohydrodynamics*, arXiv:0910.4973v1.

[28] R. J. RYHAM, C. LIU AND L. ZIKATANOV, *Mathematical models for the deformation of electrolyte droplets*, Discrete Contin. Dyn. Syst. Ser. B, **8**, 3 (2007), 649–661.

[29] M. SCHMUCK, *Analysis of the Navier-Stokes-Nernst-Planck-Poisson system*, Math. Models and Methods Appl. Sci., **19**, 6 (2009), 993–1015.

[30] M. SHAHINPOOR AND K. J. KIM, *Ionic polymer-metal composites: III. Modeling and simulation as biomimetic sensors, actuators, transducers, and artificial muscles*, Smart Mater. Struct., **13** (2004), 1362–1388.

[31] R. TEMAM, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, (2001). Theory and numerical analysis, Reprint of the 1984 edition.

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