

## INDIRECT LOCALLY INTERNAL OBSERVABILITY AND CONTROLLABILITY OF WEAKLY COUPLED WAVE EQUATIONS

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*Abstract.* In this paper, we study the exact controllability of a system of weakly coupled wave equations with an internal locally control acted on only one equation. Using a piecewise multiplier method, we show that, for a sufficiently large time  $T$ , the observation of the velocity of the first component of the solution on a neighborhood of a part of the boundary allows us to get back a weakened energy of initial data of the second component of the solution, this if the coupling parameter is sufficiently small, but non vanishing. This result leads, by the HUM method, to prove that the total system is exactly controllable by means of one locally distributed control.

### 1. Introduction and statement of the main results

Let  $\Omega$  be a non-empty bounded open domain of  $\mathbb{R}^N$  with smooth boundary  $\Gamma$  of class  $C^2$  such that  $\Gamma = \Gamma_0 \cup \Gamma_1$  and  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$  (the case  $\Gamma_0 = \emptyset$  is not excluded) and let  $\omega$  be a neighborhood of  $\Gamma_1$  in  $\Omega$ . We consider the following weakly coupled wave equations with Dirichlet condition:

$$\begin{cases} u_1'' - \Delta u_1 + \alpha u_2 = 0 & \text{in } \Omega \times (0, T), \\ u_2'' - \Delta u_2 + \alpha u_1 = 0 & \text{in } \Omega \times (0, T), \\ u_1 = u_2 = 0 & \text{on } \Gamma \times (0, T), \\ u_i(0) = u_i^0, u_i'(0) = u_i^1 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $i = 1, 2$  and  $\alpha \neq 0$  is a small constant.

In [1] and [2], Alabau-Boussouira studied the indirect boundary observability of the system (1.1). In particular, using a multiplier method, she proved that, for sufficiently large time  $T$ , the observation of the trace of the normal derivative of the first component of the solution on a part of the boundary allows us to get back a weakened energy of the initial data. Then the system (1.1) is exactly controllable by means of one boundary control. In addition, in [20] Liu and Rao studied the indirect boundary controllability of a system of two weakly coupled one-dimensional wave equations. Using the non harmonic analysis, they established a weak observability inequalities and

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proved the indirect exact controllability of the system. But the problem seems still open in the case of locally internal control.

Our purpose in this paper is to study the indirect internal observability and exact controllability of the system (1.1). Using a piecewise multiplier method, we prove that, for sufficiently large time  $T$ , the locally observation, in  $\omega$ , of the velocity of the first component of the total solution of (1.1) allows us to get back a weakened energy of initial data of the second component. This result leads, by the HUM method, to establish the exact controllability of the system (1.1) by means of only one locally internal control. To more precise, let  $U = (u_1, u'_1, u_2, u'_2)$  be a regular solution of the system (1.1). We define the associated partial energies by

$$e_i(t) = \frac{1}{2} \int_{\Omega} (|u'_i|^2 + |\nabla u_i|^2) dx, \quad i = 1, 2 \tag{1.2}$$

and the associated total (natural) energy by

$$E(U(t)) = e_1(t) + e_2(t) + \alpha \int_{\Omega} u_1 u_2 dx. \tag{1.3}$$

Moreover, in what follows we will also need to define the associated partial weakened energies by

$$\tilde{e}_i(t) = \frac{1}{2} (\|u'_i\|_{H^{-1}(\Omega)}^2 + |u_i|_{L^2(\Omega)}^2) dx, \quad i = 1, 2 \tag{1.4}$$

and the total weakened energy by

$$\tilde{E}(U(t)) = \tilde{e}_1(t) + \tilde{e}_2(t) + \alpha \int_{\Omega} \nabla(\Delta^{-1} u_1) \cdot \nabla(\Delta^{-1} u_2) dx. \tag{1.5}$$

First, using a piecewise multiplier method, we establish the following indirect internal observability inequality

$$\int_0^T \int_{\omega} |u'_1|^2 \geq c_2 (e_1(0) + \tilde{e}_2(0)). \tag{1.6}$$

Next, we consider the following system:

$$\begin{cases} y_1'' - \Delta y_1 + \alpha y_2 = v(t) 1_{\omega} & \text{in } \Omega \times (0, T), \\ y_2'' - \Delta y_2 + \alpha y_1 = 0 & \text{in } \Omega \times (0, T), \\ y_1 = y_2 = 0 & \text{on } \Gamma \times (0, T), \\ y_i(0) = y_i^0, \quad y'_i(0) = y_i^1 & \text{in } \Omega, \end{cases} \tag{1.7}$$

where  $1_{\omega}$  is the characteristic function of  $\omega$ . The solution of system (1.7) can be defined by the transposition method. Then we consider the indirect locally internal exact controllability problem: For given  $T > 0$  (sufficiently large) and initial data  $(y_1^0, y_1^1, y_2^0, y_2^1)$ , does there exists a suitable control  $v$  that brings back the solution to equilibrium at time  $T$ , that is such that the solution of (1.7) satisfies  $y_i(T) = y'_i(T) = 0$  for  $i = 1, 2$ ? Indeed, applying the HUM method introduced by Lions (see [16], [17] and [12]) we establish the indirect locally internal controllability result.

Finally, for partially damped linear systems, the transmission of the dissipation from one equation to others plays an important role for the control and stabilization. In [16], Lions studied the complete and partial observability and controllability of coupled systems of either hyperbolic-hyperbolic type or hyperbolic-parabolic type. These results assume that the coupling parameter is sufficiently small. They have been extended in [14] to the case of arbitrary coupling parameters. Complete observability and controllability results have also been obtained in [15] for systems of coupled second order hyperbolic equations containing first order terms in both the original and the coupled unknowns. In [3], Alabau-Boussouira studied the boundary stabilization of an abstract system of two coupled second order evolution equations wherein only one of the equations is damped (this called indirect boundary stabilization). Under a condition on the operators of each equation and on the boundary feedback operator, she proved that the energy of smooth solutions of there system decays polynomially at infinity. In [4], Alabau, Cannarsa and Komornik studied the the indirect internal stabilization of weakly coupled systems. In [21], using a frequency domain approach, Liu and Rao established the optimal polynomial energy decay rate of a system of coupled wave equations damped by one boundary feedback. In [32], Zhang and Zuazua obtained the exact controllability for one-dimensional system of coupled heat-wave equations by Riesz basis approach. In [22], Loreti and Rao show that a weaker damping can provide a stronger decay rate by means of spectral compensation. we recall some results existing in literature which are related to the indirect control and stabilization: [26], [27], [5], [6], [25], [32], [11], [10], [28], [7], [30], [31].

The results of this paper are mentioned in [29] and organized here as follows. In section 2, we first established the well-posedness of the system (1.1). Next, we give the proof of the observability inequality (1.6). In section 3, we proof that the total system (1.7) is exactly controllable by means of one locally distributed feedback.

## 2. Locally internal observability results

### 2.1. Well-Posedness of the problem

We first define the energy space  $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$  equipped with the usual product norm. We identify  $L^2(\Omega)$  with its dual space, then the imbeddings  $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$  are continuous, dense and compact. The scalar products on  $\mathcal{H}$  and  $L^2(\Omega)$  are, respectively, denoted by  $(\cdot, \cdot)_{\mathcal{H}}$  and  $(\cdot, \cdot)$ , whereas the corresponding norms are, respectively denoted by  $\|\cdot\|_{\mathcal{H}}$  and  $\|\cdot\|$ . We define the following bilinear form

$$\begin{aligned} (U, \tilde{U})_{\alpha} &= (U, \tilde{U})_{\mathcal{H}} + \alpha(u_2, \tilde{u}_1) + \alpha(u_1, \tilde{u}_2), \\ U &= (u_1, v_1, u_2, v_2), \tilde{U} = (\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{H}. \end{aligned}$$

Now, let  $\alpha_0 = \frac{1}{c_0}$  where  $c_0$  is the Poincaré’s constant, then it easy to see that, for  $0 < |\alpha| < \alpha_0$ , the mapping

$$U \mapsto \|U\|_{\alpha} = (U, U)_{\alpha}^{1/2}$$

defines a norm on  $\mathcal{H}$  which equivalent to the usual product norm i.e there exist  $c > 0$  and  $\tilde{c} > 0$  such that  $\tilde{c}\|U\|_\alpha \leq \|U\|_{\mathcal{H}} \leq c\|U\|_\alpha$ . In addition, we have

$$2E(U(t)) = (U, U)_\alpha = \|U(t)\|_\alpha^2.$$

Next, we define the unbounded operator  $\mathcal{A}_\alpha : D(\mathcal{A}) \rightarrow \mathcal{H}$  by

$$\begin{aligned} D(\mathcal{A}_\alpha) &= [(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)]^2, \\ \mathcal{A}_\alpha U &= (-v_1, -\Delta u_1 + \alpha u_2, -v_2, -\Delta u_2 + \alpha u_1), \\ \forall U &= (u_1, v_1, u_2, v_2) \in D(\mathcal{A}_\alpha). \end{aligned}$$

Then setting  $U = (u_1, u_1', u_2, u_2')$ , we convert the system (1.1) into an evolutionary equation:

$$U' + \mathcal{A}_\alpha U = 0, \quad U(0) = U^0 \in \mathcal{H}. \tag{2.1}$$

It easy to prove that  $\mathcal{A}_\alpha$  is a skew adjoint and maximal monotone on  $\mathcal{H}$  and therefore generates a strongly continuous group of isometries  $S_\alpha(t) = \exp(-t\mathcal{A}_\alpha)$ ,  $t \in \mathbb{R}$  on  $\mathcal{H}$  (see [24], [8]). Then we establish the well-posedness result:

**THEOREM 2.1.** *Let  $\alpha \in (0, \alpha_0)$ . Then for all  $U^0 = (u_1^0, u_1^0, u_2^0, u_2^0) \in \mathcal{H}$ , the system (2.1) has a unique solution  $U$  satisfies  $U \in \mathcal{C}(\mathbb{R}_+; \mathcal{H})$ . Moreover, if  $U^0 \in D(\mathcal{A}_\alpha^k)$  for  $k \in \mathbb{N}^*$ , then the solution  $U$  is more regular and satisfies*

$$\mathcal{C}^{k-j}(\mathbb{R}_+; D(\mathcal{A}_\alpha^j)) \text{ for } j = 0, \dots, k.$$

In addition, we have

$$E(U(t)) = E(U(0)), \quad \tilde{E}(U(t)) = \tilde{E}(U(0)), \quad t \geq 0.$$

**2.2. Observability results**

In this part, using a piecewise multiplier method, we establish the inverse indirect observability inequality. In order to use the piecewise multiplier method we need define, for  $0 < \varepsilon_0 < \varepsilon_1$ , the neighborhoods of  $\Gamma_1$  as follows (see [19] and [23])

$$\omega_{\varepsilon_i} = \{x \in \Omega; \quad d(x, \Gamma_1) < \varepsilon_i\} \quad i = 0, 1.$$

It easy to see that, for small  $\varepsilon_1$ , we have  $\omega_{\varepsilon_0} \subset \omega_{\varepsilon_1} \subset \omega$ . First, using Theorem 2.1 and the definition of the total energy, we have the following direct inequality:

$$\int_0^T \int_\omega |u_1'|^2 \leq 2 \int_0^T \|U\|_{\mathcal{H}}^2 dt \leq 2c^2 \int_0^T \|U\|_\alpha^2 dt = 2Tc^2 E(U(0)) \tag{2.2}$$

for all solution  $U$  of system (1.1). We deduce that  $u_1$  is an element of  $H^1(0, T; L^2(\omega))$ . Next, we will establish the main indirect observability inequality:

**THEOREM 2.2.** (main theorem) *Assume that there exist a constant  $\delta > 0$  and a point  $x_0 \in \mathbb{R}^N$  such that, putting  $m(x) = x - x_0$ , we have  $(m \cdot \nu) \leq 0$  on  $\Gamma_0$  and  $(m \cdot \nu) \geq \delta > 0$  on  $\Gamma_1$ . Then there exists  $\alpha^* > 0$  such that for all  $0 < |\alpha| < \alpha^*$ , there exists  $T_0 = T_0(\alpha) > 0$  such that for all  $T > T_0$  and all  $U^0 = (u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{H}$ , the solution  $(u_1, u_1', u_2, u_2')$  of system (1.1) satisfies*

$$\int_0^T \int_{\omega} |u_1'|^2 \geq c_1 (e_1(0) + \tilde{e}_2(0)), \tag{2.3}$$

where  $\omega$  be a neighborhood of  $\Gamma_1$  in  $\Omega$ ,  $T_0$  is an explicit constant and where  $c_1$  an explicit positive constant which depends only on  $\alpha$  and  $T$ .

*Proof.* Let  $\alpha_1 = \min(\alpha_0, 2^{-1}\sqrt{\alpha_0})$  and  $\alpha_2 \in (0, \alpha_1)$ .

*Step 1.* Recall that (see [1]), for all  $0 < |\alpha| < \alpha_2$  for all  $U^0 = (u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{H}$ , the solution  $(u_1, u_2)$  of system (1.1) satisfies:

$$e_1(T) + e_1(0) \leq c_3 (e_1(0) + \tilde{e}_2(0)) + c_4 \alpha \int_0^T e(t) dt, \tag{2.4}$$

$$\int_0^T |u_2|^2 dt \leq \frac{2c_5}{\alpha} (e_1(0) + \tilde{e}_2(0)) + c_6 \int_0^T e_1(t) dt, \tag{2.5}$$

$$\int_0^T \tilde{e}_2(t) dt \leq \frac{c_7}{\alpha} (e_1(0) + \tilde{e}_2(0)) + c_8 \alpha \int_0^T e_1(t) dt, \tag{2.6}$$

$$\int_0^T e_1(t) dt \geq \frac{c_9 T}{2(1 + \alpha T)} (e_1(0) - \tilde{e}_2(0)), \tag{2.7}$$

$$\int_0^T (e_1(t) + \tilde{e}_2(t)) dt \geq \frac{c_{10} T}{2} (\tilde{e}_1(0) + \tilde{e}_2(0)), \tag{2.8}$$

where  $c_i, i = 3, \dots, 10$  are independent on  $\alpha, T$ , and  $U^0$ .

*Step 2.* Recall that for all  $u \in H^2(\Omega)$ , we have the following well-know Rellich's identity:

$$\begin{aligned} 2 \int_{\Omega} \Delta u (m \cdot \nabla u) dx &= (N - 2) \int_{\Omega} |\nabla u|^2 dx \\ &+ 2 \int_{\Gamma} \partial_{\nu} u (m \cdot \nabla u) d\Gamma - \int_{\Gamma} (m \cdot \nu) |\nabla u|^2 d\Gamma. \end{aligned} \tag{2.9}$$

Multiplying the first wave equation of (1.1) by the classical multiplier  $Mu_1 = 2(m \cdot \nabla u_1) + (N - 1)u_1$  and using Rellich's identity, we get

$$\int_0^T \int_{\Gamma} (m \cdot \nu) |\partial_{\nu} u_1|^2 = 2 \int_0^T e_1(t) dt + \alpha \int_0^T \int_{\Omega} u_2 Mu_1 + \left[ \int_{\Omega} u_1' Mu_1 \right]_0^T.$$

We know that  $\|Mu_1\|_{L^2(\Omega)} \leq 2R \|\nabla u_1\|_{L^2(\Omega)}$  (see [12]), then we have

$$\left| \int_{\Omega} u_1' Mu_1 \right| \leq 2Re_1(t), \quad \forall t \in \mathbb{R}.$$

Then using the geometrical condition  $(m \cdot \nu) \leq 0$  on  $\Gamma_0$ , Cauchy-Schwartz and Poincaré’s inequalities, we deduce that

$$R \int_0^T \int_{\Gamma_1} |\partial_\nu u_1|^2 \geq 2 \int_0^T e_1(t) dt - \alpha R \int_0^T \int_{\Omega} |u_2|^2 - \alpha R \int_0^T \int_{\Omega} |\nabla u_1|^2 - 2R(e_1(0) + e_1(T)). \quad (2.10)$$

The main problem is to estimate, and more precisely to majoring the first boundary integral in (2.10). To overcome this difficulty we will consider a special vector field  $h$  in the following step.

*Step 3.* Let  $h \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^N)$ . Multiplying the first wave equation of (1.1) by  $2h \cdot \nabla u_1$  and integrating by parts, we get

$$\begin{aligned} & \int_0^T \int_{\Omega} \operatorname{div} h (|u_1'|^2 - |\nabla u_1|^2) \\ & \quad + 2 \int_0^T \int_{\Omega} \partial_i h_k \partial_i u_1 \partial_k u_1 + 2\alpha \int_0^T \int_{\Omega} u_2 (h \cdot \nabla u_1) \\ & \quad + \left[ 2 \int_{\Omega} u_1' (h \cdot \nabla u_1) \right]_0^T = \int_0^T \int_{\Gamma} (h \cdot \nu) |\nabla u_1|^2, \end{aligned} \quad (2.11)$$

where we used the convention summation of repeated indices. Now, consider a special vector field  $h$  verifying the following conditions:

$$h \cdot \nu = 1 \quad \text{on } \Gamma_1, \quad h \cdot \nu \geq 0 \quad \text{and} \quad \operatorname{supp} h \subset \tilde{\omega} \subset \omega_{\varepsilon_0}. \quad (\text{H})$$

See [16] for the proof of existence of such field vector. First since  $h$  is of class  $C^1$ , then there exists a positive constant  $c_h$  such that

$$|h(x)| \leq c_h \quad \text{and} \quad \sum_{i,j=1}^N |\partial_i h_j(x)| \leq c_h, \quad \forall x \in \overline{\Omega}.$$

On the other hand, since  $u_1 = 0$  on  $\Gamma$  then  $\nabla u_1 = (\partial_\nu u_1)$ . It follows from (2.11) and condition (H) that

$$\begin{aligned} & \int_0^T \int_{\Gamma_1} |\partial_\nu u_1|^2 \leq c_{11} \int_0^T \int_{\omega_{\varepsilon_0}} (|u_1'|^2 + |\nabla u_1|^2) \\ & \quad + \alpha^2 \int_0^T \int_{\Omega} |u_2|^2 + 2c_h(e_1(0) + e_1(T)), \end{aligned} \quad (2.12)$$

where  $c_{11}$  is independent of  $T$ ,  $\alpha$  and  $U^0$ . Finally, combining (2.10) and (2.12), we obtain

$$2 \int_0^T e_1(t) dt - c_{12} \alpha \int_0^T \int_{\Omega} |u_2|^2 - c_{13} \alpha \int_0^T \int_{\Omega} |\nabla u_1|^2$$

$$-c_{14}(e_1(0) + e_1(T)) \leq c_{11}R \int_0^T \int_{\omega_{\varepsilon_0}} (|u_1'|^2 + |\nabla u_1|^2), \quad (2.13)$$

where  $c_{12} = R(1 + \alpha_2)$ ,  $c_{13} = R$  and  $c_{14} = 2R(1 + c_h)$ .

*Step 4.* Define the cut-off function  $\xi \in C_0^\infty(\Omega)$  by

$$\begin{cases} 0 \leq \xi \leq 1 & \text{on } \Omega, \\ \xi = 1 & \text{on } \omega_{\varepsilon_0}, \\ \xi = 0 & \text{on } \Omega \setminus \omega_{\varepsilon_1}. \end{cases}$$

Multiplying the first equation of (1.1) by  $\xi u_1$  and integrating by parts, we obtain

$$\int_0^T \int_{\Omega} (-\xi |u_1'|^2 + \xi |\nabla u_1|^2 - \frac{1}{2} |u_1|^2 \Delta \xi + \alpha \xi u_1 u_2) dxdt + \left[ \int_{\Omega} u_1' \xi u_1 \right]_0^T = 0.$$

Since  $\text{supp } \xi \subset \omega_{\varepsilon_1}$ , we deduce that

$$\begin{aligned} \int_0^T \int_{\omega_{\varepsilon_0}} |\nabla u_1|^2 &\leq \int_0^T \int_{\Omega} \xi |\nabla u_1|^2 \\ &= \int_0^T \int_{\Omega} \left( \xi |u_1'|^2 + \frac{1}{2} \Delta \xi |u_1|^2 - \alpha \xi u_1 u_2 \right) dxdt - \left[ \int_{\Omega} u_1' \xi u_1 \right]_0^T \\ &\leq c_{15} \int_0^T \int_{\omega_{\varepsilon_1}} (|u_1'|^2 + |u_1|^2) \\ &\quad + \alpha^2 \int_0^T \int_{\omega_{\varepsilon_1}} |u_2|^2 + c_0(e_1(0) + e_1(T)), \end{aligned} \quad (2.14)$$

where  $c_{15}$  is independent of  $T$ ,  $\alpha$  and  $U^0$ . Substituting (2.14) into (2.13), we get

$$\begin{aligned} 2 \int_0^T e_1(t) dt - c_{16} \alpha \int_0^T \int_{\Omega} |u_2|^2 - c_{13} \alpha \int_0^T \int_{\Omega} |\nabla u_1|^2 - c_{17}(e_1(0) + e_1(T)) \\ \leq c_{18} \int_0^T \int_{\omega_{\varepsilon_1}} |u_1'|^2 + c_{19} \int_0^T \int_{\omega_{\varepsilon_1}} |u_1|^2, \end{aligned} \quad (2.15)$$

where  $c_{16} = c_{12} + \alpha_2 c_{11} R$ ,  $c_{17} = c_{14} + c_0 c_{11} R$ ,  $c_{18} = c_{11}(1 + c_{15})R$  and  $c_{19} = c_{11} c_{15} R$ . The main problem in (2.15) is the constant  $c_{19}$  is not sufficiently small and it is independent of  $\alpha$ .

*Step 5.* In order to estimate the last integral in (2.15), we need a particular multiplier called internal multiplier (boundary multiplier introduced in [9]). Then, define the function  $\zeta \in C_0^\infty(\Omega)$  by

$$\begin{cases} 0 \leq \zeta \leq 1 & \text{on } \Omega, \\ \zeta = 1 & \text{on } \omega_{\varepsilon_1}, \\ \zeta = 0 & \text{on } \Omega \setminus \omega. \end{cases} \quad (2.16)$$

Fix  $t$  and consider the solution  $z$  of the following elliptic problem:

$$\begin{cases} -\Delta z = \zeta(x)u_1 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \quad (2.17)$$

Multiplying (2.17) by  $z$  and integrating by parts, we deduce that there exists  $c_{20} > 0$  and  $c_{21} > 0$  such that

$$\int_{\Omega} |\nabla z|^2 \leq c_{20} \int_{\omega} |u_1|^2,$$

and

$$\int_{\Omega} |z|^2 \leq c_{21} \int_{\omega} |u_1|^2. \quad (2.18)$$

On the other hand, deriving (2.17) with respect to  $t$ , we deduce that  $z'$  is solution of the following problem:

$$\begin{cases} -\Delta z' = \zeta(x)u_1' & \text{in } \Omega, \\ z' = 0 & \text{on } \Gamma, \end{cases} \quad (2.19)$$

and we have the following inequality

$$\int_{\Omega} |z'|^2 \leq c_{21} \int_{\omega} |u_1'|^2. \quad (2.20)$$

Now, multiplying the first equation of (1.1) by  $z$  and we integrating by parts, we get

$$\int_0^T \int_{\Omega} (-u_1' z' - u_1 \Delta z + \alpha u_2 z) dx dt + \left[ \int_{\Omega} u_1' z dx \right]_0^T = 0. \quad (2.21)$$

Then using (2.16), (2.17), (2.18), (2.20) and Cauchy-Schwartz inequality, we deduce from (2.21) that, for all  $\varepsilon > 0$ , the following estimation holds

$$\begin{aligned} \int_0^T \int_{\omega_{\varepsilon_1}} |u_1|^2 &\leq \frac{\varepsilon}{2} \int_0^T \int_{\Omega} |u_1'|^2 + \frac{c_{21}}{2\varepsilon} \int_0^T \int_{\omega} |u_1'|^2 + \frac{\alpha}{2} \int_0^T \int_{\Omega} |u_2|^2 \\ &\quad + \frac{\alpha c_{21}}{2} \int_0^T \int_{\Omega} |u_1|^2 + c_{22}(e_1(0) + e_1(T)). \end{aligned} \quad (2.22)$$

Combining (2.22) and (2.15) we obtain

$$\begin{aligned} 2 \int_0^T e_1(t) dt - c_{23} \alpha \int_0^T \int_{\Omega} |u_2|^2 - c_{24} \alpha \int_0^T \int_{\Omega} |\nabla u_1|^2 \\ - \frac{\varepsilon c_{19}}{2} \alpha \int_0^T \int_{\Omega} |u_1'|^2 - c_{25}(e_1(0) + e_1(T)) \leq c_{26} \int_0^T \int_{\omega} |u_1'|^2, \end{aligned} \quad (2.23)$$

where:

$$\begin{aligned} c_{23} &= c_{16} + \frac{c_{19}}{2}, \quad c_{24} = c_{13} + \frac{c_{21}c_{19}c_0}{2}, \\ c_{25} &= c_{17} + c_{22}c_{19} \quad \text{and} \quad c_{26} = c_{18} + \frac{c_{21}c_{19}}{2\varepsilon}. \end{aligned}$$



Step 6. We set  $\varepsilon = \frac{1}{c_{19}}$ . Inserting (2.4) and (2.5) into (2.23) we obtain

$$(1 - \alpha c_{27}) \int_0^T e_1(t) dt - c_{28}(e_1(0) + \tilde{e}_2(0)) \leq c_{29} \int_0^T \int_{\omega} |u'_1|^2 \quad \forall \alpha \in (0, \alpha_2),$$

where  $c_{27} = 2c_4c_{25} + 2c_{24} + 2c_6c_{23}$ ,  $c_{28} = 2c_3c_{25} + 4c_{23}c_5$  and  $c_{29} = c_{18} + \frac{c_{21}c_{19}^2}{2}$ . This implies that

$$\int_0^T e_1(t) dt - c_{28}(e_1(0) + \tilde{e}_2(0)) \leq c_{29} \int_0^T \int_{\omega} |u'_1|^2.$$

Now let  $\tilde{\varepsilon} > 0$ . Then we have

$$(1 - \tilde{\varepsilon}) \int_0^T e_1(t) dt + \tilde{\varepsilon} \int_0^T (e_1(t) + \tilde{e}_2(t)) dt - \tilde{\varepsilon} \int_0^T \tilde{e}_2(t) dt - c_{28}(e_1(0) + \tilde{e}_2(0)) \leq c_{29} \int_0^T \int_{\omega} |u'_1|^2. \quad (2.24)$$

Inserting (2.6), (2.7) and (2.8) into (2.24), we obtain

$$\begin{aligned} & \left[ (1 - \tilde{\varepsilon}c_{29}) \frac{c_9T}{2(1 + \alpha T)} - c_{30} \frac{\tilde{\varepsilon} + \alpha}{\alpha} \right] e_1(0) \\ & + \left[ \tilde{\varepsilon} \frac{c_{10}T}{2} - (1 - \tilde{\varepsilon}c_{29}) \frac{c_9T}{2(1 + \alpha T)} - c_{30} \frac{\tilde{\varepsilon} + \alpha}{\alpha} \right] \tilde{e}_2(0) \\ & + \tilde{\varepsilon} \frac{c_{10}T}{2} \tilde{e}_1(0) \leq c_{29} \int_0^T \int_{\omega} |u'_1|^2, \end{aligned} \quad (2.25)$$

where  $c_{29} = 1 + \alpha_2c_8$  and  $c_{30} = \max(c_7, c_{28})$ . Let  $\varepsilon^* > 0$  such that

$$0 < \varepsilon^* < c_{29}^{-1}.$$

Then for  $\tilde{\varepsilon} \in (0, \varepsilon^*)$  we define

$$a_1 = (1 - \varepsilon c_{29}) \frac{c_9}{2} > 0, \quad a_2 = \varepsilon \frac{c_{10}}{2},$$

this implies, from (2.25), that

$$\begin{aligned} & \left[ \frac{a_1T}{1 + \alpha T} - c_{30} \frac{\tilde{\varepsilon} + \alpha}{\alpha} \right] e_1(0) + \left[ (a_2 - \frac{a_1}{1 + \alpha T})T - c_{30} \frac{\tilde{\varepsilon} + \alpha}{\alpha} \right] \tilde{e}_2(0) \\ & + a_2T \tilde{e}_1(0) \leq c_{29} \int_0^T \int_{\omega} |u'_1|^2. \end{aligned} \quad (2.26)$$

We remark that  $\frac{a_1}{a_2}$  goes to  $+\infty$  as either  $\tilde{\varepsilon}$  goes to zero. Then, for  $\alpha \in (0, \alpha_2)$ , we can define

$$T_1 = T_1(\alpha) = \left( \frac{a_1}{a_2} - 1 \right) \alpha^{-1} > 0. \quad (2.27)$$

This implies that for any  $T \geq T_1$  we have

$$a_2 - \frac{a_1}{1 + \alpha T} > 0.$$

Now we will prove that the coefficients of  $e_1(0)$  and  $\tilde{e}_2(0)$  in (2.26) which depend only on  $T$ ,  $\alpha$  and  $\tilde{\varepsilon}$  are positive for sufficiently large  $T$  and small  $\alpha$  and  $\varepsilon^*$ . For this, we denote by  $Q_\alpha$  the second order polynomial with respect to  $T$  defined by

$$Q_\alpha(T) = \alpha a_2 T^2 + [a_2 - a_1 - c_{30}(\tilde{\varepsilon} + \alpha)]T - \frac{c_{30}(\tilde{\varepsilon} + \alpha)}{\alpha}.$$

We see that the coefficient of  $\tilde{e}_2(0)$  can be written under the following form

$$\frac{Q_\alpha(T)}{1 + \alpha T}.$$

The polynomial  $Q_\alpha$  has two real roots. Moreover, using the definition of  $a_2$  the coefficient of  $T$  in  $Q_\alpha(T)$  is negative for sufficiently small  $\varepsilon$  independently on  $\alpha$ . Hence, one root  $T_2^-(\alpha)$  is negative whereas the other one  $T_2^+(\alpha)$  is positive. We remark that  $T_2^+(\alpha) > T_1(\alpha)$ , this implies that, for  $T \geq T_2^+(\alpha)$  the coefficient of  $\tilde{e}_2(0)$  is positive. It is given by

$$\alpha \frac{a_2(T - T_2^+(\alpha))(T - T_2^-(\alpha))}{1 + \alpha T}.$$

Finally, it is easy to see that, for sufficiently large  $T$  and small  $\alpha$  and  $\tilde{\varepsilon}$ , the coefficient of  $e_1(0)$  in (2.26) is positive. In fact, we set  $0 < \alpha^* < c_9(2c_{30})^{-1}$  and we define  $\hat{\varepsilon} = \min(\varepsilon^*, \bar{\varepsilon})$ , where

$$\bar{\varepsilon} = \frac{c_9 - 2\alpha^*c_{30}}{c_{29}c_9 + 2c_{30}}.$$

This implies that, for  $\alpha \in (0, \alpha^*)$  and  $\varepsilon \in (0, \hat{\varepsilon})$ , the real  $a_1 - c_{30}(\tilde{\varepsilon} + \alpha)$  is positive and we can define the real  $T_3(\alpha)$  by

$$T_3(\alpha) = \frac{c_{30}(\tilde{\varepsilon} + \alpha)}{a_1 - c_{30}(\tilde{\varepsilon} + \alpha)} > 0$$

such that for  $T > T_3$  the coefficient of  $e_1(0)$  is positive. It is given by

$$\frac{a_1(T - T_3)}{(1 + \alpha T)(1 + \alpha T_3)}.$$

The proof is thus complete.

### 3. Indirect exact controllability

In this section, we study the exact controllability of a system of two weakly coupled wave equations with locally internal control acted on only one equation. We consider the following system:

$$\begin{cases} y_1'' - \Delta y_1 + \alpha y_2 = v(t)1_\omega & \text{in } \Omega \times (0, T), \\ y_2'' - \Delta y_2 + \alpha y_1 = 0 & \text{in } \Omega \times (0, T), \\ y_1 = y_2 = 0 & \text{on } \Gamma \times (0, T), \\ y_i(0) = y_i^0, \quad y_i'(0) = y_i^1 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where  $\omega$  is a neighborhood of  $\Gamma_1$  in  $\Omega$  and  $1_\omega$  is the characteristic function of  $\omega$ . First, thanks to the direct inequality (2.2), the solution of the system (3.1) as usual by the method of transposition (see [16], [17], [12]). Let  $v_0 \in L^2(0, T; L^2(\omega))$ , we choose the control

$$v(t) = -\frac{d}{dt}v_0(t) \in H^1(0, T; L^2(\omega))', \tag{3.2}$$

where the derivative  $\frac{d}{dt}$  is not taken within the meaning of the distributions but within the meaning of the duality between  $H^1(0, T; L^2(\omega))$  and its dual  $[H^1(0, T; L^2(\omega))]'$ , i.e.

$$-\int_0^T \frac{d}{dt}v_1(t)\mu(t)dt = \int_0^T v_1(t)\frac{d}{dt}\mu(t)dt, \quad \forall \mu \in H^1(0, T; L^2(\omega)). \tag{3.3}$$

Then we have the following result.

**THEOREM 3.1.** *Let  $0 < |\alpha| < \alpha_0$ . For all*

$$Y^0 = (y_1^0, y_1^1, y_2^0, y_2^1) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$$

and  $v = -\frac{d}{dt}v_0 \in [H^1(0, T; L^2(\omega))]'$ , the controlled system (3.1) has a unique solution.

Next, we consider the indirect locally internal exact controllability problem: For given  $T > 0$  (sufficiently large) and initial data  $Y^0$ , does there exists a suitable control  $v$  that brings back the solution to equilibrium at time  $T$ , that is such that the solution of (1.4) satisfies  $y_i(T) = y_i'(T) = 0$  for  $i = 1, 2$ ? Indeed, applying the HUM method introduced by Lions (see [16], [17], [12], [13]) we obtain the following result.

**THEOREM 3.2.** *We assume the same hypothesis as in Theorem 2.1. There exists  $\alpha^* > 0$  such that for all  $0 < |\alpha| < \alpha^*$ , there exists  $T_0 = T_0(\alpha) > 0$  such that for all  $T > T_0$  and all  $Y^0 = (y_1^0, y_1^1, y_2^0, y_2^1) \in L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ , there exists  $v \in [H^1(0, T; L^2(\omega))]'$  such that the solution of the system (3.1) satisfies*

$$y_1(T) = y_1'(T) = y_2(T) = y_2'(T) = 0.$$

*Proof.* We will apply the HUM method. The idea is to seek a suitable control in the special form  $v = \partial_t(u_1')$ , where  $(u_1, u_1', u_2, u_2')$  solves the system (1.1) for some appropriate choice of the initial data  $(u_1^0, u_1^1, u_2^0, u_2^1)$ . It is sufficient to show that if  $(u_1^0, u_1^1, u_2^0, u_2^1)$  runs over an appropriate Hilbert space  $\mathcal{F}$  and if  $(\psi_1, \psi_1', \psi_2, \psi_2')$  denotes the solution (defined by the transposition method) of the following retrograde problem

$$\begin{cases} \psi_1'' - \Delta\psi_1 + \alpha\psi_2 = v(t)1_\omega & \text{in } \Omega \times (0, T), \\ \psi_2'' - \Delta\psi_2 + \alpha\psi_1 = 0 & \text{in } \Omega \times (0, T), \\ \psi_1 = \psi_2 = 0 & \text{on } \Gamma \times (0, T), \\ \psi_i(T) = \psi_i'(T) = 0, i = 1, 2 & \text{on } \Omega, \end{cases} \tag{3.4}$$

then  $\Psi = (\psi_1(0), \psi_1'(0), \psi_2(0), \psi_2'(0))$  runs over  $\mathcal{F}'$ . Indeed, then it is sufficient to choose  $v(t) = \frac{\partial}{\partial t}(u_1')$  in (3.1) with  $(u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{F}$  such that  $\Psi(0) = Y^0$ , where

the derivative  $\frac{\partial}{\partial t}$  is defined by (3.3). Equivalently, it is sufficient to show that the linear map

$$\Lambda : \mathcal{F} \rightarrow \mathcal{F}'$$

defined by the formula

$$\Lambda(u_1^0, u_1^1, u_2^0, u_2^1) = (\psi_1'(0), -\psi_1(0), \psi_2'(0), -\psi_2(0))$$

is an isomorphism. Now, Let  $U = (u_1, u_1', u_2, u_2')$  be the solution of the homogeneous problem (1.1) with the initial data  $U_0 = (u_1^0, u_1^1, u_2^0, u_2^1) \in (\mathcal{D}(\Omega))^4$ . Thanks to the observability inequalities (2.2) and (2.3) the seminorm defined by

$$\|U_0\|_{\mathcal{F}} = \left( \int_0^T \int_{\omega} |u_1'|^2 dxdt \right)^{1/2}, \tag{3.5}$$

is a norm on  $(\mathcal{D}(\Omega))^4$ . We denote by  $\mathcal{F}$  the completion of  $(\mathcal{D}(\Omega))^4$  with respect to this norm thus, we obtain an Hilbert space. Thanks to the direct and inverse observability inequalities (2.2) and (2.3), we have the following continuous and dense imbeddings:

$$\mathcal{H} \subset \mathcal{F} \subset H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega).$$

Consequently, by duality, we have the following continuous imbeddings:

$$L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \subset \mathcal{F}' \subset \mathcal{H}'.$$

On the other hand, for  $U_0 \in \mathcal{H}$ , we define the following linear form

$$\langle \Lambda U_0, \tilde{U}_0 \rangle = \int_0^T \int_{\omega} u_1' \tilde{u}_1' dxdt, \quad \forall \tilde{U}_0 \in \mathcal{H}. \tag{3.6}$$

By definition of the norm of  $\mathcal{F}$  we have the following estimate

$$|\langle \Lambda U_0, \tilde{U}_0 \rangle| \leq \|U_0\|_{\mathcal{F}} \|\tilde{U}_0\|_{\mathcal{F}}, \quad \forall U_0, \forall \tilde{U}_0 \in \mathcal{H}. \tag{3.7}$$

Hence, since  $\mathcal{H}$  is dense in  $\mathcal{F}$  by definition of  $\mathcal{F}$ , the linear map  $\Lambda U_0$  can be extended in a unique way to a continuous map on  $\mathcal{F}$  and  $\Lambda U_0 \in \mathcal{F}'$ . Moreover, using (3.7) we deduce that the linear map  $\Lambda$  that maps  $U_0 \in \mathcal{H}$  to  $\Lambda U_0 \in \mathcal{F}'$  is continuous when  $\mathcal{H}$  is equipped with the norm  $\|\cdot\|_{\mathcal{F}}$ . Hence, since  $\mathcal{H}$  is dense in  $\mathcal{F}$ , the linear map  $\Lambda$  can be extended in a unique way to a continuous linear map, still denote by  $\Lambda$ , from  $\mathcal{F}$  to  $\mathcal{F}'$ . In addition, we have

$$\langle \Lambda U_0, \tilde{U}_0 \rangle_{\mathcal{F}', \mathcal{F}} = (U_0, \tilde{U}_0)_{\mathcal{F}}, \quad \forall U_0, \tilde{U}_0 \in \mathcal{H}, \tag{3.8}$$

where  $(\cdot, \cdot)_{\mathcal{F}}$  denotes the scalar product associated with the norm  $\|\cdot\|_{\mathcal{F}}$ . The continuity of  $\Lambda$  follows from the well-posedness of the problem (1.4) and (3.4). Thanks to the time reversibility of the wave equation the well-posedness of (3.4) can be deduced from that of (1.4) by change of variable  $t \mapsto T - t$ . The coercivity of  $\Lambda$  will from

the inverse observability inequality in Theorem 2.1. Thanks to the Lax-Milgram theorem, we have that  $\Lambda$  is an isomorphism from  $\mathcal{F}$  onto  $\mathcal{F}'$ . Let  $Y^0 = (y_1^0, y_1^1, y_2^0, y_2^1) \in L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \subset \mathcal{F}'$ , and define  $\Psi(0) = Y^0$ . Then the equation

$$\Lambda(u_1^0, u_1^1, u_2^0, u_2^1) := (\psi_1'(0), -\psi_1(0), \psi_2'(0), -\psi_2(0))$$

has a unique solution  $(u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{F}$ . But, according to the nicety of the solution of the problem (3.1), we have  $y_1 = \psi_1$  and  $y_2 = \psi_2$ . Therefore

$$y_1(T) = y_{1,t}(T) = y_2(T) = y_{2,t}(T) = 0. \quad \square$$

*Remark.* These results can be generalized to other coupled equations (such as Petrowsky-Petrowsy and wave-Petrowsky).

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#### REFERENCES

- [1] F. ALABAU-BOUSSOIRA, *A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems*, SIAM J. Control Optim., **42** (2003), 871–906.
- [2] F. ALABAU-BOUSSOIRA, *Observabilité frontière indirecte de systèmes faiblement couplés*, C. R. Acad. Sci. Paris Sér. I Math., **333** (2001), 645–650.
- [3] F. ALABAU-BOUSSOIRA, *Indirect boundary stabilization of weakly coupled hyperbolic systems*, SIAM J. Control Optim., **41** (2002), 511–541.
- [4] F. ALABAU, P. CANNARSA, AND V. KOMORNIK, *Indirect internal stabilization of weakly coupled systems*, J. Evol. Equ., **2** (2002), 127–150.
- [5] F. AMMAR-KHODJAR, A. BENABDALLAH, C. DUPAIX AND I. KOSTIN, *Null-controllability of some systems of parabolic type by one control force*, ESAIM Control Optim. Calc. Var., **11** (2005), 426–448.
- [6] F. AMMAR-KHODJAR, A. BENABDALLAH AND C. DUPAIX, *Null-controllability of some reaction-diffusion systems with one control force*, J. Math. Anal. Appl., **320** (2006), 928–943.
- [7] A. BEYRATH, *Indirect internal stabilization of weakly coupled systems with locally distributed damping*, C. R. Acad. Sci. Paris Sér. I Math., **333** (2001), 451–456.
- [8] H. BREZIS, *Analyse Fonctionnelle, Théorie et Applications*, Masson, Paris, 1992.
- [9] F. CONRAD AND B. RAO, *Decay of solutions of wave equations in a star-shaped domain with nonlinear boundary feedback*, Asymptotic Anal., **7** (1993), 159–177.
- [10] T. DUYCKAERTS, *Optimal decay rates of the energy of a hyperbolic-parabolic system coupled by an interface*, Asymptot. Anal., **51** (2007), 17–45.
- [11] M. GONZÁLEZ-BURGOS AND R. PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal., **46** (2006), 123–162.
- [12] V. KOMORNIK, *Exact Controllability and Stabilization*, Masson, Paris, 1994.
- [13] V. KOMORNIK, *Rapid boundary stabilization of linear distributed systems*, SIAM J. Control Optim., **35** (1997), 1591–1613.
- [14] V. KOMORNIK AND P. LORETI, *Ingham-type theorems for vector-valued functions and observability of coupled linear systems*, SIAM J. Control Optim., **37** (1998), 461–485.
- [15] I. LASIECKA AND R. TRIGGIANI, *Carleman estimates and exact boundary controllability for a system of coupled, nonconservative second-order hyperbolic equations*, Partial differential equation methodes in control and shape analysis, Lecture Notes in Pure and Appl. Math., **188** (1997), 215–243.
- [16] J. L. LIONS, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Masson, Paris, 1988.

- [17] J.L. LIONS, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev., **30** (1988), 1–68.
- [18] J.L. Lions and E. Magenes, *Problèmes aux Limites Non-Homogènes et Applications*, Vol. 1, Dunod, Paris, 1968.
- [19] K. LIU, *Locally distributed control and damping for the conservative systems*, SIAM J. Control Optim., **35** (1997), 1574–1590.
- [20] Z. LIU AND B. RAO, *A spectral approach to the indirect boundary control of a system of weakly coupled wave equations*, Discrete Contin. Dyn. Syst., **23** (2009), 399–414.
- [21] Z. LIU AND B. RAO, *Frequency domain approach for the polynomial stability of a system of partially damped wave equations*, J. Math. Anal. Appl., **335** (2007), 860–881.
- [22] P. LORETI AND B. RAO, *Optimal energy decay rate for partially damped systems by spectral compensation*, SIAM J. Control Optim., **45** (2006), 1612–1632.
- [23] P. MARTINEZ, *A new method to obtain decay rate estimates for dissipative systems with localized damping*, Rev. Mat. Complut., **12** (1999), 251–283.
- [24] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, **44**, Springer-Verlag, 1983.
- [25] R. RAJARAM, M. NAJAFI, *Exact controllability of wave equations in  $\mathbb{R}^n$  coupled in parallel*, J. Math. Anal. Appl., **356** (2009), 7–12.
- [26] D. L. RUSSELL, *A General framework for the study of indirect damping mechanisms in elastic systems*, J. Math. Anal. Appl., **173** (1993), 339–358.
- [27] D. RUSSELL, *Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods*, J. Differential Equations, **19**, 2 (1975), 344–370.
- [28] A. SOUFYANE AND A. WEHBE, *Uniform stabilization for the Timoshenko beam by a locally distributed damping*, Electron. J. Differential Equations, **29** (2003), 1–14.
- [29] A. WEHBE, W. YOUSSEF, *Observabilité et contrôlabilité exacte indirecte interne par un contrôle localement distribué de systèmes d'équations couplées*, C. R. Math. Acad. Sci. Paris, **348** (2010), 1169–1173.
- [30] A. WEHBE AND W. YOUSSEF, *Stabilization of the uniform Timoshenko beam by one locally distributed feedback*, Appl. Anal., **88**, 7 (2009), 1067–1078.
- [31] A. WEHBE AND W. YOUSSEF, *Exponential and polynomial stability of an elastic Bresse system with two locally distributed feedback*, J. Math. Phys., **51**, 9 (2010), 17 pp.
- [32] X. ZHANG AND E. ZUAZUA, *Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system*, J. Differential Equations, **204** (2004), 380–438.

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