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A CONCAVE–CONVEX QUASILINEAR ELLIPTIC PROBLEM SUBJECT TO A NONLINEAR BOUNDARY CONDITION

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Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

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Abstract. This paper deals with the existence of a positive solution to the problem

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{r-1}, & x \in \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda u^{q-1}, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ν designates the unit outward normal to $\partial\Omega$, Δ_p is the p-Laplacian operator, $1 < q < p < r \leqslant p^*$, $p^* = Np/(N-p)$ if $p < N$, $p^* = \infty$ otherwise, while $\lambda > 0$. Our main result states the existence of $\Lambda > 0$ so that positive solutions are only possible when $0 < \lambda \leqslant \Lambda$ while the existence of a *minimal* positive solution is ensured in that range.

1. Introduction

This paper is concerned with the existence of a positive solution to the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{r-1}, & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda u^{q-1}, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ whose outward unit normal field at $\partial\Omega$ is designated by ν, Δ_p stands for the p-Laplacian operator defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ in distributional sense when acting on functions $u \in W^{1,p}(\Omega)$, and λ is regarded as a *positive* parameter. Concerning the different exponents in (1.1) it will be assumed that

$$1 < q < p < r \leqslant p^*, \tag{1.2}$$

where $p^* = Np/(N-p)$ if $1 < p < N$ (the Sobolev conjugated of p) while $p^* = \infty$ otherwise. It will be always assumed in the present work that $1 < p < N$, which is just the case worthy of study.

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The linear diffusion version of (1.1), corresponding to $p = 2$, was thoroughly studied in [10] including the issues of multiplicity of positive solutions and existence of infinitely many signed solutions. In that case observe that (1.2) reads

$$0 < q - 1 < 1 < r - 1 \leqslant 2^* - 1. \tag{1.3}$$

Thus, (1.1) couples a "convex" reaction in Ω to a "concave" source on $\partial\Omega$. The framework of the present job is reminiscent of that scenario. Here we provide a partial extension of the results of [10] to the p-Laplacian operator. On the other hand, both problem (1.1) and its linear counterpart $p = 2$ fall in the realm of diffusion problems under nonlinear boundary conditions. We refer to [17] for a general review on this active area.

The interest in the literature on concave-convex reaction-diffusion equations goes back at least to the pioneering work [1] where the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{r-1}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

q, r satisfying (1.3), $\lambda > 0$, was studied in detail. In one of the main results, the existence of a $\Lambda^* > 0$ is shown so that a positive solution to (1.4) is only possible when $0 < \lambda \leqslant \Lambda^*$. More importantly, authors succeed in getting the existence of two positive solutions for all $0 < \lambda < \Lambda^*$. This fact was later coined as "global multiplicity" of positive solutions for problem (1.4) (see [9]).

In the case $\Omega = B$ a ball of $\mathbb{R}^N$ and radially symmetric solutions, results in [1] were extended in [2] to the p-Laplacian version of (1.4). Namely,

$$\begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{r-1}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where p, q, r now satisfies (1.2). Global multiplicity of positive solutions for (1.5), when observed in a general domain Ω, was finally achieved in [9] after a hard-working to achieve $C^{1,\alpha}$ estimates in an auxiliary associated problem (see Section 5).

For immediate use in the present work it is handy to set some notation and definitions. For $s > 1$, function

$$\varphi_s(t) = |t|^{s-2}t$$

will designate the odd extension to $\mathbb{R}$ of the power function t^{s-1}. Thus, problem (1.1) is more properly written as

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \varphi_r(u), & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda \varphi_q(u), & x \in \partial\Omega. \end{cases} \tag{1.6}$$

By a weak solution $u \in W^{1,p}(\Omega)$ to (1.6) it is understood that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \varphi_p(u) v = \lambda \int_{\partial\Omega} \varphi_q(u) v + \int_\Omega \varphi_r(u) v,$$

holds for all $v \in W^{1,p}(\Omega)$. It also convenient to introduce the energy functional E_λ associated to (2.1). Namely,

$$E_\lambda(u) = \int_\Omega \frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q - \frac{1}{r}\int_\Omega |u|^r,$$

with $u \in W^{1,p}(\Omega)$.

Our main result is the following.

THEOREM 1.1. *There exists a positive number Λ such that:*

i) *problem* (1.1) *does not admit positive solutions for $\lambda > \Lambda$;*

ii) *for every $0 < \lambda < \Lambda$ problem* (1.1) *exhibits a* minimal *positive solution $u_\lambda \in C^{1,\beta}(\overline{\Omega})$ for a certain $\beta \in (0,1)$; moreover, there exists $M > 0$ such that*

$$\|u_\lambda\|_{C^{1,\beta}(\overline{\Omega})} \leqslant M, \tag{1.7}$$

for $0 < \lambda < \Lambda$; furthermore,

$$u_\lambda \to 0, \tag{1.8}$$

in $C^{1,\beta}(\overline{\Omega})$ as $\lambda \to 0$;

iii) *minimal solution u_λ is increasing with respect to $\lambda \in (0,\Lambda)$ and satisfies satisfy $E_\lambda(u_\lambda) < 0$; moreover,*

$$u_\lambda \to u^*$$

in $W^{1,p}(\Omega)$ as $\lambda \to \Lambda$ and $u^* \in W^{1,p}(\Omega)$ defines a positive solution to (1.1) *corresponding to $\lambda = \Lambda$; in addition u_λ is continuous from the left with respect to $\lambda \in (0,\Lambda]$ and u^* defines the minimal solution to* (1.1) *for $\lambda = \Lambda$.*

It follows from Theorem 1.1 that $\lambda = 0$ and $\lambda = \Lambda$ can be regarded as bifurcation values of λ with respect to the existence of positive solutions to (1.1).

This work pretends to be as self-contained as possible. All auxiliary results are shown in full detail in Section 2. Specially, the L^∞ character and further smoothness properties of weak solutions to (1.6). Section 3 is devoted to the proof of Theorem 1.1. A variant of problem (1.1) is completely studied in Section 4. Finally, some insights on the problem of the global multiplicity of positive solutions to (1.1) are contained in Section 5.

2. Preliminary results

For our purposes in what follows it is convenient to introduce the auxiliary problem,

$$\begin{cases} -\Delta_p u + \varphi_p(u) = 0, & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda\varphi_q(u), & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $1 < q < p$. Next result depicts the main features of (2.1).

THEOREM 2.1. *Problem* (2.1) *admits positive solutions only when* $\lambda > 0$. *Moreover, for every positive* λ *it possesses a unique positive weak solution* $u = \tilde{u}_\lambda \in C^{1,\alpha}(\overline{\Omega})$ *for certain* $0 < \alpha < 1$, $\tilde{u}_\lambda > 0$ *in* $\overline{\Omega}$, *where*

$$\tilde{u}_\lambda = \lambda^{\frac{1}{p-q}} \tilde{u}_1. \tag{2.2}$$

Proof. Direct integration shows that no positive solutions are possible if $\lambda \leqslant 0$. Since the scaling (2.2) reduces (2.1) to the case $\lambda = 1$ it suffices with proving that (2.1) with $\lambda = 1$ admits a unique positive solution.

To get the existence consider the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{p}\int_\Omega |u|^p,$$

just defined on the manifold $\mathscr{M} = \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^q = 1\}$. Due to the compactness of the embedding $W^{1,p}(\Omega) \subset L^q(\partial\Omega)$ it follows that $\mathscr{M}$ is weakly closed in $W^{1,p}(\Omega)$. Since J is coercive in $\mathscr{M}$, an absolute minimizer $\hat{u} \in \mathscr{M}$ of J exists, that can be chosen positive (if necessary, replace $\hat{u}$ with $|\hat{u}|$). Then, Lagrange's multiplier rule and a scaling leads to the existence of a nonnegative weak solution $u \in W^{1,p}(\Omega)$, $u \neq 0$, to (2.1) with $\lambda = 1$.

That u is a weak solution to $(2.1)_{\lambda=1}$ means of course that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \varphi_p(u)v = \int_{\partial\Omega} \varphi_q(u)v \quad \text{for all } v \in W^{1,p}(\Omega). \tag{2.3}$$

Then, Lemma 2.2 below says that $u \in L^\infty(\Omega)$. Therefore, by noticing that $|u|^{q-1} \leqslant 1 + |u|^{p-1}$ it follows from the regularity results in [15] that $u \in C^{1,\alpha}(\overline{\Omega})$ for a certain $\alpha \in (0,1)$.

On the other hand, strong maximum principle in [19] implies that every nontrivial and nonnegative weak solution u to (2.1) becomes strictly positive in the whole of $\overline{\Omega}$.

Let us finally show the uniqueness of positive solutions to $(2.1)_{\lambda=1}$. We employ the customary notation

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v, \quad u, v \in W^{1,p}(\Omega).$$

Accordingly, suppose $u_1, u_2 \in W^{1,p}(\Omega)$ are positive solutions to (2.1). Then, it follows from [16] (see also [7] in the case of a related Dirichlet problem) that

$$\begin{aligned}\Big\langle -\frac{\Delta_p u_1}{u_1^{p-1}} + \frac{\Delta_p u_2}{u_2^{p-1}}, u_1^p - u_2^p\Big\rangle \\ = \int_\Omega \{|\nabla u_1|^{p-2}\nabla u_1 \nabla v_1 - |\nabla u_2|^{p-2}\nabla u_2 \nabla v_2\} \geqslant 0,\end{aligned} \tag{2.4}$$

where

$$v_1 = \frac{u_1^p - u_2^p}{u_1^{p-1}} \quad \text{and} \quad v_2 = \frac{u_1^p - u_2^p}{u_2^{p-1}}.$$

Moreover, equality holds in (2.4) only when u_1 is a scalar multiple of u_2. On the other hand, if we set $v = v_i$, $i = 1,2$, as a test function for the integral equality (2.3) satisfied by u_i as a weak solution to (2.1), then by subtracting the resulting relations we find that

$$\int_\Omega \{|\nabla u_1|^{p-2}\nabla u_1 \nabla v_1 - |\nabla u_2|^{p-2}\nabla u_2 \nabla v_2\} \\ = \int_{\partial\Omega} \left(\frac{1}{u_1^{p-q}} - \frac{1}{u_2^{p-q}}\right)(u_1^p - u_2^p). \tag{2.5}$$

However, last integral becomes negative if $u_1 \neq u_2$. Since this contradicts (2.4), u_1 must coincide with u_2 and the uniqueness is achieved. □

The forthcoming result has to do with a further auxiliary problem stating the existence of a principal eigenvalue for a suitable "eigenvalue type" problem.

THEOREM 2.2. *Assume that* $a \in L^\infty(\Omega)$ *satisfies* $a(x) \geqslant a_0 > 0$ *a.e. in* Ω *while* $b \in L^\infty(\partial\Omega)$. *Then, the eigenvalue problem*

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \mu a(x)\varphi_p(u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = b(x)\varphi_p(u), & x \in \partial\Omega, \end{cases} \tag{2.6}$$

admits a unique *eigenvalue* $\mu = \mu_1$ *with the property of having a nonnegative eigenfunction. Moreover,* μ_1 *is the minimum eigenvalue to* (2.6) *and is simple in the sense that all eigenfunctions* ϕ *associated to* μ_1 *are a scalar multiple of a fixed normalized positive eigenfunction* ϕ_1.

Proof. A pair $(\mu, u) \in \mathbb{R} \times W^{1,p}(\Omega)$ is said to be an eigenpair (μ an eigenvalue with associated eigenfunction u) if $u \neq 0$ satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \varphi_p(u)v = \mu \int_\Omega a(x)\varphi_p(u)v + \int_{\partial\Omega} b(x)\varphi_p(u)v, \tag{2.7}$$

for all $v \in W^{1,p}(\Omega)$.

To achieve the existence of μ_1 we proceed in a standard way by looking for a minimizer of

$$J_1(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{p}\int_\Omega |u|^p - \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p,$$

on $\mathscr{M}_1 = \{u \in W^{1,p}(\Omega) : \int_\Omega a|u|^p = 1\}$ which is also a weakly closed part of $W^{1,p}(\Omega)$. By taking into account the inequality (see [11] for the case $p = 2$)

$$\int_{\partial\Omega} |u|^p \leqslant \varepsilon \int_\Omega |\nabla u|^p + C_\varepsilon \int_\Omega |u|^p,$$

which holds, for a fixed ε, for every $u \in W^{1,p}(\Omega)$ with a constant C_ε only depending on ε, it is easily found that J_1 is coercive on $\mathscr{M}_1$.

Thus, there exists a global minimizer $\phi \in \mathscr{M}_1$ of J_1 so that $\mu_1 := J_1(\phi)$ satisfies

$$\mu_1 = \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p - \int_{\partial\Omega} b(x)|u|^p}{\int_\Omega a|u|^p}.$$

Setting $\phi_1 = |\phi|$ we observe that $J(\phi_1) = \mu_1$ and so ϕ_1 defines a nonnegative eigenfunction associated to μ_1. Additionally, it is clear that μ_1 constitutes the minimum eigenvalue to (2.6).

According to Lemma 2.2 below any associated eigenfunction ϕ to μ_1 lies in $L^\infty(\Omega)$. Thus, $\phi \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ (cf. [15]) and every nonnegative eigenfunction ϕ associated to μ_1 becomes positive in $\overline{\Omega}$ ([19]).

We show now that any eigenfunction ϕ associated to μ_1 is either positive or negative in $\overline{\Omega}$. In fact, if we assume say that $\phi^+ \neq 0$, then by using $v = \phi^+$ as a test function in (2.7) we achieve $J(\phi^+) = \mu_1$. This means that ϕ^+ is an eigenfunction and so $\phi > 0$ in $\overline{\Omega}$. Similarly, one gets $\phi < 0$ in $\overline{\Omega}$ provided $\phi^- \neq 0$.

To show the simplicity we choose arbitrary eigenfunctions ϕ_1, ϕ_2 which can be both taken positive. By using equation (2.7) firstly for ϕ_1 with test function $v_1 = (\phi_1^p - \phi_2^p)/\phi_1^{p-1}$, and secondly for ϕ_2 and test function $v_2 = (\phi_1^p - \phi_2^p)/\phi_2^{p-1}$, then by subtracting the resulting equalities we achieve that

$$\int_\Omega \{|\nabla\phi_1|^{p-2}\nabla\phi_1\nabla v_1 - |\nabla\phi_2|^{p-2}\nabla\phi_2\nabla v_2\} = 0.$$

As already observed in the proof of Theorem 2.1 this implies that ϕ_1 and ϕ_2 are proportional ([16]). Hence, μ_1 is simple.

The proof of the fact that μ_1 is the unique eigenvalue with a nonnegative eigenfunction fits entirely the pattern of the corresponding one for Theorem 1 in [12] and so it is omitted. □

Our immediate goal consists in showing that weak solutions $u \in W^{1,p}(\Omega)$ to problem (1.6) are bounded provided r, q fall in the regime (1.2). We proceed in two steps. The first one is reminiscent of a well known result in [4] (cf. [18]).

LEMMA 2.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the growth conditions*

$$|f(x,u)| \leqslant a(x)(1 + |u|^{p-1}) \ \ a.e.\ x \in \Omega,\ u \in \mathbb{R},$$
$$|g(x,u)| \leqslant b(x)(1 + |u|^{p-1}) \ \ a.e.\ x \in \partial\Omega,\ u \in \mathbb{R},$$

with $a \in L^{\frac{N}{p}}(\Omega)$, $b \in L^{\frac{N-1}{p-1}}(\partial\Omega)$. Then, every weak solution $u \in W^{1,p}(\Omega)$ to

$$\begin{cases} -\Delta_p u = f(x,u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = g(x,u), & x \in \partial\Omega, \end{cases} \tag{2.8}$$

belongs to $L^q(\Omega) \cap L^q(\partial\Omega)$ for all $q \geqslant 1$.

Proof. We are showing that $|u|^{s+1} \in W^{1,p}(\Omega)$ under the assumption that $|u|^{p(s+1)} \in L^1(\Omega) \cap L^1(\partial\Omega)$. Since the latter fact holds for $s=0$ and we are of course supposing that $p<N$, we get that $u \in L^{q_m}(\Omega) \cap L^{\hat{q}_m}(\partial\Omega)$ for all $m \in \mathbb{N}$ with

$$q_m = \left(\frac{N}{N-p}\right)^m \qquad \text{and} \qquad \hat{q}_m = \left(\frac{N-1}{N-p}\right)^m,$$

what shows the Lemma. To prove the former assertion we choose, as usual,

$$v = u(|u| \wedge L)^{ps},$$

as a test function in the weak formulation of (2.8), where $L>0$ and $|u| \wedge L := \min\{|u|, L\}$. Then, we get

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v &\leqslant \int_\Omega a(1+|u|^{p-1})|v| + \int_{\partial\Omega} b(1+|u|^{p-1})|v| \\ &\leqslant c + 4\int_\Omega a\{|u|(|u| \wedge L)^s\}^p + 4\int_{\partial\Omega} b\{|u|(|u| \wedge L)^s\}^p := A, \end{aligned} \tag{2.9}$$

where constant c depends on $\|a\|_{L^1(\Omega)}$ and $\|b\|_{L^1(\partial\Omega)}$. We now choose $K>0$ and continuing the estimate in (2.9) we obtain

$$A \leqslant c + 4\int_{\{a \geqslant K\}} a\{|u|(|u| \wedge L)^s\}^p + 4\int_{\{b \geqslant K\}} b\{|u|(|u| \wedge L)^s\}^p, \tag{2.10}$$

c depending now in addition on the norms $\|u\|_{L^{p(s+1)}(\Omega)}$ and $\|u\|_{L^{p(s+1)}(\partial\Omega)}$. Taking into account the embeddings $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $W^{1,p}(\Omega) \subset L^{p_*}(\partial\Omega)$, where $p_* = p(N-1)/(N-p)$, we observe that

$$\begin{aligned} \int_{\{a \geqslant K\}} a\{|u|(|u| \wedge L)^s\}^p &\leqslant \left(\int_{\{a \geqslant K\}} a^{\frac{N}{p}}\right)^{\frac{p}{N}} \left(\int_\Omega \{|u|(|u| \wedge L)^s\}^{p^*}\right)^{\frac{p}{p^*}} \\ &\leqslant c + C\|a\|_{L^{\frac{N}{p}}(\{a \geqslant K\})} \|\nabla(|u|(|u| \wedge L)^s)\|^p_{L^p(\Omega)}, \end{aligned}$$

and similarly,

$$\begin{aligned} \int_{\{b \geqslant K\}} b\{|u|(|u| \wedge L)^s\}^p &\leqslant \left(\int_{\{b \geqslant K\}} b^{\frac{N-1}{p-1}}\right)^{\frac{p-1}{N-1}} \left(\int_{\partial\Omega} \{|u|(|u| \wedge L)^s\}^{p_*}\right)^{\frac{p}{p_*}} \\ &\leqslant c + C\|b\|_{L^{\frac{N-1}{p-1}}(\{a \geqslant K\})} \|\nabla(|u|(|u| \wedge L)^s)\|^p_{L^p(\Omega)}, \end{aligned}$$

where constant c has the same status as in (2.10).

By combining the latter estimates with (2.10) we arrive to

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \leqslant c + \varepsilon \|\nabla(|u|(|u| \wedge L)^s)\|^p_{L^p(\Omega)}, \tag{2.11}$$

where ε can be chosen as small as desired by choosing K large enough, and constant c does not depend on L. On the other hand we easily find that

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \geqslant \| \, (|u|\wedge L)^s \nabla |u| \, \|^p_{L^p(\Omega)} \\ + \frac{p}{s^{p-1}} \| \, |u| \nabla ((|u|\wedge L)^s) \, \|^p_{L^p(\Omega)} \\ \geqslant \eta \| \nabla (|u|(|u|\wedge L)^s)\|^p_{L^p(\Omega)},\end{aligned}$$

with a non depending on L constant η. This estimate, together with (2.11) ensure us that $|u|(|u|\wedge L)^s$ is bounded in $W^{1,p}(\Omega)$ as $L\to\infty$. Therefore, $|u|^{s+1}\in W^{1,p}(\Omega)$. This finishes the proof. □

LEMMA 2.2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain, while $p<N$ together with*

$$1<r\leqslant p^* \quad \text{and} \quad 1<q\leqslant p_*, \tag{2.12}$$

where $p_* = (N-1)/(N-p)$. Then, every possible weak solution $u\in W^{1,p}(\Omega)$ to problem (1.6) *satisfies $u\in L^\infty(\Omega)$.*

Proof. As a first remark, observe that we fall under the assumptions of Lemma 2.1 since

$$|u|^{r-1}\leqslant a(x)(1+|u|^{p-1}), \qquad |u|^{q-1}\leqslant b(x)(1+|u|^{p-1})$$

where $a(x)=\max\{1,|u(x)|^{p^*-p}\}$ and $b(x)=\max\{1,|u(x)|^{p_*-p}\}$. Thus $a\in L^{\frac{N}{p}}(\Omega)$, $b\in L^{\frac{N-1}{N-p}}(\partial\Omega)$ and then, it follows from Lemma 2.1 that $u\in L^q(\Omega)\cap L^q(\partial\Omega)$ for all $q>1$.

We show now that $u^+\in L^\infty(\Omega)$. By employing Lemma 5.1 of Chapter 2 in [14] it suffices with getting an estimate of the form

$$\int_\Omega (u-k)^+\leqslant C|A_k|^{1+\varepsilon}, \tag{2.13}$$

for all $k>0$ large and certain constant C, where $A_k=\{x\in\Omega : u\geqslant k\}$. To this purpose if $u\in W^{1,p}(\Omega)$ is a weak solution to (1.6) we set $v=(u-k)^+$ and use it as a test function. Thus we obtain

$$\int_\Omega |\nabla v|^p+|v|^p\leqslant c\left\{\int_\Omega |u|^{r-1}v+\int_{\partial\Omega}|u|^{q-1}v\right\}, \tag{2.14}$$

where in the sequel c will stand for a certain constant whose explicit value is not relevant for the proof. We now regard $|u|^{r-1}$ as a function in $L^{s_1}(\Omega)$ and $|u|^{q-1}$ as belonging to $L^{s_1}(\partial\Omega)$ where $s_1>1$ is chosen so large as to have,

$$\frac{1}{s_1}\frac{1}{p-1}<\frac{1}{p}-\frac{1}{p^*}. \tag{2.15}$$

By setting $s=s_1/(s_1-1)$ we deduce from (2.14),

$$\|v\|^p_{W^{1,p}(\Omega)}\leqslant c\{\|v\|_{L^s(\Omega)}+\|v\|_{W^{1,s}(\Omega)}\}$$

$$\leqslant c\left\{|A_k|^{\frac{1}{s}-\frac{1}{p^*}}+|A_k|^{\frac{1}{s}-\frac{1}{p}}\right\}\|v\|_{W^{1,p}(\Omega)}$$
$$\leqslant c|A_k|^{\frac{1}{s}-\frac{1}{p}}\|v\|_{W^{1,p}(\Omega)}, \tag{2.16}$$

since $|A_k|\to 0$ as $k\to\infty$. Thus,

$$\|v\|_{W^{1,p}(\Omega)}\leqslant c|A_k|^{\frac{1}{p-1}\left(\frac{1}{s}-\frac{1}{p}\right)},$$

for $k\geqslant k_0$ and certain positive constants k_0, c.

On the other hand,

$$\int_\Omega v\leqslant |A_k|^{1-\frac{1}{p^*}}\|v\|_{W^{1,p}(\Omega)}\leqslant c|A_k|^{1-\frac{1}{p^*}+\frac{1}{p}-\frac{1}{s_1(p-1)}}.$$

Taking into account (2.15) the proof of relation (2.13) is completed and so $u^+\in L^\infty(\Omega)$. That $u^-\in L^\infty(\Omega)$ is shown in a entirely similar way. Thus, the proof of the Lemma is finished. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We proceed separately on each of the several statements of the theorem.

PROOF OF THEOREM 1.1–i). We are showing the existence of a positive number Λ^* so that no nonnegative nontrivial solutions to (1.1) exist for $\lambda>\Lambda^*$.

Assume $u\in W^{1,p}(\Omega)$, $u\geqslant 0$, is a nontrivial solution to (1.1). Then, in view of Lemma 2.2 one has that $u\in L^\infty(\Omega)$ while Theorem 2 in [15] implies that $u\in C^{1,\beta}(\overline{\Omega})$ for a certain $0<\beta<1$. In addition, it follows from the strong maximum principle ([19]) that $u>0$ in $\overline{\Omega}$.

On the other hand $\overline{u}=u$ defines a strict positive supersolution to (2.1). Since the solution $\tilde{u}_\lambda$ to (2.1) satisfies $\tilde{u}_\lambda\to 0$ in $C^1(\overline{\Omega})$ as $\lambda\to 0+$, some $0<\hat{\lambda}\leqslant\lambda$ can be found so that $\tilde{u}_{\hat{\lambda}}\leqslant u$ in Ω. Therefore, $\underline{u}=\tilde{u}_{\hat{\lambda}}$ constitutes a subsolution to (2.1) comparable with $\underline{u}=u$. Thus, we have shown that every nonnegative and nontrivial solution $u\in W^{1,p}(\Omega)$ to (1.1) must satisfy

$$\tilde{u}_\lambda(x)\leqslant u(x),\ x\in\Omega. \tag{3.1}$$

We can now consider u as solving the auxiliary eigenvalue problem (2.6) under the choices $\mu=1$, $a=u^{r-p}$ and $b=\lambda u^{q-p}$. The uniqueness assertion concerning the main eigenvalue to (2.6) in Theorem 2.2 shows that $\mu_1=1$ is the first eigenvalue to (2.6) meanwhile, the variational characterization of μ_1 entails that

$$1\leqslant\frac{\int_\Omega|\nabla w|^p+|w|^p-\int b|w|^p}{\int_\Omega a|w|^p}, \tag{3.2}$$

for every $w\in W^{1,p}(\Omega)$. By combining this relation with (3.1) and (2.2) we conclude that

$$\lambda^{\frac{r-p}{p-q}}\leqslant\frac{\int_\Omega|\nabla w|^p+|w|^p}{\int_\Omega|\tilde{u}_1|^{r-p}|w|^p},$$

for all $w \in W^{1,p}(\Omega)$. Therefore,

$$\lambda \leqslant \mu^{*\frac{p-q}{r-p}}, \tag{3.3}$$

where $\mu = \mu^*$ is the first eigenvalue to the eigenvalue type problem,

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \mu |\tilde{u}_1|^{r-p} \varphi_p(u), & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.4}$$

The existence, uniqueness and positiveness of such an eigenvalue are a consequence of Theorem 2.2.

Estimate (3.3) shows that the set

$$\mathscr{S} := \{\lambda > 0 : (1.1) \text{ admits a positive solution}\} \tag{3.5}$$

is bounded above. In other words, the existence of the announced value Λ^*. We are next showing that an optimum election Λ of Λ^* can be performed. □

PROOF OF THEOREM 1.1)–ii). We begin by showing the existence of a small $\bar{\lambda}$ so that a positive solution to (1.1) exists for all $0 < \lambda \leqslant \bar{\lambda}$. We are employing sub and supersolutions (see [12] for and adaptation of such technique to problems in the format (2.8)).

First observe that for all $\lambda > 0$, problem (1.1) possesses subsolutions which are as small as desired. In fact $\underline{u} = \tilde{u}_{\lambda'}$ is a subsolution to (1.1) for all $0 < \lambda' \leqslant \lambda$. On the other hand, this fact together with estimate (3.1) entail that, whenever (1.1) admits a positive solution $u \in W^{1,p}(\Omega)$, then a minimal positive solution u_λ exists and so

$$u_\lambda \leqslant u.$$

In fact, one can take $\underline{u} = \tilde{u}_\lambda$ and $\overline{u} = u$ as a comparable pair of sub and supersolutions to (1.1) and then u_λ is the limit in $C^1(\overline{\Omega})$ of the sequence $u_n \in W^{1,p}(\Omega)$, being $v = u_n$ the solution to the scheme

$$\begin{cases} -\Delta_p v + \varphi_p(v) = u_{n-1}^{r-1}, & x \in \Omega, \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu} = \lambda u_{n-1}^{q-1}, & x \in \partial\Omega, \end{cases} \tag{3.6}$$

with $u_0 = \underline{u}$.

To get a supersolution we consider the parameterized eigenvalue problem (see [11] for related ideas)

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \theta \varphi_p(u), & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda \varphi_p(u), & x \in \partial\Omega, \end{cases} \tag{3.7}$$

whose principal eigenvalue $\theta = \theta(\lambda)$ is furnished by, say, Theorem 2.2. Proceeding by arguments similar to those in Lemma 8 in [11] for the linear case $p = 2$, it can be shown that $\theta(\lambda)$ is concave, decreasing, $\theta(0) > 0$ ($\theta(0)$ being the first Neumann eigenvalue for $-\Delta_p + \varphi_p(\cdot)$ in Ω) and $\theta(\lambda_1) = 0$ for a unique positive $\lambda = \lambda_1$ (the first Steklov eigenvalue for $-\Delta_p + \varphi_p(\cdot)$ in Ω).

Choose $0 < \lambda_0 < \lambda_1$ and set $\phi_1 := \phi_1(\lambda_0)$ the positive eigenfunction associated to $\theta_0 := \theta(\lambda_0)$ which is normalized so that $\sup_\Omega \phi_1 = 1$. Then,

$$\overline{u} = A\phi_1, \ \ A = \theta_0^{\frac{1}{r-p}},$$

defines a supersolution to the equation in (1.1). On the other hand,

$$|\nabla\overline{u}|^{p-1}\frac{\partial\overline{u}}{\partial\nu} \geqslant \lambda|\overline{u}|^{q-1}, \ \ 0 < \lambda \leqslant \bar{\lambda},$$

where

$$\bar{\lambda} = \lambda_0\theta_0^{\frac{p-q}{r-p}} \inf_{\partial\Omega} \phi_1^{p-q}.$$

Therefore, (1.1) admits a positive solution for all $0 < \lambda \leqslant \bar{\lambda}$. In particular, the minimal positive solution u_λ satisfies

$$u_\lambda \leqslant \overline{u} = \theta_0^{\frac{1}{r-p}}\phi_1.$$

We notice now that choosing $\lambda_{0n} \to \lambda_1$ then $\theta_{0n} \to 0$ and the corresponding $\bar{\lambda}_n \to 0$. Therefore,

$$u_\lambda \to 0,$$

uniformly in $\overline{\Omega}$ as $\lambda \to 0$ what shows (1.8).

At this level, we can already assert that the set $\mathscr{S}$ in (3.5) is nonempty. In fact, in view of part i) $\Lambda = \sup\mathscr{S}$ is finite meanwhile, by the previous remarks $(0,\Lambda) \subset \mathscr{S}$. This completes the proof of ii). $\square$

PROOF OF THEOREM 1.1–iii). That the minimal solution u_λ is increasing with λ follows from the fact that $u_{\lambda'}$ defines a supersolution to (1.1) for values λ lesser than λ' and the existence of subsolutions to (1.1) with an amplitude as small as desired.

We define,

$$u^*(x) = \lim_{\lambda\to\Lambda} u_\lambda(x)$$

and are first showing that $u^* \in W^{1,p}(\Omega)$. To this purpose it is enough with proving that u_λ keeps bounded in $W^{1,p}(\Omega)$ as $\lambda \to \Lambda$. Define

$$\widetilde{E}_\lambda(u) = \frac{1}{p}\int_\Omega\{|\nabla u|^p + |u|^p\} - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q, \quad u \in W^{1,p}(\Omega),$$

the energy functional associated to the auxiliary problem (2.1). Since the solution $\tilde{u}_\lambda$ to (2.1) satisfies,

$$\int_\Omega\{|\nabla\tilde{u}_\lambda|^p + |\tilde{u}_\lambda|^p\} = \lambda\int_{\partial\Omega}|\tilde{u}_\lambda|^q.$$

Then,

$$\widetilde{E}_\lambda(\tilde{u}_\lambda) = \left(\frac{\lambda}{p} - \frac{\lambda}{q}\right)\int_{\partial\Omega}|\tilde{u}_\lambda|^q < 0,$$

for every $\lambda > 0$. Thus,

$$\begin{aligned} E_\lambda(\tilde{u}_\lambda) &= \frac{1}{p}\int_\Omega \{|\nabla \tilde{u}_\lambda|^p + |\tilde{u}_\lambda|^p\} - \frac{\lambda}{q}\int_{\partial\Omega} |\tilde{u}_\lambda|^q - \frac{1}{r}\int_\Omega |\tilde{u}_\lambda|^r \\ &= \widetilde{E}_\lambda(\tilde{u}_\lambda) - \frac{1}{r}\int_\Omega |\tilde{u}_\lambda|^r < 0, \end{aligned} \tag{3.8}$$

for every $\lambda > 0$. If we define

$$H = \{u \in W^{1,p}(\Omega) : \tilde{u}_\lambda \leqslant u(x) \leqslant u_\lambda(x) \quad \text{a. e. in } \Omega\},$$

then we know from the variational version of the method of sub and super solutions provided in [12] that

$$E_\lambda(u_\lambda) = \inf_{v \in H} E(v). \tag{3.9}$$

Hence,

$$E_\lambda(u_\lambda) < 0 \quad \text{for all } 0 < \lambda < \Lambda. \tag{3.10}$$

By taking into account (3.10) together with

$$\|u_\lambda\|^p_{W^{1,p}(\Omega)} = \int_\Omega \{|\nabla u_\lambda|^p + |u_\lambda|^p\} = \lambda \int_{\partial\Omega} |u_\lambda|^q + \int_\Omega |u_\lambda|^r,$$

then we achieve

$$\left(\frac{1}{p} - \frac{1}{r}\right)\int_\Omega |u_\lambda|^r \leqslant \left(\frac{\lambda}{q} - \frac{\lambda}{p}\right)\int_{\partial\Omega} |u_\lambda|^q.$$

Thus, there exists a constant $C > 0$ such that

$$\|u_\lambda\|^p_{W^{1,p}(\Omega)} \leqslant C\int_{\partial\Omega} |u_\lambda|^q \leqslant C|\partial\Omega|^{1-\frac{1}{p}}_{N-1}\|u_\lambda\|^q_{L^p(\partial\Omega)} \leqslant C|\partial\Omega|^{1-\frac{1}{p}}_{N-1}\|u_\lambda\|^q_{W^{1,p}(\Omega)}$$

for $0 < \lambda < \Lambda$ (in previous relations C stands for a generic constant whose precise value is irrelevant for the proof). This implies that $\|u_\lambda\|_{W^{1,p}(\Omega)} \leqslant M$ for a certain $M > 0$ as $\lambda \to \Lambda$.

Since there exists $\lambda_n \to \Lambda$ so that $u_{\lambda_n} \to u^*$ weakly in $W^{1,p}(\Omega)$ we find that $u^* \in W^{1,p}(\Omega)$ (in particular, $u^*(x)$ keeps finite a. e. in Ω). Furthermore, since for every $\lambda_n \to \Lambda$ it is possible to extract a subsequence $\lambda_{n'}$ such that $u_{\lambda_{n'}} \to u^*$ weakly in $W^{1,p}(\Omega)$, then we conclude that the whole $u_\lambda \to u^*$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega)$. Thus $u_\lambda \to u^*$ strongly in $L^q(\partial\Omega)$ (q satisfying (1.2)).

We are further showing that $u_\lambda \to u^*$ in $L^r(\Omega)$ as $\lambda \to \Lambda$. This is in fact clear if $r < p^*$ due to Sobolev's embedding. In the case of $r = p^*$ we observe that

$$\int_\Omega u_\lambda^{p^*} \leqslant \|u_\lambda\|^p_{W^{1,p}(\Omega)} \leqslant M^p.$$

Thus, by monotone convergence, $\|u^*\|_{L^{p^*}(\Omega)} \leqslant M^{p/p^*}$. Moreover, that $u^* \in L^{p^*}(\Omega)$ permits concluding in addition (see [3]) that

$$u_\lambda \to u^* \qquad \text{in } L^{p^*}(\Omega),$$

as $\lambda \to \Lambda$.

Our next step is proving that u^* solves (1.1) for $\lambda = \Lambda$. For this goal we choose $\lambda_n \to \Lambda$, put $u_n = u_{\lambda_n}$ and are showing that $u_n \to u^*$ (strongly) in $W^{1,p}(\Omega)$. After this fact is achieved, that u^* solves (1.1) with $\lambda = \Lambda$ follows immediately by taking limits in the equality

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v + u_n^{p-1} v = \lambda_n \int_{\partial\Omega} u_n^{q-1} v + \int_\Omega u_n^{r-1} v, \tag{3.11}$$

for $v \in W^{1,p}(\Omega)$ arbitrary.

We now employ an argument which is standard in the theory of monotone operators (see the so called condition S_+ in [8]). First recall that, for $u, v \in W^{1,p}(\Omega)$ arbitrary, the following inequalities hold

$$\langle -\Delta_p u - (-\Delta_p v), u - v\rangle \geqslant C_p \int_\Omega |\nabla u - \nabla v|^p, \tag{3.12}$$

if $p \geqslant 2$, and

$$\langle -\Delta_p u - (-\Delta_p v), u - v\rangle \geqslant \\ C_p \left\{ \frac{\|\nabla u - \nabla v\|_{L^p(\Omega)}}{\|\nabla u - \nabla v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}} \right\}^{2-p} \int_\Omega |\nabla u - \nabla v|^p, \tag{3.13}$$

for $1 < p < 2$, being C_p a positive constant only depending on p.

Thanks to (3.11) sequence u_n satisfies

$$\begin{aligned}\langle -\Delta_p u_n, u_n - u^*\rangle &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u^*) \\ &= \lambda_n \int_{\partial\Omega} u_n^{q-1}(u_n - u^*) + \int_\Omega u_n^{r-1}(u_n - u^*) - \int_\Omega u_n^{p-1}(u_n - u^*).\end{aligned}$$

So we obtain that $\langle -\Delta_p u_n, u_n - u^*\rangle \to 0$. Since $u_n \to u^*$ weakly in $W^{1,p}(\Omega)$ we additionally find that $\langle -\Delta_p u^*, u_n - u^*\rangle \to 0$. Thus,

$$\langle -\Delta_p u_n - (-\Delta_p u^*), u_n - u^*\rangle \to 0.$$

In view of relations (3.12) and (3.13) this means that $\nabla u_n \to \nabla u^*$ in $L^p(\Omega)$ and so $u_n \to u^*$ in $W^{1,p}(\Omega)$. Furthermore, being u_n arbitrary we conclude that the whole $u_\lambda \to u^*$ in $W^{1,p}(\Omega)$.

To complete the proof we observe that the continuity from the left of u_λ in λ (when observed as taking values in $W^{1,p}(\Omega)$) is a consequence of the minimality of u_λ. On the other hand, since $0 < u_\lambda \leqslant u^*$ and $u^* \in L^\infty(\Omega)$ (Lemma 2.2) then u_λ is uniformly bounded in L^∞ and both the $C^{1,\beta}$ smoothness of u_λ and the uniform bound (1.7) follow from Theorem 2 in [15]. □

REMARKS 1. a) Proof of assertion i) in Theorem 1.1 somehow simplifies the corresponding one in the case $p = 2$ given in [10] (see Theorem 1.2). In fact, our argument avoids reasoning with the associated evolution problem (see Section 5 in [10]). On the other hand, such approach is less transparent and harder to carry out in the case of the p-Laplacian operator.

b) It should be pointed out that the Gidas-Spruck approach employed in [10] to obtain uniform L^∞ estimates can not be used here in the context of the p-Laplacian. It should be also remarked that a uniform bound for u_λ in $H^1(\Omega)$ as $\lambda \to \Lambda$ is achieved in [1] in the linear case $p=2$ and Dirichlet conditions by getting (3.10). To this purpose, authors linearize around $u=u_\lambda$ and show that the first eigenvalue associated to the linearized problem is always nonnegative. This treatment is, of course, out of use in the present framework. In fact, the linearization of the p-Laplacian at u_λ becomes singular ($p<2$) or degenerate ($p>2$) at the critical points of u_λ. Here, we succeed in showing (3.10) by both employing the energy associated to the auxiliar problem (2.1) and the variational side of the method of sub and super solutions.

c) For every fixed $0<\lambda_0 \leqslant \Lambda$ and $0<\beta'<\beta$ arbitrary, it can be shown, via the estimates in [15], that $u_\lambda \to u_{\lambda_0}$ in $C^{1,\beta'}(\overline{\Omega})$ as $\lambda \to \lambda_0+$.

4. A related problem

In the case of linear diffusion $p=2$, the following alternative version of problem (1.1),

$$\begin{cases} -\Delta u + u = u^{q-1}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u^{r-1}, & x \in \partial\Omega, \end{cases}$$

was also addressed in [10]. Its p-Laplacian counterpart reads as follows

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \varphi_q(u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda \varphi_r(u), & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda>0$ and q,r still fall in the regime (1.2). Problem (4.1) is just (1.1) but terms $\varphi_r(u)$ and $\varphi_q(u)$ have been interchanged.

In the present section it will be proved that all conclusions in Theorem 1.1 hold also true for (4.1). We are reviewing step by step, the arguments leading to the proof of Theorem 1.1 and we are proceeding to the required adaptations when necessary.

By a simple scaling, problem (4.1) can be transformed in the equivalent version

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \lambda \varphi_q(u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \varphi_r(u), & x \in \partial\Omega. \end{cases} \tag{4.2}$$

As exponents q,r lie in range (1.2), Lemma 4 and Lemma 5 show that any weak nonnegative solution $u \in W^{1,p}(\Omega)$ belongs to $L^\infty(\Omega)$. Then, provided u is nontrivial, it is positive and satisfies $u \in C^{1,\beta}(\overline{\Omega})$ for some $0<\beta<1$.

Auxiliary problem (2.1) must be now replaced with

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \lambda\,\varphi_q(u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

where $1 < q < p$. The same kind of arguments as those employed in Theorem 2.1 allows showing that (4.3) admits, for each $\lambda > 0$, a unique positive solution $u = \hat{u}_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ which satisfies in addition

$$\hat{u}_\lambda = \lambda^{\frac{1}{p-q}}\hat{u}_1, \;\; \lambda > 0.$$

Such solutions furnish a family of subsolutions to (4.2) that can be chosen as small as desired. By arguing as in Section 3 one finds that if $u \in W^{1,p}(\Omega)$, $u \neq 0$, is a nonnegative solution to (4.2), then necessarily

$$\hat{u}_\lambda(x) \leqslant u(x), \;\; x \in \Omega.$$

Two consequences can be extracted from this estimate. First, the existence of a positive solution to (4.2) is ensured by the mere existence of a positive supersolution. Second, once one has a solution u then a minimal solution $u_\lambda \leqslant u$ can be obtained as $u_\lambda = \lim u_n$, $u = u_n$ being defined by the iterative scheme

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \lambda\, u_{n-1}^{q-1}, & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = u_{n-1}^{r-1}, & x \in \partial\Omega, \end{cases}$$

beginning at $u_0 = \hat{u}_\lambda$.

Next step consists in achieving the existence of some $\Lambda^* > 0$ such that no positive solutions to (4.2) exist when $\lambda \geqslant \Lambda^*$. A proof of this fact was given in [10], for the case $p = 2$, by recurring to the associated evolution problem. A direct argument, in the lines of Section 3, is given now. In the present case, the relevant eigenvalue problem is

$$\begin{cases} -\Delta_p u + \varphi_p(u) = b(x)\varphi_p(u), & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \mu a(x)\varphi_p(u), & x \in \partial\Omega, \end{cases} \tag{4.4}$$

which is the p-Laplacian version of an eigenvalue problem of Steklov-type. Here it is assumed that $a \in L^\infty(\partial\Omega)$, $a(x) \geqslant a_0 > 0$ a. e. on $\partial\Omega$, while $b \in L^\infty(\Omega)$.

By a variational approach entirely similar to the one in Theroem 2.2 (see Lemma 9 in [11] for the linear case $p = 2$) it can be shown that (4.4) admits an eigenvalue μ_1 with a positive associated eigenfunction $\phi \in W^{1,p}(\Omega)$, if and only if, the first eigenvalue $\tilde{\lambda} = \lambda_1^D$ to the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_p u + \varphi_p(u) = b(x)\varphi_p(u) + \tilde{\lambda}\,\varphi_p(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.5}$$

satisfies

$$\lambda_1^D > 0. \tag{4.6}$$

Under this condition μ_1 is unique and fulfills similar properties as those in Theorem 2.2 for problem (2.6). Particularly,

$$\mu_1 = \inf_{w \in W^{1,p}(\Omega)} \frac{\|w\|_{W^{1,p}(\Omega)}^p - \int_\Omega b|w|^p}{\int_{\partial\Omega} a|w|^p}.$$

If the existence of a positive solution $u \in W^{1,p}(\Omega)$ to (4.2) is assumed, then u can be regarded as a positive eigenfunction to (4.4) with $b = \lambda u^{q-p}$, $a = u^{r-p}$ and $\mu = 1$. Therefore, by uniqueness $\mu_1 = 1$. It should be remarked that condition (4.6) is fulfilled since, under the elections of a and b performed before, the own solution u has the status of strict positive supersolution to (4.5) with $\tilde{\lambda} = 0$ (see Theorem 2 in [13]). Thus,

$$1 \leqslant \inf_{w \in W^{1,p}(\Omega)} \frac{\|w\|_{W^{1,p}(\Omega)}^p}{\int_{\partial\Omega} a|w|^p} \leqslant \inf_{w \in W^{1,p}(\Omega)} \frac{\|w\|_{W^{1,p}(\Omega)}^p}{\lambda^{\frac{r-p}{p-q}} \int_{\partial\Omega} \hat{u}_1^{r-p}|w|^p}.$$

Hence,

$$\lambda^{\frac{r-p}{p-q}} \leqslant \mu_1^*,$$

where $\mu = \mu_1^*$ is the first eigenvalue of the Steklov-type problem,

$$\begin{cases} -\Delta_p u + \varphi_p(u) = 0, & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \mu \hat{u}_1^{r-p} \varphi_p(u), & x \in \partial\Omega. \end{cases}$$

Therefore, no positive solutions to (4.2) are possible for $\lambda > {\mu_1^*}^{\frac{p-q}{r-p}}$.

We now prove the existence of $\bar{\lambda} > 0$ such that a minimal positive solution to (4.2) exists for all $0 < \lambda \leqslant \bar{\lambda}$. Indeed, it is enough with finding $\bar{\lambda} > 0$ such that (4.2) with $\lambda = \bar{\lambda}$ possesses a positive supersolution $\overline{u}$. This is accomplished by taking $0 < \lambda_0 < \lambda_1$, θ_0 and ϕ_1 ($\theta_0 = \theta(\lambda_0)$, $\phi_1 = \phi_1(\lambda_0)$) as in Section 3, by defining

$$\overline{u} = \lambda_0^{\frac{1}{r-p}} \phi_1,$$

and choosing $\bar{\lambda} = \theta_0 \lambda_0^{\frac{p-q}{r-p}} \inf_\Omega \phi_1^{p-q}$. Similarly, since the minimal solution u_λ satisfies $0 < u_\lambda \leqslant \overline{u}$ as $0 < \lambda \leqslant \bar{\lambda}$ and both $\overline{u}$ and $\bar{\lambda}$ vanish as $\lambda_0 \to 0$, then $u_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to 0$.

Finally, by reasoning as in Section 3 one finds $0 < \Lambda_1 \leqslant \Lambda^*$ such that a minimal positive solution u_λ to (4.2) exists for $0 < \lambda < \Lambda_1$ and no positive solution is possible when $\lambda > \Lambda_1$. The same argument as that employed in Section 3 yields the existence of a further minimal solution u^* corresponding to $\lambda = \Lambda_1$.

5. Final remarks

In this section we collect together some reflections on the existence of a second positive solution to (1.1) for all values of $0 < \lambda < \Lambda$, Λ given in Theorem 1.1 ("global" multiplicity of positive solutions).

According to the plan to achieve global multiplicity in [1] (see also [10]) a first step would consist in finding a pair of positive sub and supersolutions $\underline{u}$, $\overline{u}$ so that $0 < \underline{u}(x) < u_\lambda(x) < \overline{u}(x)$ for all $x \in \Omega$, where u_λ in the minimal solution to (1.1). This would mean that u_λ becomes a local minimizer for the energy functional E_λ in the $C(\overline{\Omega})$ topology.

In the context of our problem (1.1), natural candidates are $\underline{u} = u_{\lambda_1}$, $\overline{u} = u_{\lambda_2}$ where $0 < \lambda_1 < \lambda < \lambda_2$. By taking λ_1 small the inequality $\underline{u} < u_\lambda$ is easily obtained. On the other hand, due to the fact that none of the gradients ∇u_λ, ∇u_{λ_2} vanish near the boundary, it can be checked that the difference $w = u_{\lambda_2} - u_\lambda$ satisfies an inequality $\mathscr{L}(x,w,\nabla w) \geqslant 0$ in an inner neighborhood of $\partial\Omega$ together with $B(x,w,\nabla w)\nabla w > 0$ on $\partial\Omega$, where $\mathscr{L}$ stands for a nondegenerate elliptic operator and B is the conormal outer field associated to $\mathscr{L}$ on $\partial\Omega$. Therefore, Hopf's principle implies that the strict inequality $u_\lambda(x) < u_{\lambda_2}(x)$ holds for all x such that $\operatorname{dist}(x,\partial\Omega) \leqslant \delta$ for some $\delta > 0$. However, the problem arises when trying to propagate the strict inequality to the whole of Ω, specially when $p > 2$ in the diffusion operator. Recall that the strong comparison principle for operators involving the p-Laplacian only holds true under quite restricted circumstances ([5], [6]).

Provided that one succeeds in getting $\underline{u} < u_\lambda < \overline{u}$ in Ω for *all* $0 < \lambda < \Lambda$, a second step is showing that u_λ constitutes a local minimizer of E_λ in the $W^{1,p}$ topology. To show this fact, the existence of a sequence $u_n \in W^{1,p}(\Omega)$, $u_n \to u_\lambda$ in $W^{1,p}(\Omega)$, satisfying $E_\lambda(u_n) < E_\lambda(u_\lambda)$ must be discarded. To this purpose, a uniform C^α estimate of the sequence u_n should be obtained. Then, after passing to a subsequence, it follows that $u_n \to u_\lambda$ in $C(\overline{\Omega})$ what would contradict that u_λ is a local minimum in $C(\overline{\Omega})$. The experience in the Dirichlet problem (1.4) reveals that proving the C^α estimate is also a delicate enterprise (see [9]).

In a future work we will come back to these questions, in special the issue of uniform estimates. Nevertheless, the problem of "global" multiplicity of positive solutions to (1.1) when Ω is a ball of $\mathbb{R}^N$ and we restrict ourselves to radial solutions, can be answered affirmatively. Author's results on this case will be soon published.

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