

ON ABSTRACT BARENBLATT EQUATIONS

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*Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday*

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Abstract. In this paper we are interested in abstract problems of Barenblatt's type. In a first part, we investigate the problem $f(\partial_t u) + Au = g$ where f and A are maximal monotone operators and by assuming that A derives from a potential J . With general assumptions on the operators, we prove the existence of a solution. In the second part of the paper, we examine a stochastic version of the above problem: $f[\partial_t(u - \int_0^t h dw)] + Au = 0$, with some restrictive assumptions on the data due principally to the framework of the Itô integral.

1. Introduction

In this paper we are interested in the deterministic and the stochastic abstract problems of Barenblatt's type:

$$(P_1) : \begin{cases} f(\partial_t u) + Au = g, \\ u(t=0) = u_0, \end{cases} \quad (P_2) : \begin{cases} f(\partial_t(u - \int_0^t h dw)) + Au = 0, \\ u(t=0) = u_0. \end{cases}$$

In the deterministic case, such a problem has been investigated by G. Díaz and J. I. Díaz in [9], where the authors were interested in the asymptotic behavior of the solution of the problem $\partial_t u - \Delta \beta(u) = 0$ where β is a maximal monotone graph in \mathbb{R}^2 . The essential tool was to consider the "dual" problem $\partial_t v + \beta(-\Delta v) = 0$ of type $(P_1) : f(\partial_t v) - \Delta v = 0$ where $f = [-\beta(-\cdot)]^{-1}$. The study of such a problem was based on the work of K. S. Ha [12] where the author was interested in the existence of solutions to a class of quasilinear Barenblatt equations of type $f \in \partial_t u + \beta A(u)$ when A is assumed to be a m-accretive operator in $L^\infty(\Omega)$. Let us also mention the work of H. Konishi where the author studies in [15] the properties of the nonlinear semi-groups associated with $f(\partial_t u) = \Delta u$.

Next, more recently, G. Schimperna *et al.* [18] have been interested in the differential inclusion: $f \in \alpha(\partial_t u) - \operatorname{div}(b(x, \nabla u)) + W'(u)$ where, among other things, $\alpha \subset \mathbb{R}^2$

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was a maximal monotone graph, W a λ -convex function and $f \in W^{1,1}(0, T, H)$. Presented in an abstract way, this work was in connection with the modeling of phase change phenomena and gas flow in porous media.

Concerning such kind of modelings, problems of type (P_1) were originally considered by G. I. Barenblatt in [4] in the theory of fluids in elasto-plastic porous medium. Written in following way: $F(\partial_t u) - \Delta u = 0$ where $F(x) = x + \gamma|x|$ ($0 < |\gamma| < 1$), existence of regular and self-similar solutions have been investigated by S. Kamin - L. A. Peletier - J. L. Vázquez in [14]. Formal solutions given by expansions of a suitable new variable $\chi = \chi(t, x)$ is also proposed in [7] concerning nonlinear diffusive process with a non-conservative mass.

For nonlinear operators A , the existence of self-similar solutions has been proposed by J. Hulshof - J. L. Vázquez in [13] for the so called "modified porous medium equation": $F(\partial_t u) - \Delta u^m = 0$. For the "modified p -Laplace equation": $F(\partial_t u) - \Delta_p u = 0$, a result of existence of self-similar solutions has been proposed by N. Igbida in [16], and the existence of weak solutions by C. Bauzet *et al.* in [5]. Then, in an abstract setting, our approach revisit the one of P. Colli in [8].

A first approach of the stochastic case has been proposed by Adimurthi *et al.* in [2] concerning the existence of a solution to the stochastic pseudoparabolic Barenblatt problem:

$$f\left(\partial_t(u - \int_0^t hdw)\right) - \Delta u - \varepsilon \Delta \partial_t u = 0, \quad \varepsilon > 0.$$

Then, C. Bauzet *et al.* in [5] has envisaged the case $\varepsilon = 0$, where strong solutions are considered, see also J. I. Díaz *et al.* [10].

In the present work, we propose to extend the previous cited results concerning (P_1) by weakening the assumptions on the data and we propose to study the abstract stochastic parabolic-Barenblatt problem (P_2) with additive noise, then with a multiplicative one.

In the first part of the paper, H is a Hilbert space and V is a reflexive separable Banach space such that V is embedded in H with a dense and compact injection and one will identify H with its dual space H' . One denotes by (\cdot, \cdot) , resp. $|\cdot|$, the scalar product of H , resp. the norm in H , by $\langle \cdot, \cdot \rangle$ the dual product $V' - V$ and by $\|\cdot\|$ the norm in V .

$f : H \rightarrow H' \equiv H$ and $A : V \rightarrow V'$ are maximal monotone operators, A derives from a potential J , and general assumptions are made to prove the existence of a solution. In particular, we assume neither strong monotonicity for f , nor a control from below of $J(u)$ by a power of the norm of u in V . This allows us to apply our results in the case of Orlicz spaces (See [1], Chap. VIII, p. 227) when the problem allows easily a control of the modulus given by the N-function, rather than the Luxembourg norm. One can cite for example the case of the Musielak-Orlicz spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces (See [11]).

In the second part of the paper, we will be interested in stochastic problems. Because of the theory of the stochastic integration, V needs to be a Hilbert space and for technical reasons, one assumes that $(u, v) \mapsto \langle Au, v \rangle$ is a scalar product whose associated norm is equivalent to the one of V .

This paper is organized as follows. The next section contains some classical notations used throughout the paper and the statements of our main results. The deterministic case is established in Section 3 and the stochastic one in Section 4.

2. Notations, assumptions and results

Denote by (P_1) the following problem:

$$(P_1) : \begin{cases} f(\partial_t u) + Au = g, \\ u(t = 0) = u_0, \end{cases}$$

and assume that:

- H is a Hilbert space and V is a reflexive separable Banach space such that $V \hookrightarrow H$ with a dense and compact injection. Thus, one has the classical Gelfand-Lions triplet: $V \hookrightarrow H \equiv H' \hookrightarrow V'$;
- $f : H \rightarrow H' \equiv H$ is a demicontinuous (univoque) maximal monotone operator.

REMARK 2.1. This is the case for example if f is the subdifferential ∂F of a continuous, Gâteaux differentiable and proper convex function $F : H \rightarrow \mathbb{R}$.

Assume moreover that:

- $\exists \alpha > 0, \lambda \in \mathbb{R}, \forall x \in H, (f(x), x) \geq \alpha|x|^2 - \lambda,$
- $\exists C_1, C_2 \geq 0, \forall x \in H, |f(x)| \leq C_1|x| + C_2;$
- $J : V \rightarrow \mathbb{R}$ is a continuous, Gâteaux differentiable and proper convex function. Since J can be defined modulo a constant value, assuming that $J(0) = 0$ does not affect the generality. One denotes by A its subdifferential $\partial J : V \rightarrow V'$. We recall that it is a demicontinuous (univoque) maximal monotone operator.

One assumes moreover that:

- J is bounded above on bounded subsets of V (therefore, A maps bounded subsets of V into bounded nonempty subsets of V' (e.g. [6] Prop. 4.1.25),
- either $\exists \delta > 0, \varphi_1 : u \mapsto \delta|u|^2 + J(u)$ is coercive over V in the sense:

$$\frac{\delta|u_n|^2 + J(u_n)}{\|u_n\|} \text{ goes to } +\infty \text{ if } \|u_n\| \text{ goes to } +\infty.$$

Or, f derives from the potential F (see Remark 2.1), and $\varphi_2 : u \mapsto F(u) + J(u)$ is coercive over V ;

- $u_0 \in V$ and $g \in L^2(0, T, H)$.

The main results in that case are the following.

THEOREM 2.2. *There exists $u \in W^{1,\infty,2}(0, T, V, H)$ ¹ solution of (P_1) . Moreover, for a.e. t ,*

$$Au(t) = g(t) - f(\partial_t u(t)) \in H, \quad J(u) \in W^{1,1}(0, T)$$

¹ $W^{1,p,q}(0, T, V, H)$ denotes the space of functions $u \in L^p(0, T, V)$ such that $\partial_t u \in L^q(0, T, H)$.

and, for any t ,

$$\int_0^t (f(\partial_t u), \partial_t u) ds + J(u(t)) = J(u_0) + \int_0^t (g, \partial_t u) ds,$$

$$\alpha \int_0^t |\partial_t u|^2 ds + 2J(u(t)) \leq 2J(u_0) + \frac{1}{\alpha} \int_0^t |g|^2 ds + 2\lambda.$$

COROLLARY 2.3. *The following three statements hold.*

- i) *If A is linear and f strictly monotone, then the solution is unique.*
- ii) *If A is linear and $J(0) < J(w)$ for any $w \neq 0$, then the solution is unique and it belongs to $C([0, T], V)$. Moreover, the application $(u_0, g) \mapsto u$ is continuous from $V \times L^2(0, T, H)$ to $C([0, T], V)$.*
- iii) *If A is linear and f strongly monotone, the application $(u_0, g) \mapsto \partial_t u$ is continuous from $V \times L^2(0, T, H)$ to $L^2(0, T, H)$.*

Denote by (P_2) the following problem:

$$(P_2) : \begin{cases} f(\partial_t(u - \int_0^t h dw)) + Au = 0, \\ u(t=0) = u_0. \end{cases}$$

In addition to the above hypotheses, assume moreover that:

- H and V are separable Hilbert spaces;
- $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on some complete probability space (Ω, \mathcal{F}, P) , with the property that $w_0 = 0$;
- $A = \partial J$ is a linear operator, $J(u) > 0 = J(0)$ if $u \neq 0$ and f is strongly monotone;
- $u_0 \in V$ and $h \in N_w^2(0, T, V)$ where, for a separable Hilbert space X , $N_w^2(0, T, X)$ denotes the set of predictable processes of $L^2((0, T) \times \Omega, X)$ (Cf. [17] for example).

The main result in that case is:

THEOREM 2.4. *There exists a unique $u \in N_w^2(0, T, V)$, such that $\partial_t(u - \int_0^t h dw) \in L^2((0, T) \times \Omega, H)$, solution of (P_2) . Moreover, $u \in C([0, T], L^2(\Omega, V))$ and, for any $u_0, \hat{u}_0 \in V$, any $h, \hat{h} \in N_w^2(0, T, V)$ and any t ,*

$$E \int_0^t (f(\partial_t U) - f(\partial_t \hat{U}), \partial_t [U(t) - \hat{U}(t)]) ds + E \|(u - \hat{u})(t)\|_A^2$$

$$\leq E \|u_0 - \hat{u}_0\|_A^2 + \int_0^t E \|h - \hat{h}\|_A^2 ds,$$

where U (resp. \hat{U}) denotes $u - \int_0^t h dw$ (resp. $\hat{u} - \int_0^t \hat{h} dw$).

COROLLARY 2.5. Assume that $\mathcal{H} : V \rightarrow V$ is a Lipschitz-continuous mapping. Then there exists a unique $u \in N_w^2(0, T, V)$ such that

$$\partial_t [u - \int_0^t \mathcal{H}(u)dw] \in L^2((0, T) \times \Omega, H)$$

solution of Problem

$$(P_{\mathcal{H}}) : f\left(\partial_t [u - \int_0^t \mathcal{H}(u)dw]\right) + Au = 0, \quad \text{with } u(0, \cdot) = u_0.$$

3. The deterministic case

The aim of this section is to prove Theorem 2.2 and Corollary 2.3. We propose to prove the existence of a solution by passing to the limit in a time-discretization scheme. For any positive integers N and any $n \leq N$, we denote by

$$\Delta t = \frac{T}{N}, \quad t_n = n\Delta t \quad \text{and} \quad g^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} g(s)ds.$$

3.1. Existence of the approximation sequence

LEMMA 3.1. For any sequence $(g^n) \subset H$, there exists a sequence $(u^n) \subset V$ such that $u^0 = u_0$ and

$$f\left(\frac{u^{n+1} - u^n}{\Delta t}\right) + Au^{n+1} = g^{n+1}. \tag{3.1}$$

Proof. Since $H' \hookrightarrow V'$, $M : u \in V \mapsto f\left(\frac{u - u^n}{\Delta t}\right) \in V'$ is a monotone operator. If one denotes by S a bounded subset of V , then, for any $s \in S$,

$$\|M(s)\|_{V'} \leq C \left| f\left(\frac{s - u_n}{\Delta t}\right) \right| \leq C \left| \frac{s - u_n}{\Delta t} \right| + C \leq C \left\| \frac{s - u_n}{\Delta t} \right\| + C$$

and M is a bounded operator.

If one considers that u_k converges weakly to u in V , then, it converges to u in H and Mu_n converges weakly to Mu in H since f is demicontinuous in H . Thus, for any $v \in V$, $\lim_k (Mu_k, u_k - v) = (Mu, u - v)$ and M is pseudomonotone in V .

For any $u \in V$, one has that

$$\begin{aligned} \left(f\left(\frac{u - u^n}{\Delta t}\right), u\right) &= \Delta t \left(f\left(\frac{u - u^n}{\Delta t}\right), \frac{u - u^n}{\Delta t}\right) + \left(f\left(\frac{u - u^n}{\Delta t}\right), u^n\right) \\ &\geq \alpha \Delta t \left|\frac{u - u^n}{\Delta t}\right|^2 - C(\Delta t)(|u| + c(u^n)) \\ &\geq \frac{\alpha}{\Delta t} |u|^2 - C(\Delta t)(|u| + c(u^n)). \end{aligned}$$

Thus, for small values of Δt , one gets that

$$\frac{\left(f\left(\frac{u - u^n}{\Delta t}\right), u\right) + \langle Au, u \rangle}{\|u\|} \geq \frac{\frac{\alpha}{\Delta t} |u|^2 - C(\Delta t)(|u| + c(u^n)) + J(u) - J(0)}{\|u\|}$$

$$\geq \frac{\delta |u|^2 + J(u)}{\|u\|} - C(\Delta t, u^n).$$

Then, since by assumption φ_1 is coercive, the result of the lemma holds thanks to classical arguments on pseudomonotone operators (Cf. [19] : Cor. 7.1 p.84 for example).

If one assumes that f derives from a potential F (see Remark 2.1).

Then, the convex function $\varphi_3 : V \mapsto \mathbb{R}$, defined for any $u \in V$ by

$$\varphi_3(u) = \Delta t F\left(\frac{u - u^n}{\Delta t}\right) + J(u) - (g^{n+1}, u),$$

is continuous and Gâteaux differentiable. Moreover,

$$\begin{aligned} \langle d\varphi_3(u), v \rangle &= \left\langle f\left(\frac{u - u^n}{\Delta t}\right), v \right\rangle + \langle Au, v \rangle - (g^{n+1}, v), \\ F(u) &= F\left(\Delta t \frac{u - u^n}{\Delta t} + u^n\right) \leq \Delta t F\left(\frac{u - u^n}{\Delta t}\right) + (1 - \Delta t) F\left(\frac{u^n}{1 - \Delta t}\right) \end{aligned}$$

and

$$\begin{aligned} \varphi_3(u) &\geq F(u) + J(u) - (g^{n+1}, u) - (1 - \Delta t) F\left(\frac{u^n}{1 - \Delta t}\right) \\ &\geq F(u) + J(u) - |g^{n+1}| |u| - C(\Delta t) \\ &= \left[\frac{F(u) + J(u)}{|u|} - |g^{n+1}| \right] |u| - C(\Delta t). \end{aligned}$$

The coercivity of φ_2 yields the existence a critical point to φ_3 which corresponds to a solution u^{n+1} for the lemma.

REMARK 3.2. Note that if f , or A , is strictly monotone, then the solution is unique. Indeed, if u and \hat{u} are two given solutions, one has

$$\Delta t \left(f\left(\frac{u - u^n}{\Delta t}\right) - f\left(\frac{\hat{u} - u^n}{\Delta t}\right), \frac{u - u^n}{\Delta t} - \frac{\hat{u} - u^n}{\Delta t} \right) + \langle Au - A\hat{u}, u - \hat{u} \rangle = 0.$$

3.2. A priori estimates

Let us test Equation (3.1) with $v = \frac{u^{n+1} - u^n}{\Delta t}$. Then,

$$\left(f\left(\frac{u^{n+1} - u^n}{\Delta t}\right), \frac{u^{n+1} - u^n}{\Delta t} \right) + \left\langle Au^{n+1}, \frac{u^{n+1} - u^n}{\Delta t} \right\rangle = \left(g^{n+1}, \frac{u^{n+1} - u^n}{\Delta t} \right),$$

yields

$$\Delta t \frac{\alpha}{2} \left| \frac{u^{n+1} - u^n}{\Delta t} \right|^2 + J(u^{n+1}) \leq J(u^n) + \frac{\Delta t}{\alpha} |g^{n+1}|^2 + \lambda \Delta t.$$

Thus, there exists a constant C such that

$$\sum_{k=0}^n \Delta t \frac{\alpha}{2} \left| \frac{u^{k+1} - u^k}{\Delta t} \right|^2 + J(u^{n+1}) \leq J(u_0) + T\lambda + \frac{\Delta t}{\alpha} \sum_{k=0}^n |g^{k+1}|^2 \leq C,$$

and

LEMMA 3.3. *There exists a constant C such that*

$$\|\partial_t \tilde{u}^{\Delta t}\|_{L^2(0,T,H)} + \|J(u^{\Delta t})\|_{L^\infty(0,T)} + \frac{1}{\Delta t^2} \|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{L^2(0,T,H)}^2 \leq C.$$

Since,

$$\left| f\left(\frac{u^{k+1} - u^k}{\Delta t}\right) \right| \leq C_1 \left| \frac{u^{k+1} - u^k}{\Delta t} \right| + C_2,$$

one has that

LEMMA 3.4. *There exists a constant C such that $\|f(\partial_t \tilde{u}^{\Delta t})\|_{L^2(0,T,H)} \leq C$.*

If one assumes that f derives from a potential F (see Remark 2.1), then

$$\begin{aligned} F(u^{n+1}) &\leq F(u_0) + |f(u^{n+1})| |u_0 - u^{n+1}| \\ &\leq F(u_0) + [C_1 |u^{n+1}| + C_2] |u_0 - u^{n+1}| \\ &\leq Cte \end{aligned}$$

and there exists a constant C such that $\varphi_2(u^n) \leq C$.

Since φ_1 (resp. φ_2) is coercive, this yields

LEMMA 3.5. *There exists a constant C such that*

$$\|u^{\Delta t}\|_{L^\infty(0,T,V)} + \|\tilde{u}^{\Delta t}\|_{L^\infty(0,T,V)} \leq C.$$

Finally, since for any $v \in V$,

$$\langle Au^{n+1}, v \rangle = \left\langle g^{n+1} - f\left(\frac{u^{n+1} - u^n}{\Delta t}\right), v \right\rangle,$$

one gets that

$$\sup_{v \neq 0} \frac{\langle Au^{n+1}, v \rangle}{\|v\|} \leq C \left| g^{n+1} - f\left(\frac{u^{n+1} - u^n}{\Delta t}\right) \right|$$

and

LEMMA 3.6. *There exists a constant C such that $\|Au^{\Delta t}\|_{L^2(0,T,V')} \leq C$.*

3.3. At the limit

Let us recall that, by construction, almost everywhere in $(0, T)$, one has the discretization:

$$\forall v \in V, \quad (f(\partial_t \tilde{u}^{\Delta t}), v) + \langle Au^{\Delta t}, v \rangle = (g^{\Delta t}, v). \tag{3.2}$$

As the sequence $\tilde{u}^{\Delta t}$ is bounded in $W^{1,\infty,2}(0, T, V, H)$, up to a subsequence denoted similarly, Simon’s compactness argument ensures the existence of $u \in W^{1,\infty,2}(0, T, V, H)$ such that $\tilde{u}^{\Delta t}$ converges weakly to u in $L^\infty(0, T, V)$ weak-* and strongly in $C([0, T], H)$. Moreover, $u^{\Delta t}$ converges to u in $L^\infty(0, T, V)$ weak-* and strongly in $L^2(0, T, H)$, and $\partial_t \tilde{u}^{\Delta t}$ converges weakly to $\partial_t u$.

Concerning the nonlinear terms, one denotes by f_u and A_u the weak limits, respectively in $L^2(0, T, H)$ and $L^2(0, T, V')$, of $f(\partial_t \tilde{u}^{\Delta t})$ and $Au^{\Delta t}$. Thus,

$$\int_0^T \langle Au^{\Delta t}, u^{\Delta t} - u \rangle dt = \int_0^T (g^{\Delta t} - f(\partial_t \tilde{u}^{\Delta t}), u^{\Delta t} - u) dt \rightarrow 0 \quad \text{when } \Delta t \rightarrow 0,$$

and,

$$\int_0^T \langle Au^{\Delta t}, u^{\Delta t} \rangle dt \rightarrow \int_0^T \langle A_u, u \rangle dt \quad \text{when } \Delta t \rightarrow 0.$$

By assumption, the application $u \in V \mapsto \langle Au, v \rangle$ is continuous. Thus, if $w : (0, T) \rightarrow V$ is a measurable function, Aw is a weak-* measurable one. Since by assumption V is a separable reflexive Banach space, Aw is firstly weakly measurable, then measurable.

Set $v \in L^\infty(0, T, V)$ and $|\lambda| \leq 1$. By monotonicity of A and thanks to the previous convergence, one gets that

$$0 \leq \lambda \int_0^T \langle A_u - A(u - \lambda v), v \rangle dt.$$

For $t \in (0, T)$ a.e., one gets that

$$\|u(t) - \lambda v(t)\| \leq C = \|u\|_{L^\infty(0, T, V)} + \|v\|_{L^\infty(0, T, V)}.$$

Since J is bounded above on bounded subsets of V , $A(\overline{B}_V(0, C))$ is bounded (e.g. [6] Prop. 4.1.25 p.137) and M exists such that,

$$\|A(u(t) - \lambda v(t))\|_{V'} \leq M, \quad t \text{ a.e. in } (0, T).$$

Since A is demi-continuous, $\langle A_u - A(u - \lambda v), v \rangle$ converges to $\langle A_u - Au, v \rangle$ when λ goes to 0. Then, Lebesgue’s theorem yields:

$$\int_0^T \langle A_u - A(u - \lambda v), v \rangle dt \rightarrow \int_0^T \langle A_u - A(u), v \rangle dt \quad \text{when } \lambda \rightarrow 0$$

and one concludes that

$$0 = \int_0^T \langle A_u - A(u), v \rangle dt \quad \text{and} \quad A_u = Au.$$

By passing to the limit, one gets that $f_u + Au = g$ in $L^2(0, T, H)$, or similarly, that $\partial_t u + Au = h := g - f_u + \partial_t u$ where $h \in L^2(0, T, H)$ and $u_0 \in V$.

Then, thanks to Section 5, for any t , the following equality holds:

$$\int_0^t (f_u, \partial_t u) ds + J(u(t)) = J(u_0) + \int_0^t (g, \partial_t u) ds.$$

Coming back to the discrete formulation, adding

$$\left(f \left(\frac{u^{n+1} - u^n}{\Delta t} \right), \frac{u^{n+1} - u^n}{\Delta t} \right) + \langle Au^{n+1}, \frac{u^{n+1} - u^n}{\Delta t} \rangle = \left(g^{n+1}, \frac{u^{n+1} - u^n}{\Delta t} \right)$$

over n , yields

$$\begin{aligned} \Delta t \sum_{n=0}^{N-1} \left(f \left(\frac{u^{n+1} - u^n}{\Delta t} \right), \frac{u^{n+1} - u^n}{\Delta t} \right) + J(u^N) \\ \leq J(u_0) + \Delta t \sum_{n=0}^{N-1} \left(g^{n+1}, \frac{u^{n+1} - u^n}{\Delta t} \right), \end{aligned}$$

that is,

$$\int_0^T (f(\partial_t \tilde{u}^\Delta), \partial_t \tilde{u}^\Delta) dt + J(\tilde{u}^\Delta(T)) \leq J(u_0) + \int_0^T (g^\Delta, \partial_t \tilde{u}^\Delta) dt.$$

Since \tilde{u}^Δ converges to u in $C([0, T], H)$ and as $\tilde{u}^\Delta(T)$ is bounded in V , one gets that $\tilde{u}^\Delta(T)$ converges weakly to $u(T)$ in V (Note that the same can be told for any t , *i.e.* $\tilde{u}^\Delta(t)$ converges weakly to $u(t)$ in V), and we get back the initial condition $u(t=0) = u_0$ in V . Thus,

$$\begin{aligned} \limsup_{\Delta t} \int_0^T (f(\partial_t \tilde{u}^\Delta), \partial_t \tilde{u}^\Delta) dt + J(u(T)) \\ \leq \limsup_{\Delta t} \int_0^T (f(\partial_t \tilde{u}^\Delta), \partial_t \tilde{u}^\Delta) dt + \liminf_{\Delta t} J(\tilde{u}^\Delta(T)) \\ \leq \limsup_{\Delta t} \left[\int_0^T (f(\partial_t \tilde{u}^\Delta), \partial_t \tilde{u}^\Delta) dt + J(\tilde{u}^\Delta(T)) \right] \\ \leq J(u_0) + \int_0^T (g, \partial_t u) dt = \int_0^T (f_u, \partial_t u) ds + J(u(T)). \end{aligned}$$

Then, an argument of Minty's type in $L^2(0, T, H)$, similar to one used above with A , leads to $f_u = f(\partial_t u)$ and to the existence of a solution.

Note in particular that $Au = g - f(\partial_t u) \in H$ and that, for any t ,

$$\int_0^t (f(\partial_t u), \partial_t u) ds + J(u(t)) = J(u_0) + \int_0^t (g, \partial_t u) ds \tag{3.3}$$

and

$$\alpha \int_0^t |\partial_t u|^2 ds + 2J(u(t)) \leq 2J(u_0) + \frac{1}{\alpha} \int_0^t |g|^2 ds + 2\lambda. \tag{3.4}$$

3.4. If A is linear

If A is a linear operator and if u and \hat{u} are given solutions associated to the initial conditions u_0, \hat{u}_0 and the right hand side members g, \hat{g} , one gets that

$$\begin{aligned} (f(\partial_t u) - f(\partial_t \hat{u}), v) + \langle A(u - \hat{u}), v \rangle &= (g - \hat{g}, v), \\ (u - \hat{u})(t = 0) &= u_0 - \hat{u}_0, \end{aligned}$$

that is, by denoting $w = u - \hat{u}$,

$$\begin{aligned} (\partial_t w, v) + \langle Aw, v \rangle &= ([g - \hat{g}] - [f(\partial_t u) - f(\partial_t \hat{u})] + \partial_t w, v), \\ (u - \hat{u})(t = 0) &= u_0 - \hat{u}_0. \end{aligned}$$

Then, thanks to Section 5, for any t ,

$$\begin{aligned} \int_0^t |\partial_t w|^2 ds + J(w(t)) &= J(u_0 - \hat{u}_0) \\ &+ \int_0^t ([g - \hat{g}] - [f(\partial_t u) - f(\partial_t \hat{u})] + \partial_t w, \partial_t w) ds, \end{aligned}$$

and

$$\int_0^t (f(\partial_t u) - f(\partial_t \hat{u}), \partial_t w) ds + J(w(t)) = J(u_0 - \hat{u}_0) + \int_0^t (g - \hat{g}, \partial_t w) ds. \tag{3.5}$$

Since A is linear, $A(0) = 0$ and $0 \in \partial J(0)$, i.e. $J(0) = \min J$ (Prop. 4.1.8 p.130 [6] et al., for example), and

PROPOSITION 3.7. If moreover A is a linear operator and assuming that either f is strictly monotone, or the optimal value of J is only satisfied at 0 , then the solution is unique.

If A is linear and $J(0) = 0$, then, for any $u, v \in V$, one gets that $J(u) = \frac{1}{2} \langle Au, u \rangle$, $\langle Au, v \rangle = \langle Av, u \rangle$ and $\|\cdot\|_A : u \in V \mapsto \sqrt{\langle Au, u \rangle}$ is a norm on V associated to the scalar product $(u, v) \mapsto \langle Au, v \rangle$.

Note that assuming that $J(v) > 0$ if $v \neq 0$ yields that $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent norms over V . Indeed, the first inequality holds since A is bounded on the bounded sets, A is a continuous linear operator.

Assume that the second one doesn't hold. Then, there exists a sequence $(v_n) \in V$ such that $\|v_n\| = 1$, v_n converges weakly (resp. strongly) to a given v in V (resp. H) and $J(v_n) = 2\|v_n\|_A^2$ goes to 0 . Since J is a continuous convex function, one gets that $0 = J(0) \leq J(v) \leq 0$, and since $J(v) > 0$ if $v \neq 0$, one concludes that $v = 0$.

As φ_1 is a bilinear coercive mapping, there exists a positive constant α such that, for any $u \in V$, $\varphi_1(u) = \delta|u|^2 + \|u\|_A^2 \geq \alpha\|u\|^2$. Since $\varphi_1(v_n)$ tends to 0 , one has that v_n goes to 0 in V , one gets a contradiction and the norms are equivalent.

The solution u belongs to $W^{1,\infty,2}(0, T, V, H)$. Then, it belongs to $C_w([0, T], V)$, the V -valued scalar continuous functions. Since (3.3) yields the continuity of the norm, u

is a V -valued continuous function. Since (3.5) and (3.4) yield the existence of a positive constant $C = C(u_0, \hat{u}_0, g, \hat{g})$ such that, for any t ,

$$\int_0^t (f(\partial_t u) - f(\partial_t \hat{u}), \partial_t w) ds + \frac{1}{2} \|w(t)\|_A^2 \leq J(u_0 - \hat{u}_0) + C \|g - \hat{g}\|_{L^2(0,T,H)},$$

one gets the continuity of the infinity-norm of the solution with respect to u_0 and g .

If f is assumed to be strongly monotone, then the time derivative of the solution is continuous with respect to u_0 and g in $L^2(0, T, H)$. This finishes the proof of the corollary.

4. The stochastic case

In this section, we are interested in the stochastic version of Barenblatt’s equations. So, we need first to precise the sense we wish to give to the stochastic version of an equation with such a nonlinear term. For this, remark that the homogeneous deterministic equation writes : $\partial_t u \in f^{-1}(-Au)$. Then, the stochastic version of the problem would be: $du \in f^{-1}(-Au)dt + hdw$ where $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on a complete probability space (Ω, \mathcal{F}, P) , with the property that $w_0 = 0$; and $h \in \mathcal{N}_w^2(0, T, V)$, the set of predictable functions of $L^2((0, T) \times \Omega, V)$ ([17] p.28 for example).

Following [2], [3] section 44 p. 183, G. Vallet [20] or G. Vallet and P. Wittbold [21] for example, the equation can be understood in the following way:

$$\partial_t \left[u - \int_0^t hdw(s) \right] \in f^{-1}(-Au), \quad \text{i.e.} \quad f \left(\partial_t \left[u - \int_0^t hdw(s) \right] \right) + Au = 0,$$

where $\int_0^t hdw(s)$ denotes the Itô integration of h . Then $U = u - \int_0^t hdw(s)$ is a solution to the random equation

$$f(\partial_t U) + A(t)U = 0, \quad \text{where} \quad A(t)U = A \left[U + \int_0^t hdw(s) \right].$$

Since we are interested in strong solutions, standard argumentations do not suit and additional assumptions are needed.

In the sequel, A is assumed to be linear and $J(v) > J(0) = 0$ for any $v \neq 0$. Thus, as explained in the end of the previous section, $J(u) = \frac{1}{2} \langle Au, u \rangle$ and $(\cdot, \cdot)_A : (u, v) \mapsto \langle Au, v \rangle$ is a scalar product. One denotes by $\| \cdot \|_A$ the associated norm; it is equivalent to the one of V .

Thanks to the continuity and the linearity of A , the problem is equivalent to

$$(\tilde{P}_1) : f(\partial_t U) + AU = -A \int_0^t hdw = - \int_0^t Ahdw, \quad U(t=0) = u_0,$$

where $Ah \in \mathcal{N}_w^2(0, T, V')$.

LEMMA 4.1. *There exists at most one solution of Problem (P_1) .*

Indeed, if u and \hat{u} are two solutions, associated with $U = u - \int_0^t h dw$ and $\hat{U} = \hat{u} - \int_0^t \hat{h} dw$ and if one denotes by $w = u - \hat{u} = U - \hat{U}$, one gets:

$$\partial_t w + Aw = G := \partial_t w + f(\partial_t \hat{U}) - f(\partial_t U), \quad w(t = 0) = 0.$$

Then, thanks to Section 5, the following energy equality holds for any t :

$$\int_0^t |\partial_t w|^2 ds + J(w(t)) = J(0) + \int_0^t (G, \partial_t w) ds,$$

that is,

$$\int_0^t (f(\partial_t U) - f(\partial_t \hat{U}), \partial_t [U - \hat{U}]) ds + J(w(t)) = 0,$$

and the solution is unique.

We wish, in the sequel, to use the previous section. So, in a first step, we assume that $h \in N_w^2(0, T, V)$ and $Ah \in N_w^2(0, T, H)$. Thus, a.s. in Ω , $\int_0^t Ah dw \in L^2(0, T, H)$ and there exists a unique solution to (\hat{P}_1) . Moreover, the result of continuity of Corollary 2.3 ensures that $U \in N_w^2(0, T, V)$, thus $u \in N_w^2(0, T, V)$ as well, and that $\partial_t U \in L^2(\Omega \times (0, T), H)$. In particular, for any t ,

$$\int_0^t (f(\partial_t U), \partial_t U) ds + \frac{1}{2} \|U(t)\|_A^2 = \frac{1}{2} \|u_0\|_A^2 - \int_0^t \left(\int_0^s Ah dw, \partial_t U \right) ds,$$

and

$$\alpha \int_0^t |\partial_t U|^2 ds + \|U(t)\|_A^2 \leq \|u_0\|_A^2 + \frac{1}{\alpha} \int_0^T \left| \int_0^s Ah dw \right|^2 ds + 2\lambda.$$

The same corollary asserts that, a.s., $U \in C([0, T], V)$. Thus, for any fixed time t and any sequence $(t_n) \in [0, T]$ such that t_n converges to t , one gets that $\|U(t_n) - U(t)\|$ goes to 0 a.s.

Thanks to the above inequality, Lebesgue's theorem yields $E\|U(t_n) - U(t)\|^2$ goes to 0 and leads to the continuity of U from $[0, T]$ to $L^2(\Omega, V)$. Then, thanks to the properties of the Ito integral, it is the same for u .

Consider two solutions u and \hat{u} , associated with

$$U = u - \int_0^t h dw \quad \text{and} \quad \hat{U} = \hat{u} - \int_0^t \hat{h} dw$$

and with the initial conditions u_0 and \hat{u}_0 . For convenience, set

$$W = u - \hat{u} - \int_0^t [h - \hat{h}] dw \quad \text{and} \quad w = u - \hat{u}$$

and note that for any $t > \Delta t > 0$,

$$\begin{aligned} (f(\partial_t U) - f(\partial_t \hat{U}), W(t) - W(t - \Delta t)) + (w(t), w(t) - w(t - \Delta t))_A \\ = \left(w(t), \int_{t-\Delta t}^t (h - \hat{h}) dw \right)_A. \end{aligned}$$

Then, an integration from Δt to t gives :

$$\begin{aligned} & \int_{\Delta t}^t \left(f(\partial_t U) - f(\partial_t \hat{U}), \frac{W(s) - W(s - \Delta t)}{\Delta t} \right) ds \\ & \quad + \frac{1}{2\Delta t} \int_{\Delta t}^t \|w(s) - w(s - \Delta t)\|_A^2 ds \\ & \quad + \frac{1}{2\Delta t} \int_{t-\Delta t}^t \|w(s)\|_A^2 ds \\ & \leq \frac{1}{2\Delta t} \int_0^{\Delta t} \|w(s)\|_A^2 ds \\ & \quad + \frac{1}{\Delta t} \int_{\Delta t}^t \left(w(s) - w(s - \Delta t), \int_{s-\Delta t}^s (h - \hat{h}) dw \right)_A ds \\ & \quad + \frac{1}{\Delta t} \int_{\Delta t}^t \left(w(s - \Delta t), \int_{s-\Delta t}^s (h - \hat{h}) dw \right)_A ds \end{aligned}$$

and, by taking the expectation, the following inequalities hold

$$\begin{aligned} & E \int_{\Delta t}^t \left(f(\partial_t U) - f(\partial_t \hat{U}), \frac{W(s) - W(s - \Delta t)}{\Delta t} \right) ds \\ & \quad + \frac{1}{2\Delta t} E \int_{t-\Delta t}^t \|w(s)\|_A^2 ds \\ & \leq \frac{1}{2\Delta t} E \int_0^{\Delta t} \|w(s)\|_A^2 ds + \frac{1}{2\Delta t} \int_{\Delta t}^t E \left\| \int_{s-\Delta t}^s (h - \hat{h}) dw \right\|_A^2 ds \\ & \leq \frac{1}{2\Delta t} E \int_0^{\Delta t} \|w(s)\|_A^2 ds + \frac{1}{2\Delta t} \int_{\Delta t}^t \int_{s-\Delta t}^s E \|(h - \hat{h})(\sigma)\|_A^2 d\sigma ds. \end{aligned}$$

At the limit, one gets that for any t

$$\begin{aligned} & E \int_0^t (f(\partial_t U) - f(\partial_t \hat{U}), \partial_t [U(t) - \hat{U}(t)]) ds + \frac{1}{2} E \|(u - \hat{u})(t)\|_A^2 \\ & \leq \frac{1}{2} E \|u_0 - \hat{u}_0\|_A^2 + \frac{1}{2} \int_0^t E \|h - \hat{h}\|_A^2 ds. \quad (4.1) \end{aligned}$$

Consider $h \in N_w^2(0, T, V)$ and $(h_n) \subset N_w^2(0, T, V)$ such that $Ah_n \in N_w^2(0, T, H)$ and (h_n) converges to h in $N_w^2(0, T, V)$. Thanks to the previous inequality, the sequence (u_n) of the corresponding solutions is a Cauchy sequence in $C([0, T], L^2(\Omega, V))$. As the same kind of calculations leads to the boundedness of $(\partial_t(u_n - \int_0^t h_n dw))$ in $L^2((0, T) \times \Omega, H)$, the uniqueness of the possible limit-point for the weak convergence yields the weak convergence of the sequence to $\partial_t(u - \int_0^t h dw)$ in $L^2((0, T) \times \Omega, H)$. Moreover, up to a subsequence, $f(\partial_t(u_{n_k} - \int_0^t h_{n_k} dw))$ converges weakly to a given element χ in $L^2((0, T) \times \Omega, H)$. Using again (4.1), one gets that

$$\limsup_{n,m} E \int_0^T (f(\partial_t U^n) - f(\partial_t \hat{U}^m), \partial_t [U^n(t) - \hat{U}^m(t)]) dt \leq 0,$$

and thanks to the assumptions on f , one concludes that $\chi = f(\partial_t(u - \int_0^t h dw))$, that a solution exists and that (4.1) holds for any h and $\hat{h} \in N_w^2(0, T, V)$.

4.1. The multiplicative case

Assume in this section that $H : V \rightarrow V$ is a Lipschitz-continuous mapping. Then $H(h) \in \mathcal{N}_w^2(0, T, V)$ if $h \in \mathcal{N}_w^2(0, T, V)$ ([17] lemma 2.41 p.35), and the result of this section is the following.

Denote by

$$\Phi : \mathcal{N}_w^2(0, T, V) \rightarrow \mathcal{N}_w^2(0, T, V)$$

the map defined for any $h \in \mathcal{N}_w^2(0, T, V)$ by $\Phi(h) = u$ where u is the solution of the Barenblatt’s problem

$$f\left(\partial_t[u - \int_0^t H(h)dw]\right) + Au = 0$$

for the initial condition $u(0, \cdot) = u_0$. Then, u is a solution to Problem (P_H) , if and only if, u is a fixed point to Φ . Then, for any positive α , (4.1) yields

$$\begin{aligned} \int_0^T e^{-\alpha t} E\|\Phi(h) - \Phi(\hat{h})(t)\|_A^2 dt &\leq 2 \int_0^T e^{-\alpha t} \int_0^t E\|[H(h) - H(\hat{h})](s)\|^2 ds dt \\ &\leq \frac{C}{\alpha} \int_0^T e^{-\alpha s} E\|[h - \hat{h}](s)\|_A^2 ds. \end{aligned}$$

Since the exponential weight in time provides an equivalent norm in $\mathcal{N}_w^2(0, T, V)$, if $\alpha > C$, Φ is a contractive mapping, it has an unique fixed-point and the result holds.

5. Annexe

Let us consider the following nonlinear parabolic problem:

$$(P) : \begin{cases} \partial_t u + Au = h \in L^2(0, T, H), \\ u(t = 0) = u_0 \in V. \end{cases}$$

It is a classical result that there exists a unique weak solution u and that this solution is the mild solution.

With the hypothesis on the data, $u \in W^{1,\infty,2}(0, T, V, H)$ and for any t ,

$$\int_0^t |\partial_t u|^2 ds + J(u(t)) \leq J(u_0) + \int_0^t (h, \partial_t u) ds.$$

Moreover, following [3] p.158 for example, one gets that for t in $(0, T)$ a.e., $u(t) \in D(A)$ and $J(u) \in W^{1,1}(0, T)$. By testing the equation with $u(\cdot + \Delta t) - u$, one has that

$$\begin{aligned} (\partial_t u(s), u(s + \Delta t) - u(s)) + \langle Au(s), u(s + \Delta t) - u(s) \rangle \\ = (h(s), u(s + \Delta t) - u(s)) \end{aligned}$$

and,

$$\begin{aligned} (\partial_t u(s), u(s + \Delta t) - u(s)) + J(u(s + \Delta t)) \\ \geq J(u(s)) + (h(s), u(s + \Delta t) - u(s)). \end{aligned}$$

By dividing by $\Delta t > 0$ and integrating in time from 0 to $t - \Delta t$, the continuity of $s \mapsto J(u(s))$ yields

$$\int_0^t |\partial_t u|^2 ds + J(u(t)) \geq J(u_0) + \int_0^t (h, \partial_t u) ds.$$

In conclusion, for any t ,

$$\int_0^t |\partial_t u|^2 ds + J(u(t)) = J(u_0) + \int_0^t (h, \partial_t u) ds.$$

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