Differential
Equations
& Applications
Volume 3, Number 4 (2011), 503–525

WAVE EQUATION WITH $p(x,t)$–LAPLACIAN AND DAMPING TERM: EXISTENCE AND BLOW–UP

STANISLAV ANTONTSEV

Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

(*Communicated by C. O. Alves*)

Abstract. In this work, we consider the Dirichlet problem for equation

$$u_{tt} = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) + \alpha\Delta u_t + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t).$$

Under suitable conditions on the functions a, b, f, p, σ the local, global and blow up solutions have been discussed.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary Γ and $Q_T = \Omega \times (0,T]$. We consider the following initial boundary value problem

$$u_{tt} = Lu + f(x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

$$Lu = \operatorname{div}\left(a(x,t)\,|\nabla u|^{p(x,t)-2}\,\nabla u + \alpha\nabla u_t\right) + b(x,t)\,|u|^{\sigma(x,t)-2}\,u,$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{2}$$

$$u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega \times (0,T). \tag{3}$$

Here $\alpha > 0$ is a constant. The coefficients $a(x,t), b(x,t),$ the exponents $p(x,t)$, $\sigma(x,t)$ and the source term $f(x,t)$ are given measurable functions of their arguments. We assume that

$$u_0 \in L^2(\Omega) \cap W^{1,p(\cdot,0)},\ u_1 \in L^2(\Omega),\ f \in L^2(Q_T). \tag{4}$$

We discuss existence and blow up of solutions to problem (1)-(3), concentrating our attention on difficulties caused by variable exponents $p(x,t)$, $\sigma(x,t)$. It should be

Mathematics subject classification (2010): 35B40, 35L70, 35L45.
Keywords and phrases: nonlinear wave equations, energy estimates, global existence, blow up, non-standard growth conditions.
This work was partially supported by the Research Project PTDC/MAT/110613/2009, FCT, Portugal and by the Research Project MTM2011-26119, MICINN, Spain.

Paper DEA-03-32

mentioned that questions of existence, uniqueness and regularity of weak solutions for parabolic and elliptic equations in the forms

$$u_t = \left(a_i(x,t,u)|u_{x_i}|^{p_i(x,t)-2}u_{x_i} + b_i(x,t,u)\right)_{x_i} + d(x,t,u), \tag{5}$$

$$\left(a_i(x,u)|u_{x_i}|^{p_i(x)-2}u_{x_i} + b_i(x,u)\right)_{x_i} + d(x,u) = 0, \tag{6}$$

have been studied by many authors under various conditions on the data and by different methods- (see, e.g., [4, 8, 9, 10, 11, 12, 13, 32, 37], and the further references therein). These equations are usually referred to as parabolic and elliptic equations with nonstandard growth conditions. Also the localization (vanishing) and blow up properties of energy weak solutions for elliptic and parabolic equations of the type (5) and (6) have been investigated sufficiently completely (see, e.g., [4, 6, 7, 12]).

Such elliptic, parabolic and hyperbolic equations occur in the mathematical modelling of various physical phenomena, e.g., the flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and the image processing (see, e.g., [5, 6, 8, 35, 36]) and the further references therein).

For hyperbolic equations in the form (1) with constant exponents p, σ local and global existence and blow up have been investigated in many papers - see, e.g.,[18, 19, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 33, 37, 40, 41, 42, 43, 44, 45, 46, 47] and the further references therein.

The hyperbolic equations with nonstandard growth conditions, to the best of our knowledge, were considered only in papers [14, 15, 22, 34].

In [22], the existence result was proved for the problem (1)-(3) with $a \equiv 1$, $p \equiv p(x)$, $b \equiv 0$. In the paper [34], the existence and blow of solutions were studied for the problem (1)-(3) with $a \equiv 1$, $p \equiv 2$, $b \equiv 0$ and the source term either is a power, $f(u) = b(x)u^{p(x)}$, or is nonlocal $f(u) = b(x)\int_\Omega u^{q(y)}(y,t)dy$. Existence and blow-up results for complete equation (1) were announced in [1, 2, 3].

The present paper is organized as follows. In Section 2 we introduce the function spaces of Orlicz-Sobolev type and a brief description of their main properties. Section 3 is devoted to proof the existence of local and global energy weak solutions to problem (1)-(3). The weak solution is obtained as the limit of the sequence of Galerkin's approximations. First we derive estimates for an energy functional. As in last section we will consider the blow-up for energy weak solutions with nonpositive energy functional.

Next under suitable conditions we obtain estimates for Galerkin's approximations in any finite time or only for small time. Further, we pass to the limit, using a standard monotonicity argument. Finally we prove existence theorems for small and any finite time. Section 4 is devoted to the investigation of the blow up of energy weak solutions. We consider separately two cases. First, we take $p(x,t) = p(x)$, $\sigma(x,t) = \sigma(x)$, $\alpha > 0$. Second, we take $p = p(x,t)$, $\sigma = \sigma(x,t)$ and $\alpha > 0$. Also we consider the case $\alpha = 0$, $p = p(x)$, $\sigma = \sigma(x)$ and establish conditional results, assuming that the problem (1)-(3) has at least one local energy solution for suitable initial data.

2. The function spaces

2.1. Spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$

The definitions of the function spaces used throughout the paper and a brief description of their properties follow [17].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ be Lipschitz-continuous, and let $p(x) \in [p^-, p^+] \subset (1,\infty)$ be log-continuous in Ω: $\forall x, y \in \Omega$ such that $|x-y| < \frac{1}{2}$,

$$|p(x) - p(y)| \leqslant \omega(|x-y|), \tag{7}$$

where

$$\overline{\lim}_{\tau \to 0+} \omega(\tau) \ln \frac{1}{\tau} = C < \infty.$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 :\ A_{p(\cdot)}(f/\lambda) \leqslant 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,p(\cdot)}(\Omega)$ with $p(x) \in [p^-, p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) : (|u|, |\nabla u|) \in L^{p(\cdot)}(\Omega)\right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega} + \|u\|_{p(\cdot),\Omega}. \end{cases} \tag{8}$$

An equivalent norm of $W_0^{1,p(\cdot)}$ is given by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega}.$$

- If condition (7) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ can be defined then as the closure of $C_0^\infty(\Omega)$ with respect to the norm (8) – see [17, 38, 48].
- The space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that $p(x) \in C^0(\overline{\Omega})$.
- Let

$$1 < q(x) \leqslant \sup_\Omega q(x) < \inf_\Omega p_*(x),$$

$$\text{with} \quad p_*(x) = \begin{cases} \dfrac{p(x)n}{n - p(x)} & \text{if } p(x) < n, \\ \infty & \text{if } p(x) > n. \end{cases}$$

Then

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leqslant C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

and the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

- It follows directly from the definition that

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \leqslant A_{p(\cdot)}(f) \leqslant \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right).$$

- Hölder's inequality. For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1,\infty), \quad p' = \frac{p}{p-1},$$

the following inequality holds:

$$\int\limits_{\Omega} |f g|\, dx \leqslant \left(\frac{1}{p^-} + \frac{1}{(p)'^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leqslant 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} .$$

2.2. Spaces $L^{p(\cdot,\cdot)}(Q_T)$ and $\mathbf{W}(Q_T)$

Let us assume that the exponent $p(x,t) \in [p_-, p_+] \subset (1,\infty)$ is continuous in $\overline{Q}_T$ with logarithmic module of continuity:

$$|p(x,t) - p(y,\tau)| \leqslant \omega(|x-y| + |t-\tau|), \tag{9}$$

where

$$\overline{\lim_{s\to+0}}\, \omega(s) \ln \frac{1}{s} \leqslant C < \infty.$$

Under condition (9) the space $C^\infty(0,T,C_0^\infty(\Omega))$ is dense in $L^{p(\cdot)}(Q_T)$ and the last one can be defined then as the closure of $C^\infty(0,T,C_0^\infty(\Omega))$ (see, [17, 38, 48]).

We introduce the Banach space

$$\mathbf{V}_t(\Omega) = \left\{u : u \in L^2(\Omega) \cap W_0^{1,p_-}(\Omega) \cap W_0^{1,2}(\Omega),\, |\nabla u|^{p(\cdot,t)} \in L^1(\Omega)\right\},$$
$$\|u\|_{\mathbf{V}_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(\cdot,t),\Omega},$$

and denote by $\mathbf{V}_t'(\Omega)$ it's dual. For every $t \in [0,T]$ the inclusion

$$\mathbf{V}_t(\Omega) \subset \mathbf{X} = W_0^{1,p^-}(\Omega) \cap L^2(\Omega) \cap W_0^{1,2}(\Omega)$$

holds, which is why $\mathbf{V}_t(\Omega)$ is reflexive and separable as a closed subspace of $\mathbf{X}$.

By $\mathbf{W}(Q_T)$ we denote the Banach space

$$\mathbf{W}(Q_T) = \left\{u :\, [0,T] \mapsto \mathbf{V}_t(\Omega) |\, u, u_t, |\nabla u|^{p(\cdot)/2} \in L^2(Q_T),\, u = 0 \text{ on } \Gamma_T\right\},$$
$$\|u\|_{\mathbf{W}(Q_T)} = \|\nabla u\|_{p(\cdot),Q_T} + \|u\|_{2,Q_T} + \|u_t\|_{2,Q_T} + \|\nabla u_t\|_{2,Q_T} .$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$. Set

$$\mathbf{V}_+(\Omega)=\left\{u|\; u\in L^2(\Omega)\cap W_0^{1,1}(\Omega),\; |\nabla u|\in L^{p^+}(\Omega)\right\}.$$

By analogy with [11] we prove the following propositions.

(a) Let us prove first that every function $u\in C^\infty(0,T;C_0^\infty(\Omega))$ can be approximated in the norm of $\mathbf{W}(Q_T)$ by the functions $\sum_{k=1}^m d_k(t)\psi_k(x)$ with $d_k(t)\in C^2[0,T]$. We denote the set of such functions by P_m.

Let us take for $\{\psi_k\}$ the orthonormal basis of the Hilbert space $H_0^s(\Omega)$ with $s\geqslant 1$ so big that $H_0^s(\Omega)$ is dense in $\mathbf{V}_+(\Omega)$. According to ([27], Ch.6), we can take a special basis $\psi_k(x)$ such that

$$(v,\,\psi_k)_{H_0^s(\Omega)}=\lambda_k(v,\psi_k),\;\forall v\in H_0^s(\Omega),\;\lambda_k>0.$$

Let us represent

$$u=\sum_{i=1}^\infty u_i(t)\psi_i(x),\quad u_i(t)=(u(x,t),\,\psi_i(x))_{H_0^s(\Omega)}. \tag{10}$$

For every $t\in[0,T]$,

$$\|u_t\|^2_{H_0^s(\Omega)}(t)+\|\nabla u_t\|^2_{H_0^s(\Omega)}(t)+\|u\|^2_{H_0^s(\Omega)}(t)$$
$$=\sum_{i=1}^\infty((u_i'(t))^2(1+\lambda_i)+u_i^2(t))<\infty. \tag{11}$$

Let us consider the sequence $\{u^{(m)}\}$, $u^{(m)}=\sum_{i=1}^m u_i(t)\psi_i(x)$. For every $t\in[0,T]$,

$$\|u_t-u_t^{(m)}\|^2_{H_0^s(\Omega)}(t)+\|\nabla u_t-\nabla^{(m)}u_t\|^2_{H_0^s(\Omega)}(t)+\|u-u^{(m)}\|^2_{H_0^s(\Omega)}(t)$$
$$=\sum_{i=m+1}^\infty((u_i'(t))^2(1+\lambda_i)+u_i^2(t))\to 0$$

as $m\to\infty$, because of (11). Since for every $t\in[0,T]$ the sequence

$$\phi_m(t)\equiv\|u-u^{(m)}\|^2_{H_0^s(\Omega)}(t)+\|u_t-u_t^{(m)}\|^2_{H_0^s(\Omega)}(t)+\|\nabla u_t-\nabla^{(m)}u_t\|^2_{H_0^s(\Omega)}(t)$$

is monotone decreasing, nonnegative, and tends to zero as $m\to\infty$, by the Beppo Levi theorem

$$\int_0^T\phi_m(\tau)\,d\tau\to 0\quad\text{as } m\to\infty.$$

Thus,

$$\|u-u^{(m)}\|_{\mathbf{W}(Q)}\leqslant C\Big(\|\big(|u-u^{(m)}|+|u_t-u_t^{(m)}|$$
$$+|\nabla(u_t-u_t^{(m)})|\big)\|^2_{L^2(0,T;H_0^s(\Omega))}\Big)\to 0$$

as $m\to\infty$.

(b) Let now $u\in\mathbf{W}(Q)$ and $\{u_\delta\}$ be the sequence of mollifiers such that $u_\delta\in C^\infty(0,T;C_0^\infty(\Omega))$. Given ε, we take δ such that $\|u-u_\delta\|_{\mathbf{W}(Q)}<\varepsilon$ and approximate u_δ using (a):

$$\left\|u-u_\delta^{(m)}\right\|_{\mathbf{W}(Q)}\leqslant\|u-u_\delta\|_{\mathbf{W}(Q)}+\|u_\delta-u_\delta^{(m)}\|_{\mathbf{W}(Q)}<2\varepsilon,$$

with

$$u_\delta^{(m)}=\sum_{i=1}^m d_i(t)\psi_i(x),\qquad d_i(t)=(u_\delta,\psi_i)_{H_0^s(\Omega)}\in C^\infty[0,T].$$

PROPOSITION 2.1. *For every $u\in\mathbf{W}(Q_T)$ there is a sequence $\{d_k(t)\}$, $d_k(t)\in C^2[0,T]$, such that*

$$\|u-\sum_{k=1}^m d_k(t)\psi_k(x)\|_{\mathbf{W}(Q_T)}\to 0\quad \textit{as } m\to\infty.$$

We introduce also a subset of functions $u\in W(Q_T)$ such that

$$W_\infty=\left\{u:u\in W(Q_T),\left(u,u_t,\ |\nabla u|^{p/2},\ |u|^{\sigma/2}\right)\in L^\infty(0,T;L^2(\Omega))\right\}.$$

3. Local and global existence

3.1. Definition. Main results

DEFINITION 3.1. A function $u\colon\Omega_T\to\mathbb{R}$ is called a energy weak solution to (1)-(3) if:

$$u\in W(Q_T)\cap W_\infty(Q_T);\tag{12}$$

$$u(\cdot,t)\to u_0\ \text{ in } W_0^{1,2}(\Omega)\cap W^{1,p(\cdot,0)}(\Omega),\ u_t(\cdot,t)\to u_1\ \text{ in } L^2(\Omega);\tag{13}$$

$$\int_{Q_T}\left(-u_t\varphi_t+\left(a\,|\nabla u|^{p(\cdot)-2}\nabla u+\alpha\nabla u_t\right)\cdot\nabla\varphi-b\,|u|^{\sigma(\cdot)-2}u\varphi\right)dxdt$$
$$=\int_\Omega u_1\varphi(\cdot,0)dx+\int_{Q_T}f\varphi dx,\tag{14}$$

for all $\varphi\in C^\infty\left(0,T;C_0^\infty(\Omega)\right)$, $\varphi(x,T)=0, x\in\Omega$.

Let us assume that the coefficients of the problem (1)-(3), in addition to (9), satisfy

$$0<a_-\leqslant a(x,t)\leqslant a_+<\infty,\ |a_t|\leqslant C_a,\ 0<\alpha,\tag{15}$$

$$1 < p_- \leqslant p(x,t) \leqslant p_+ < \infty,\ |p_t| = -p_t \leqslant C_p, \tag{16}$$

$$1 < \sigma_- \leqslant \sigma(x,t) \leqslant \sigma_+ < \infty,\ 0 \leqslant \sigma_t \leqslant C_\sigma, \tag{17}$$

and that one of the following conditions be true: either

$$0 < b_- \leqslant -b(x,t) \leqslant b_+ < \infty,\ 0 \leqslant b_t, \tag{18}$$

or

$$0 < b_- \leqslant b(x,t) \leqslant b_+ < \infty,\ 0 \leqslant b_t \ \text{ and } \ \sigma_+ \leqslant 2 \ \text{ or } \ \sigma_+ < p_-. \tag{19}$$

Also it is assumed that

$$u_0 \in L^2(\Omega) \cap W_0^{1,2}(\Omega) \cap W^{1,p(\cdot,0)}(\Omega),\ u_1 \in L^2(\Omega),\ f \in L^2(Q_T). \tag{20}$$

In this section we prove global and local in time existence theorems.

THEOREM 3.1. (Global existence in time) *Under conditions* (9), (15)-(20), *the problem* (1)-(3) *has at least one energy weak solution in the sense of Definition* 3.1 *which is global in time (for any $t \in [0,T]$, $T < \infty$).*

Let us assume that

$$0 < a_- \leqslant a(x,t) \leqslant a_+ < \infty,\ |a_t| \leqslant C_a, \tag{21}$$

$$0 < b_- \leqslant b(x,t) = b(x,t) \leqslant b_+ < \infty,\ |b_t| \leqslant C_b, \tag{22}$$

$$p_t \leqslant 0, |p_t| \leqslant C_p,\ 0 \leqslant \sigma_t \leqslant C_\sigma, \tag{23}$$

and

$$2 < \sigma_- \leqslant \sigma_+ < \frac{n+2}{n} p_-, \quad \frac{2n}{n+2} < p_-. \tag{24}$$

THEOREM 3.2. (Local existence in a small time) *Under conditions* (9), (21)-(24), *the problem* (1)-(3) *has at least one weak solution in the sense of Definition* 3.1 *for a small time $t \in [0,T_0)$, ($T_0 > 0$ is small).*

3.2. Step 1. Galerkin's approximations

The Galerkin's approximations of solutions to problem (1)-(3) are sought in the form

$$u^{(m)} \equiv \sum_{k=1}^{m} u_k(t)\psi_k(x),\ u_k(t) = (u(x,t),\ \psi_k(x))_{H_0^s(\Omega)}. \tag{25}$$

We assume also

$$u_1^{(m)} \to u_1 \ \text{ strongly in } L^2(\Omega), \quad u_0^{(m)} \to u_0 \ \text{ strongly in } W_0^{1,2}(\Omega). \tag{26}$$

The coefficients $u_k(t)$ are defined from the relations

$$\int_\Omega (u_{tt}^{(m)} - Lu^{(m)} - f)\psi_k = 0,\ k = 1,...,m. \tag{27}$$

Last equalities and the initial conditions lead us to the Cauchy problem for the system of m ordinary differential equations of the second order for the coefficients $u_k(t)$

$$u_k'' = F_k(t, u_1(t), ..., u_m(t)), \tag{28}$$

$$u_k(0) = \int_\Omega u_0 \psi_k,\ u_k'(0) = \int_\Omega u_1 \psi_k,\ \ k = 1, ..., m, \tag{29}$$

where

$$F_k = \int_\Omega \left[-\left(\left(a \left| \nabla u^{(m)}(\cdot,t) \right|^{p(\cdot,t)-2} \right) \nabla u^{(m)} + \alpha \nabla u_t^{(m)} \right) \nabla \psi_k \right] dx \\ + \int_\Omega \left[b \left| u^{(m)} \right|^{\sigma(\cdot,t)-2} u^{(m)} \psi_k + f \psi_k \right] dx.$$

By Peano's Theorem, for every finite m the problem (28), (29) has a solution $u_k(t)$, $k = 1, ..., m$ on an interval $(0, T_m)$ for each m. The estimates below allow one to take $T_m = T$ for all m.

3.3. Step 2. A priori estimates

Here we derive estimates for an energy functional and for approximated solutions which do not depend on m.

3.3.1. Energy relation

Multiplying each of equations (28) by $c_k'(t)$ and summing over $k = 1, ..., m,$ we arrive at the relation

$$\frac{d}{dt} \int_\Omega \left[\frac{\left|u_t^{(m)}\right|^2}{2} + \frac{a\left|\nabla u^{(m)}\right|^p}{p} - \frac{b\left|u^{(m)}\right|^\sigma}{\sigma} \right] dx + \alpha \int_\Omega \left|\nabla u_t^{(m)}\right|^2 dx \\ = \int_\Omega \left[a_t \frac{|\nabla u^{(m)}|^p}{p} + \frac{a|\nabla u^{(m)}|^p}{p^2} \left(1 - p \ln \left|\nabla u^{(m)}\right| \right) \left|p_t\right| \right] dx \\ - \int_\Omega \left(\frac{b_t |u^{(m)}|^\sigma}{\sigma} + \frac{b|u^{(m)}|^\sigma}{\sigma^2} \left(1 - \sigma \ln |u^{(m)}| \right) \sigma_t \right) dx + \int_\Omega f u_t^{(m)} dx. \tag{30}$$

Omitting the index m for simplicity and introducing the energy functional

$$E(t) = \int_\Omega \left[\frac{|u_t(\cdot,t)|^2}{2} + a(\cdot,t) \frac{|\nabla u|^{p(\cdot,t)}}{p(\cdot,t)} - b(\cdot,t) \frac{|u|^{\sigma(\cdot,t)}}{\sigma(\cdot,t)} \right] dx, \tag{31}$$

we can rewrite (30) in the form

$$E'(t) + \alpha \int_\Omega |\nabla u_t(\cdot,t)|^2 \, dx = \Lambda, \tag{32}$$

where

$$\Lambda(t) = \Lambda_1 + \Lambda_2 + \Lambda_3, \tag{33}$$

$$\Lambda_1 = \int_\Omega \left[a_t \frac{|\nabla u|^p}{p} + \frac{a|\nabla u|^p}{p^2}(1 - p\ln|\nabla u|)|p_t| \right] dx, \tag{34}$$

$$\Lambda_2 = -\int_\Omega \left(\frac{b_t|u|^\sigma}{\sigma} + \frac{b|u|^\sigma}{\sigma^2}(1-\sigma\ln|u|)\,\sigma_t \right) dx, \tag{35}$$

$$\Lambda_3 = \int_\Omega f u_t dx\,. \tag{36}$$

3.3.2. Estimates of the energy functional

In this Section we analyze different conservation laws for energy functional and solution estimates which do not depend on m.

In particular, we derive some estimates for energy functional which will be important to prove the blow-up.

Let us assume that

$$1 < p_- \leqslant p(x,t) \leqslant p_+ < \infty,\ 1 < \sigma_- \leqslant \sigma(x,t) \leqslant \sigma_+ < \infty, \tag{37}$$

$$0 < a_- \leqslant a(x,t) \leqslant a_+ < \infty,\ |a_t| \leqslant C_a, \tag{38}$$

$$0 < b_- \leqslant b(x,t) = b(x,t) < b_+ < \infty,\ |b_t| \leqslant C_b, \tag{39}$$

$$a_t \leqslant 0,\ 0 \leqslant b_t,\ p_t \leqslant 0,\ 0 \leqslant \sigma_t,\ |p_t| \leqslant C_p, |\sigma_t| \leqslant C_\sigma. \tag{40}$$

LEMMA 3.1. *Let* (37)-(39) *be fulfilled and in addition*

$$p_t = \sigma_t = f = 0. \tag{41}$$

Then

$$E(t) + \alpha \int_0^t \int_\Omega |\nabla u_t(x,s)|^2 dxds \leqslant E(0),\ \forall t \geqslant 0. \tag{42}$$

The inequality (42) *transforms to the equality if* $a_t = b_t = 0$.

Proof. To prove this Lemma it is enough to apply the formulas (32)-(35). □

LEMMA 3.2. *Let* (37)-(40) *be fulfilled and* $f = 0$. *Then*

$$E(t) + \int_0^t \int_\Omega \alpha |\nabla u_t(x,s)|^2 dxds \leqslant E(0) + Ct, \tag{43}$$

with the constant $C = e(a_+C_p + b_+C_\sigma)|\Omega|$.

Proof. We evaluate Λ_1, Λ_2 in the following way:

$$\begin{aligned}\Lambda_1 &= \int_\Omega \left[a_t \frac{|\nabla u|^p}{p} + \frac{a|\nabla u|^p}{p^2}\big(1 - p\ln|\nabla u|\big)|p_t| \right] dx \\ &\leqslant \int_{\Omega\cap(p\ln|\nabla u|\leqslant 1)} \frac{a|\nabla u|^p}{p^2}\big(1 - p\ln|\nabla u|\big)|p_t| dx\end{aligned}$$

$$\leqslant ea_+ C_p |\Omega| = C_1, \tag{44}$$

and

$$\begin{aligned}\Lambda_2 &= -\int_\Omega \Big[\frac{b_t |u\,|^\sigma}{\sigma} + \frac{b|u\,|^\sigma}{\sigma^2}\big(1-\sigma \ln|u\,|\big)\sigma_t\Big]dx \\ &\leqslant \int_{\Omega\cap(\sigma ln|u|\leqslant 1)} \frac{b|u\,|^\sigma}{\sigma^2}\big(1-\sigma\ln|u\,|\big)\sigma_t dx \\ &\leqslant eb_+ C_\sigma |\Omega| = C_2. \end{aligned}\tag{45}$$

Integrating the energy relation (32) with respect to t, we obtain that

$$E(t) + \int_0^t \int_\Omega \alpha\, |\nabla u_t|^2\, dxds \leqslant E(0) + tC, C = C_1 + C_2. \tag{46}$$

$\square$

3.3.3. A priori estimates of solutions

LEMMA 3.3. (Global estimates of solutions ($b(x,t) \leqslant 0$)). *Let the conditions* (9), (15)-(20) *be fulfilled. Then for any finite* $T < \infty$*,*

$$\Psi(T) = \sup{}_{0\leqslant t\leqslant T} \int_\Omega \Big[|u_t|^2 + |\nabla u|^{p(\cdot)} + |u|^{\sigma(\cdot)}\Big]dx \\ + \alpha \int_0^T \int_\Omega |\nabla u_t|^2 dxds \leqslant C \tag{47}$$

with a constant C *which depends on:*

$$\|f\|^2_{2,Q_T},\ \|u_1\|_{L^2(\Omega)},\ \|u_0\|_{L^{\sigma(\cdot,0)}(\Omega)},\ \|u_0\|_{W^{1,2}(\Omega)},\ |\Omega|,\ T$$

and does not depend on m*.*

Proof. Recall that in this case all terms of the energy functional E are nonnegative. Then we obtain

$$\begin{aligned}\Lambda_1 &= \int_\Omega \Big[a_t \frac{|\nabla u\,|^p}{p} + \frac{a|\nabla u\,|^p}{p^2}(1-p\ln|\nabla u\,|)|p_t|\Big]dx \\ &\leqslant \frac{C_a}{a_-}E(t) + \int_{\Omega\cap(p\ln|\nabla u|\leqslant 1)} \frac{a|\nabla u\,|^p}{p^2}\big(1-p\ln|\nabla u\,|\big)|p_t|dx \\ &\leqslant \frac{C_a}{a_-}E(t) + \frac{a_+}{p_-^2}e\,|\Omega| \leqslant C(E(t)+1),\end{aligned}$$

and

$$\Lambda_2 = -\int_\Omega \Big(\frac{b_t|u\,|^\sigma}{\sigma} + \frac{b|u\,|^\sigma}{\sigma^2}\big(1-\sigma\ln|u\,|\big)\sigma_t\Big)dx$$

$$\leqslant \frac{C_b}{\sigma_-}E(t)+\int_{\Omega\cap(\sigma ln|u|\leqslant 1)}\frac{b|u\,|^{\sigma}}{\sigma^2}\big(1-\sigma\ln|u\,|\big)\sigma_t dx$$
$$\leqslant \frac{C_b}{\sigma_-}E(t)+\frac{b_+}{\sigma_-^2}e\,|\Omega|\leqslant C(E(t)+1),$$

and

$$|\Lambda_3|\leqslant E(t)+\|f\|^2_{2,\Omega}.$$

Finally,

$$\Lambda\leqslant C(E(t)+\|f\|^2_{2,\Omega}+1).$$

Hence we arrive at the inequality

$$E'(t)+\alpha\int_\Omega |\nabla u_t(\cdot,t)|^2\,dx\leqslant C(E(t)+\|f\|^2_{2,\Omega}+1).$$

Applying Gronwall's lemma, we conclude the proof of the Lemma. □

LEMMA 3.4. (Global estimates of solutions ($0\leqslant b(x,t)$, $\sigma_+\leqslant 2$ or $2<\sigma_+<p_-$.)) *The estimate* (47) *remains valid if the condition* $b(x,t)\leqslant 0$ *is replaced by the following*

$$0\leqslant b(x,t),\ \sigma_+\leqslant 2\quad or\quad 2<\sigma_+<p_-.$$

Proof. In this case, we rewrite the energy relation (32) in the form

$$\widetilde{E}'(t)+\alpha\int_\Omega |\nabla u_t(\cdot,t)|^2\,dx=\left(\int_\Omega b\frac{|u|^\sigma}{\sigma}dx\right)'+\Lambda,$$

where

$$\widetilde{E}(t)=\int_\Omega\left[\frac{|u_t|^2}{2}+a(\cdot,t)\frac{|\nabla u|^{p(\cdot,t)}}{p(\cdot,t)}\right]dx.$$

Integrating last one with respect to t, we obtain

$$\widetilde{E}(t)+\alpha\int_0^t\int_\Omega |\nabla u_t(\cdot,s)|^2\,dxds=\left(\int_\Omega b\frac{|u|^\sigma}{\sigma}\right)dx\Big|_0^t+\int_0^t\Lambda ds+\widetilde{E}(0), \tag{48}$$

with

$$\left|\left(\int_\Omega b\frac{|u|^\sigma}{\sigma}dx\right)\Big|_0^t\right|\leqslant C\Big(\int_\Omega |u|^\sigma\,dx+1\Big),\quad |\Lambda|\leqslant C\widetilde{E}(t)+C\int_\Omega |u|^\sigma dx.$$

Now it is enough to evaluate the term $\int_\Omega |u|^\sigma\,dx$. If $\sigma_+\leqslant 2$ we use the chain of inequalities

$$\Big|\int_\Omega b(\cdot,t)\frac{|u|^\sigma}{\sigma}dx\Big|\leqslant C\int_\Omega |u|^\sigma\,dx$$
$$\leqslant C\Big(1+\int_\Omega |u|^2\,dx\Big)$$

$$\leqslant 2C\left(1+t\int_0^t\int_\Omega |u_t|^2\,dxds+\int_\Omega |u_0|^2\,dx\right)$$
$$\leqslant \widetilde{C}\left(1+t\int_0^t \widetilde{E}(s)ds\right).$$

If $2<\sigma_+<p_-$ we use the embedding inequality

$$\int_\Omega |u\,|^{\sigma_+}dx \leqslant C\left(\int_\Omega |\nabla u\,|^{p_-}dx\right)^{\frac{\sigma_+}{p_-}}$$
$$\leqslant C\left(\int_\Omega |\nabla u\,|^{p}dx\right)^{\frac{\sigma_+}{p_-}}+C$$
$$\leqslant \varepsilon\widetilde{E}(t)+C(\varepsilon),\ \varepsilon\in(0,1),\ \sigma_+<p_-<np_-/(n-p_-).$$

For suitable $\varepsilon>0$ we come to the inequality

$$\widetilde{E}(t)+\alpha\int_0^t\int_\Omega |\nabla u_t(\cdot,s)|^2\,dxds\leqslant C\left(\int_0^t \widetilde{E}(s)ds+1\right)$$

and Gronwall's lemma and (48) lead us to the desired estimate.

LEMMA 3.5. (Local estimates $2<\sigma_-\leqslant\sigma_+<\frac{n+2}{n}p_-$, $\frac{2n}{n+2}<p_-$) *Let the conditions* (21)-(24) *be fulfilled. Then there exists a small* $T_0>0$ *such that*

$$\Psi(t)=\sup_{0\leqslant s\leqslant t}\int_\Omega\left[|u_t(x,s)|^2+|\nabla u|^{p(\cdot)}+|u|^{\sigma(\cdot)}\right]dx$$
$$+\alpha\int_0^t\int_\Omega |\nabla u_t|^2\,dxds\leqslant C,\ 0\leqslant t<T_0 \tag{49}$$

with a constant C *which depends on:*

$$T_0,\ \|f\|^2_{2,Q_T},\ \|u_1\|_{L^2(\Omega)},\ \|u_0\|_{L^{\sigma(\cdot,0)}(\Omega)},\ \|u_0\|_{W^{1,p(\cdot,0)}(\Omega)}$$

but does'nt depend on m.

Proof. We will use the energy relation (48) and inequalities

$$\left|\int_0^t\Lambda\right|\leqslant C\int_0^t\left(\widetilde{E}(s)+\int_\Omega |u|^\sigma\,dx\right)ds, \tag{50}$$

$$\int_\Omega |u|^{\sigma(\cdot,t)}\,dx\leqslant\int_\Omega |u|^{\sigma_+}\,dx+|\Omega|,\ \int_\Omega |\nabla u|^{p_-}\,dx\leqslant\int_\Omega |\nabla u|^{p}\,dx+|\Omega|, \tag{51}$$

$$\int_\Omega |u|^{\sigma_+}\,dx\leqslant C\left(\int_\Omega |\nabla u|^{p_-}\,dx\right)^{\frac{\sigma_+}{p_-}\theta}\left(\int_\Omega |u|^2\,dx\right)^{\frac{\sigma_+}{2}(1-\theta)}$$
$$\leqslant\varepsilon\int_\Omega |\nabla u|^{p(\cdot,t)}\,dx+C_\varepsilon\left(\int_\Omega |u|^2\,dx\right)^{\gamma}+C,\ \ \varepsilon\in(0,1), \tag{52}$$

where

$$\frac{\sigma_+}{p_-}\theta = \frac{(\sigma_+ - 2)n}{np_- - 2(n-p_-)} < 1, \text{ if } \sigma_+ < \frac{n+2}{n}p_-,$$
$$\gamma = \frac{\sigma_+ p_-}{2(p_- - \theta\sigma_+)} > 1.$$

On the other hand we have that

$$\int_\Omega |u|^2\,dx \leqslant C\left(\int_0^t \int_\Omega |u_t|^2\,dxds\right) + \int_\Omega |u_0|^2\,dx \leqslant C\left(\int_0^t \widetilde{E}(s)ds + 1\right).$$

Combining (48), (50)-(52) with a suitable ε , we come to the inequality

$$\widetilde{E}(t) \leqslant C\left(\left(\int_0^t \widetilde{E}(s)ds\right)^\gamma + 1\right), \ \gamma > 1,$$

which gives the estimate

$$\widetilde{E}(t) + 1 \leqslant \left(\widetilde{E}(0)+1\right)\left(1 - tC(\widetilde{E}(0)+1)^{\gamma-1}(\gamma-1)\right)^{-\frac{1}{\gamma-1}} < \infty,$$
$$t < T_0 = \left(C(\widetilde{E}(0)+1)^{\gamma-1}(\gamma-1)\right)^{-1}.$$

The Lemma is proved. □

REMARK 3.1. The estimates of the Lemmas 3.3 and 3.4 imply that

$$\Lambda(T) = \sup_{t\in[0,T]} \int_\Omega \left(|u|^2 + |u|^{\frac{2n}{n-2}} + |u|^q\right) dx \leqslant C, \tag{53}$$

where $q \leqslant np_-/(n-p_-)$ if $p_- < n$ and $q < \infty$ if $p_- \geqslant n$.

Also we have that

$$\int_\Omega |\nabla u(x,t)|^2\,dx \leqslant 2\left(\int_\Omega |\nabla u_0(x)|^2\,dx + t\int_0^T \int_\Omega |\nabla u_t|^2\,dxds\right),$$
$$\int_0^T \left(\int_\Omega |u_t|^q\,dx\right)^{\frac{2}{q}} dt \leqslant C\int_{Q_T} |\nabla u_t|^2\,dxdt,$$
$$1 \leqslant q \leqslant \frac{2n}{n-2}, \ n > 2, \ 1 \leqslant q < \infty, \ n = 2.$$

Finally we arrive at the estimate

$$\Lambda(T) + \sup_{0\leqslant s\leqslant T} \int_\Omega \left[|u_t(x,s)|^2 + |\nabla u|^2 + |\nabla u|^p + |u|^\sigma\right] dx$$
$$+ \int_0^T \int_\Omega |\nabla u_t|^2 dxds + \int_0^T \left(\int_\Omega |u_t|^q dx\right)^{\frac{2}{q}} dt \leqslant K \tag{54}$$

with a constant K independent on m. Last estimate is valid for any finite interval $[0,T]$ (under conditions of Lemmas 3.3 and 3.4) or only for a small interval of the time $[0,T_0)$ (under conditions of Lemma 3.5).

3.3.4. Compactness of $u_t^{(m)}$

LEMMA 3.6. *Assume that the estimate* (54) *is valid. Then*

$$\left|\int_{Q_T} u_{tt}^{(m)} \varphi dxdt\right| \leqslant C||\varphi||_{W(Q_T)},\ T < T_0 \tag{55}$$

with a constant C independent of m.

Proof. Let $d_k(t) \in C^2(0,T)$ be arbitrary functions. Multiplying (27) by $d_k(t)$, integrating over $[0,T]$ and summing with respect to k, we arrive at the identity

$$\int_{Q_T} \left(u_{tt}^{(m)} \varphi + \left(a|\nabla u^{(m)}|^{p(x,t)-2}\nabla u^{(m)} + \alpha \nabla u_t^{(m)}\right)\nabla\varphi\right) dxdt \\ - \int_{Q_T} \left(b|u^{(m)}|^{\sigma-2}u^{(m)} + f\right)\varphi dxdt = 0 \tag{56}$$

which is valid for any function

$$\varphi = \sum_{k=1}^{N} d_k(t)\psi_k(x),\ N \leqslant m. \tag{57}$$

We introduce

$$\overrightarrow{G}_m = \left(a\left|\nabla u^{(m)}\right|^{p(\cdot)-2}\nabla u^{(m)} + \alpha \nabla u_t^{(m)}\right).$$

Then (56) takes the form

$$\int_{Q_T} u_{tt}^{(m)} \varphi dxdt = \int_{Q_T} \left(-\overrightarrow{G}_m \nabla\varphi + \left(b|u^{(m)}|^{\sigma-2}u^{(m)} + f\right)\varphi\right) dxdt = J.$$

Using results of Section 2 it's easy to verify that

$$\begin{aligned} |J| \leqslant{}& a_+\left\|\left|\nabla u^{(m)}\right|^{p(\cdot)-1}\right\|_{\frac{p(\cdot)}{p(\cdot)-1},Q_T} \|\nabla\varphi\|_{p(\cdot),Q_T} \\ &+ \alpha\left\|\nabla u_t^{(m)}\right\|_{2,Q_T}\left\|\nabla\varphi\right\|_{2,Q_T} + b_+\left\||u^{(m)}|^{\sigma-1}\right\|_{\frac{\sigma(\cdot)}{\sigma(\cdot)-1},Q_T} ||\varphi||_{\sigma(\cdot),Q_T} \\ &+ ||f||_{2,Q_T}||\varphi||_{2,Q_T} \\ \leqslant{}& C(K)||\varphi||_{W(Q_T)}. \end{aligned}$$

The Lemma is proved. □

REMARK 3.2. Applying the known inequality

$$\int_{\Omega} \left|\frac{w(x+h)-w(x)}{h}\right|^2 dx \leqslant C||\nabla w||^2_{L^2(\Omega)},\ \forall w \in W_0^1(\Omega)$$

to the function $w = u_t^{(m)}$, we derive from (54),

$$\int_0^T \int_\Omega \left|u_t^{(m)}(x+h,t) - u_t^{(m)}(x,t)\right|^2 dxdt \leqslant C(K)|h|^2. \tag{58}$$

According to results of [39], Lemma 3.6 and estimate (58) we can conclude the compactness of the sequence u_{mt} in the following sense

$$u_t^{(m)} \to u_t \quad \text{strongly in } L^2(Q_T) \cap L^2(\Omega). \tag{59}$$

In certain papers (see, for example [43, 44]) to prove the compactness $u_t^{(m)}$, they used the following

LEMMA 3.7. ([16]) *Let Ω be any bounded domain in $\mathbb{R}^n$, $\{w_k\}_{k=1}^{\infty}$ be an orthogonal basis in $L^2(\Omega)$. Then for every $\varepsilon > 0$, there exists a positive number $N_{1\varepsilon}$ such that*

$$\|u\|_{2,\Omega} = \|u\| \leqslant \left(\sum_{k=1}^{N_\varepsilon} (u, w_k)_\Omega^2\right)^{\frac{1}{2}} + \varepsilon\|u\|_{1,q}$$

for all $u \in W_0^{1,q}(\Omega)$, $(2 \leqslant q < \infty)$.

By this Lemma and estimates (54), for every $\varepsilon > 0$, there exist positive constants $N_{1\varepsilon}$ and $N_{2\varepsilon}$ independent of m such that, as $m \to \infty$,

$$\begin{aligned} \|u^{(m)}(t) - u(t)\| \leqslant & \left(\sum_{k=1}^{N_{1\varepsilon}} \left(u^{(m)} - u, w_k\right)_\Omega^2\right)^{1/2} \\ & + \varepsilon\|u^{(m)}(t) - u(t)\|_{1,2} \leqslant C(T)\varepsilon,\ t \in [0,T], \end{aligned} \tag{60}$$

and

$$\begin{aligned} \int_0^T \|u_t^{(m)} - u_t\|^2 \leqslant 2 & \left(\sum_{k=1}^{N_{2\varepsilon}} \int_0^T \left(u_t^{(m)} - u_t, w_k\right)_\Omega^2\right) \\ & + 2\varepsilon^2 \int_0^T ||u_t^{(m)} - u_t||_{1,2}^2 \leqslant C\varepsilon^2. \end{aligned} \tag{61}$$

Since ε is arbitrary, we obtain

$$u^{(m)} \to u \ \text{ strongly in } L^\infty(0,T;L^2(\Omega) \ \text{ and } \ \text{a.e. in } Q_T, \tag{62}$$

$$u_t^{(m)} \to u_t \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{63}$$

3.4. Step 3. Passage to the limit as $m \to \infty$

A weak solution of problem (1)-(3) will be obtained as the limit of the sequence of Galerkin's approximations $u^{(m)}$ as $m \to \infty$. Let us establish the passage to the limit as $m \to \infty$ on the intervals of the time when the estimate (54) is valid. Last estimate allows us conclude that there exist u and a subsequence of $\{u^{(m)}\}$, still denoted by $\{u^{(m)}\}$, such that (see also, e.g., [39, 42, 43]).

$$
\begin{aligned}
&u^{(m)} \to u \text{ strongly in } L^\infty(0,T;L^2(\Omega)), \text{ a.e. on } Q_T\\
&u_t^{(m)} \to u_t \text{ strongly in } L^2(Q_T), \text{ a.e. on } Q_T\\
&u_t^{(m)} \rightharpoonup u_t \text{ weakly in } L^\infty(0,T;L^2(\Omega)), \text{ a.e. } t \in [0,T]\\
&\nabla u^{(m)} \rightharpoonup \nabla u \text{ weakly in } L^{p(\cdot)}(Q_T) \cap L^\infty(0,T;L^2(\Omega)),\\
&\nabla u_t^{(m)} \rightharpoonup \nabla u_t \text{ weakly in } L^2(Q_T),\\
&u_{tt}^{(m)} \rightharpoonup u_{tt} \text{ weakly in } W'(Q_T),\\
&a|\nabla u^{(m)}|^{p-2}\nabla u^{(m)} \rightharpoonup \eta \text{ weakly in } L^{p'(\cdot)}(Q_T).
\end{aligned}
\tag{64}
$$

Integrating by parts in the first term in (56), we can rewrite last one in the form

$$
\begin{aligned}
&\int_{Q_t} \Big(- u_t^{(m)}\varphi_t + \Big(a||\nabla u^{(m)}||^{p(x,t)-2}\nabla u^{(m)} + \alpha \nabla u_t^{(m)}\Big)\nabla\varphi\Big) dx d\tau\\
&\qquad - \int_{Q_t} \big(b||u^{(m)}||^{\sigma-2}u^{(m)} + f\big)\varphi dx d\tau - \big(u_t^{(m)},\varphi\big)_\Omega\Big|_0^t = 0,\ t \in [0,T].
\end{aligned}
\tag{65}
$$

Taking into account (64) and passing to the limit in (65) as $m \to \infty$, we obtain for any $\varphi \in P_m$,

$$
\begin{aligned}
&\int_{Q_t} \big(- u_t\varphi_t + (\eta + \alpha\nabla u_t)\nabla\varphi\big) dx d\tau\\
&\qquad - \int_{Q_t} \big(b||u||^{\sigma-2}u + f\big)\varphi dx d\tau - (u_t,\varphi)_\Omega\Big|_0^t = 0, \text{ a.e. } t \in [0,T].
\end{aligned}
\tag{66}
$$

Now we prove that

$$
\int_0^t (\eta,\nabla\varphi)_\Omega d\tau = \int_0^t \Big(a|\nabla u|^{p(\cdot)-2}\nabla u, \nabla v\Big)_\Omega d\tau.
$$

Substituting φ in (65) by $u^{(m)}$ and by u in (66) and integrating by parts in the terms with α we obtain

$$
\begin{aligned}
&\int_{Q_t} \Big(- \big(u_t^{(m)}\big)^2 + a||\nabla u^{(m)}||^{p(x,t)-2}\nabla u^{(m)}\nabla u^{(m)}\Big) dx d\tau\\
&\qquad + \alpha\big(\nabla u^{(m)},\nabla u^{(m)}\big)_\Omega\Big|_0^t - \int_{Q_t} \big(b||u^{(m)}||^{\sigma-2}u^{(m)} + f\big)u^{(m)} dx d\tau
\end{aligned}
$$

$$-\left(u_t^{(m)},u^{(m)}\right)_\Omega\Big|_0^t=0, \tag{67}$$

and

$$\int_{Q_t}\left(-u_t^2+\eta\nabla u\right)dxd\tau+\alpha\left(\nabla u,\nabla u\right)_\Omega\Big|_0^t \\ -\int_{Q_t}\left(b||u||^{\sigma-2}u+f\right)udxd\tau-\left(u_t,u\right)_\Omega\Big|_0^t=0,\ \text{a.e. } t\in[0,T]. \tag{68}$$

According to the monotonicity of the operator $A(u)=a\,|\nabla u|^{p(\cdot)-2}\nabla u$, we have that

$$0\leqslant\int_0^t\Big(A(u^{(m)})-A(v),\nabla u^{(m)}-\nabla v\Big)_\Omega d\tau. \tag{69}$$

Taking into account (64), (67) and (68), (69), we derive that

$$0\leqslant\int_0^t(\eta-A(v),\nabla u-\nabla v)_\Omega\,d\tau,\ \forall v\in W(Q_T). \tag{70}$$

Choosing $v=u-\lambda w,$ where $\lambda>0$ is a real number and $w\in W(Q_T),$ and substituting it into (70), we have

$$0\leqslant\int_0^t(\eta-A(u-\lambda w),\nabla w)_\Omega\,d\tau.$$

Letting $\lambda\to 0$ in last inequality and using (70), we come at the inequality

$$0\leqslant\int_0^t(\eta-A(u),\nabla w)_\Omega\,d\tau.$$

Taking into account the density of the functions w, we conclude the proof of the Theorems 3.1 and 3.2. □

4. Blow up of solutions

First we consider equation (1) with $\alpha>0$, assuming that conditions of Lemma 3.1 are fulfilled. Assume that

$$E(0)\leqslant 0,\ 0<(u_0,u_1)_{L^2(\Omega)},\ 2\leqslant p_-\leqslant p_+<\sigma_-. \tag{71}$$

THEOREM 4.1. *Let u be an energy weak solution to problem* (1)-(3). *Let the conditions of the Lemma* 3.1 *be fulfilled and* (71) *hold. Then there exists a finite time $t_{max}<\infty$ such that*

$$\Phi(t)=\|u(t)\|_{2,\Omega}^2+\alpha\int_0^t\int_\Omega|\nabla u|^2\,dxds\to\infty\ \ if\ \ t\to t_{\max}. \tag{72}$$

Proof. It's easy to verify that

$$\Phi' = 2(u,u_t)_\Omega + \alpha \int_\Omega ||\nabla u||^2 dx,$$
$$\Phi'' = 2||u_t||^2_{2,\Omega} + 2\int_\Omega \big(-a||\nabla u||^p + b||u||^\sigma\big)dx.$$

Using the inequality (42) of the Lemma 3.1, we calculate

$$\begin{aligned}\Phi'' &\geqslant 2\|u_t(t)\|^2_{2,\Omega} + 2\int_\Omega \left(-a|\nabla u|^p + b|u|^\sigma\right)dx \\ &\quad + 2\lambda\left(E(t) + \alpha\int_0^t\int_\Omega |\nabla u_t|^2 dxds\right) \\ &= (2+\lambda)\|u_t(t)\|^2 \\ &\quad + 2\int_\Omega \left(\left(\frac{\lambda}{p}-1\right)a||\nabla u||^p + b\left(1-\frac{\lambda}{\sigma}\right)||u||^\sigma\right)dx \\ &\quad + 2\lambda\alpha\int_0^t\int_\Omega ||\nabla u_t||^2 dxds > 0 \end{aligned} \tag{73}$$

for some $\lambda > 2$ and $p_+ < \lambda < \sigma_-$.

It follows that

$$\Phi'(t) > 0, \ \text{ if } \Phi'(0) \geqslant 2(u_0,u_1)_{L^2(\Omega)} > 0\,.$$

Thus we can conclude that $0<\Phi(t)$ $0<\Phi'(t)$, $0<\Phi''(t)$ and $\Phi(t)\to\infty$ as $t\to t_{\max}$.

Assume in contrast that $t_{\max} = \infty$. Using properties the Orlicz-Sobolev spaces (see Section 2), we derive the following inequalities (for any fixed t)

$$\begin{aligned}||u(\cdot,t)||_{2,\Omega} &\leqslant C||u(\cdot,t)||_{L^{\sigma(\cdot,t)}(\Omega)} \\ &\leqslant C\max\left[\left(\int_\Omega ||u(\cdot,t)||^{\sigma(\cdot,t)}dx\right)^{\frac{1}{\sigma_-}}, \left(\int_\Omega ||u(\cdot,t)||^{\sigma(\cdot,t)}\,dx\right)^{\frac{1}{\sigma_+}}\right],\end{aligned} \tag{74}$$

$$\begin{aligned}&||\nabla u(\cdot,t)||_{2,\Omega} \\ &\quad\leqslant C||\nabla u(\cdot,t)||_{L^{p(\cdot)}(\Omega)} \\ &\quad\leqslant C\max\left[\left(\int_\Omega ||\nabla u(\cdot,t)||^{p(\cdot,t)}dx\right)^{\frac{1}{p_-}}, \left(\int_\Omega ||\nabla u(\cdot,t)||^{p(\cdot,t)}dx\right)^{\frac{1}{p_+}}\right].\end{aligned} \tag{75}$$

Notice that, according to (73),

$$\int_\Omega |u_t|^2 dx \leqslant C\Phi'', \ \int_\Omega |\nabla u(\cdot,t)|^{p(\cdot,t)}dx \leqslant C\Phi'', \ \int_\Omega |u(\cdot,t)|^{\sigma(\cdot,t)dx} \leqslant C\Phi''.$$

Then we can rewrite (74), (75) in the forms

$$||u(\cdot,t)||_{2,\Omega} \leqslant C\max\left[(\Phi'')^{\frac{1}{\sigma_-}}, (\Phi'')^{\frac{1}{\sigma_+}}\right],$$

$$||\nabla u(\cdot,t)||_{2,\Omega} \leqslant C \max\left[\left(\Phi''\right)^{\frac{1}{p_-}}, \left(\Phi''\right)^{\frac{1}{p_+}}\right].$$

Hence we arrive at the inequality

$$\begin{aligned}0 \leqslant \Phi' &= 2\int_\Omega u\, u_t dx + \frac{\alpha}{2}\int_\Omega |\nabla u|^2 dx \\ &\leqslant 2\,\|u(\cdot,t)\|_{2,\Omega}\,\|u_t(\cdot,t)\|_{2,\Omega} + +\frac{\alpha}{2}\int_\Omega |\nabla u|^2 dx \\ &\leqslant C\left(\max\left[\left(\Phi''\right)^{\frac{1}{\sigma_-}+\frac{1}{2}}, \left(\Phi''\right)^{\frac{1}{\sigma_+}+\frac{1}{2}}\right] + \max\left[\left(\Phi''\right)^{\frac{2}{p_-}}, \left(\Phi''\right)^{\frac{2}{p_+}}\right]\right).\end{aligned}$$

Taking into account that $0 < \Phi''(t)$, we assume, without loss of generality, that $\Phi' \gg 1$ and come at the ordinary differential inequality

$$C\left(\Phi'\right)^{\mu} \leqslant \Phi'', \tag{76}$$

where

$$\frac{1}{\mu} = \max\left(\frac{1}{\sigma_-} + \frac{1}{2}, \frac{2}{p_-}\right) < 1 \text{ if } \sigma_- > 2,\ p_- > 2.$$

The last inequality leads us to estimate

$$\Phi'(t) \geqslant \Phi'(0)\left(1 - \frac{t(\mu-1)}{C}\left(\Phi'(0)\right)^{\mu-1}\right)^{-\frac{1}{\mu-1}} \to \infty\,, \tag{77}$$

as

$$t \to t_{\max} = \frac{C}{\mu-1}\left(\Phi'(0)\right)^{-\mu+1} < \infty.$$

The theorem is proved. □

REMARK 4.1. Let us notice that constants μ and C (and respectively t_{max}) in (77) depend only on $|\Omega|$, n, $a_\pm$, $b_\pm$, $p_\pm$, $\sigma_\pm$.

Now we assume that the exponents p, σ weakly dependent on t, that is, the constants C_p, C_σ are small. The proof the blow up is the same as in the previous Theorem if we guarantee that

$$E(t) + \alpha\int_0^t\int_\Omega |\nabla u_t|^2 dxds \leqslant 0,\ 0 \leqslant t \leqslant t_{max} \tag{78}$$

with $t_{\max}$ already defined in Theorem 4.1. According to the Lemma 3.2 (see inequality (43)), we have that

$$E(t) + \alpha\int_0^t\int_\Omega |\nabla u_t|^2 dxds \leqslant E(0) + t_{\max}e(a_+C_p + b_+C_\sigma)|\Omega|.$$

Assuming that

$$\delta = \max\left(C_p, C_\sigma\right) \leqslant |E(0)|\left(t_{\max}e(a_+ + b_+)|\Omega|\right)^{-1},\ \ E(0) < 0, \tag{79}$$

we arrive at (78). Then we come to

THEOREM 4.2. *Let u be an energy weak solution to problem* (1)-(3). *Let the conditions of Lemma* 3.2 *and* (79) *(with $t_{\max}$ defined in Theorem* 4.1*) be fulfilled. Let the conditions*

$$E(0)<0,\ 0<(u_0,u_1)_{L^2(\Omega)},\ 2\leqslant p_-\leqslant p_+<\lambda<\sigma_-$$

hold. Then the solution u blows up (in the sense that $\Phi(t)$ becomes unbounded) on the finite interval $(0,t_{max})$.

Now we consider equation (1) with $\alpha=0$, assuming that the problem (1)-(3) has at least one local energy solution. Here we follow the paper [19], where the authors proved the blow up for an abstract hyperbolic equation in a Banach space which includes, as an example, the equation of the type (1) with the $a=b=1,\ p=const.$, $\sigma=const.$

We assume that

$$E(0)\leqslant 0,\ 0<(u_0,u_1)_{L^2(\Omega)},\ \sigma_->\max\{2,p_+\}. \tag{80}$$

Repeating the arguments of the paper [19], we prove the following theorems.

THEOREM 4.3. *Let u be an energy weak solution to problem* (1)-(3) *with $\alpha=0$. Let conditions of the Lemma* 3.1 *be satisfied and assume that* (80) *holds. Then u blows up* (*in the sense that $\|u(t)\|^2_{2,\Omega}$ becomes unbounded*) *on the finite interval $(0,t_{max})$ with $t_{max}=2\|u_0\|^2_{2,\Omega}/(\lambda-2)(u_0,u_1)_\Omega$.*

Proof. Let us introduce the function

$$G(t)=\|u(t)\|^2=\|u(t)\|^2_{2,\Omega}.$$

It is very easy to verify that

$$G'(t)=2\,(u,u_t)_\Omega\,,\ G''(t)=2\,\|u_t(t)\|^2+2\int_\Omega\left(-a\,|\nabla u|^p+b\,|u|^\sigma\right)dx. \tag{81}$$

Taking into account (80), we evaluate $G''(t)$ in the following way

$$\begin{aligned}G''(t)&\geqslant 2\,\|u_t(t)\|^2+2\int_\Omega\left(-a\,|\nabla u|^p+b\,|u|^\sigma\right)dx+2\lambda E(t)\\&=(2+\lambda)\,\|u_t(t)\|^2+2\int_\Omega\left(\left(\frac{\lambda}{p}-1\right)a\,|\nabla u|^p+b\left(1-\frac{\lambda}{\sigma}\right)|u|^\sigma\right)dx\\&\geqslant(2+\lambda)\,\|u_t(t)\|^2\end{aligned} \tag{82}$$

for some λ, $p_+\leqslant\lambda\leqslant\sigma_-$. Then we literally repeat the arguments of the paper [19].

From the first identity (81) we conclude that

$$G'^2(t)\leqslant 4\,\|u_t\|^2\,\|u\|^2=4G\,\|u_t\|^2\Rightarrow\|u_t\|^2\geqslant\frac{G'^2}{4G}. \tag{83}$$

Combining (82) and (83), we come to the following ordinary differential inequality

$$G''(t) \geqslant (2+\lambda)\frac{G'^2}{4G} \Leftrightarrow \frac{G''}{G'^2} \geqslant \frac{(2+\lambda)}{4G} \geqslant 0. \tag{84}$$

Then (80), (84) imply that $G'(t) > 0$ for all $t > 0$. Integrating (84) twice as above, we come to the inequality

$$\|u_0\|_{2,\Omega}^2 \left[1 - t\frac{\lambda-2}{2}\frac{(u_0,u_1)_\Omega}{\|u_0\|_2^2}\right]^{-\frac{4}{\lambda-2}} \leqslant \|u(t)\|_{2,\Omega}^2 .$$

□

Now we assume that the exponents p,σ weakly dependent on t, that is, the constants C_p, C_σ are small.

Repeating the above mentioned arguments (see (78), (79)), we prove

THEOREM 4.4. *Let u be an energy weak solution to problem* (1)-(3) *with* $\alpha = 0$. *Let conditions of the Lemma* 3.2 *be fulfilled and* (79) (*with t_{max} defined in previous Theorem*) *and* (80) *hold. Then the solution u blows up on the finite interval* $(0, t_{\max})$.

REFERENCES

[1] S. ANTONTSEV, *Wave equation with $p(x,t)$ - Laplacian and damping term: Existence and blow-up*, in Proceedings of the International Conference "Modern Problems of Applied Mathematics and Mechanics: Theory, Experiment and Applications" devoted to the 90th anniversary of Prof. N.N Yanenko, Novosibirsk, Russia, May 30-June 4, 2011.

[2] S. ANTONTSEV, *Wave equation with $p(x,t)$ -Laplacian and damping term: Existence and blow-up*, in Abstracts, Nonlinear Models in Partial Differential Equations, An international congress on occasion of J.I. Diaz's 60th birthday, Toledo, Spain, June 14-17, 2011, p. 8.

[3] S. ANTONTSEV, *Wave equation with $p(x,t)$ -Laplacian and damping term: Blow-up of solutions*, C. R. Mecanique, (2011, in press).

[4] S. ANTONTSEV, M. CHIPOT, AND Y. XIE, *Uniqueness results for equations of the $p(x)$ -Laplacian type*, Adv. Math. Sci. Appl., **17** (2007), 287–304.

[5] S. N. ANTONTSEV, J. I. DÍAZ, AND S. SHMAREV, *Energy Methods for Free Boundary Problems: Applications to Non-linear PDEs and Fluid Mechanics*, Bikhäuser, Boston, 2002. Progress in Nonlinear Differential Equations and Their Applications, Vol. 48.

[6] S. N. ANTONTSEV AND J. F. RODRIGUES, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara, Sez.,VII. Sci. Mat., **52** (2006), 19–36.

[7] S. N. ANTONTSEV AND S. I. SHMAREV, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions*, Journal Nonlinear Analysis, **65** (2006), 722–755.

[8] S. N. ANTONTSEV AND S. I. SHMAREV, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, Elsevier, 2006. Handbook of Differential Equations. Stationary Partial Differential Equations, Elsevier, Vol. 3, Chapter 1, 1-100.

[9] S. N. ANTONTSEV AND S. I. SHMAREV, *Parabolic equations with anisotropic nonstandard growth conditions*, in Internat. Ser. Numer. Math. 154, Birkhäuser, Verlag Basel/Switzerland, 2006, 33–44.

[10] S. N. ANTONTSEV AND S. I. SHMAREV, *Extinction of solutions of parabolic equations with variable anisotropic nonlinearities*, Proceedings of the Steklov Institute of Mathematics, Moscow, Russia, **268** (2008), 2289–2301.

[11] S. N. ANTONTSEV AND S. I. SHMAREV, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Sec. Mat. Univ. Autònoma Barcelona, (2009), 355–399.

[12] S. N. ANTONTSEV AND S. I. SHMAREV, *Localization of solutions of anisotropic parabolic equations*, Nonlinear Anal., **71** (2009), e725–e737.

[13] S. N. ANTONTSEV AND S. I. SHMAREV, *Blow-up of solutions to parabolic equations with nonstandard growth conditions*, Journal of Computational and Applied Mathematics, **234** (2010), 2633–2645.

[14] G. AUTUORI, P. PUCCI, AND M. C. SALVATORI, *Asymptotic stability for anisotropic Kirchhoff systems*, J. Math. Anal. Appl., **352** (2009), 149–165.

[15] G. AUTUORI, P. PUCCI, AND M. C. SALVATORI, *Global nonexistence for nonlinear Kirchhoff systems*, Arch. Rational Mech. Anal., **196** (2011), 489–516.

[16] J. CLEMENTS, *Existence theorems for a quasilinear evolution equation*, SIAM J. Appl. Math., **26** (1974), 745–752.

[17] L. DIENING, P. HARJULEHTO, P. HASTO, AND M. RÚŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, 2011. Series: Lecture Notes in Mathematics, Vol. 2017,1st Edition.

[18] M. DREHER, *The wave equation for the p-Laplacian*, Hokkaido Math. J., **36** (2007), 21–52.

[19] V. A. GALAKTIONOV AND S. I. POHOZAEV, *Blow-up and critical exponents for nonlinear hyperbolic equations*, Nonlinear Analysis, **53** (2003), 453–466.

[20] H. GAO AND T. F. MA, *Global solutions for a nonlinear wave equation with $p-$Laplacian operator*, EJQTDE., (1999), 1–13.

[21] V. GEORGIEV AND G. TODOROVA, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations, **109** (1994), 295–308.

[22] J. HAEHNLE AND A. PROHL, *Approximation of nonlinear wave equations with nonstandard anisotropic growth conditions*, Math. Comp., **79** (2010), 189–208.

[23] M. JAZAR AND R. KIWAN, *Blow-up results for some second-order hyperbolic inequalities with a nonlinear term with respect to the velocity*, J. Math. Anal. Appl., **327** (2007), 12–22.

[24] M. JAZAR AND R. KIWAN, *Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **25** (2008), 215–218.

[25] T. KATO, *Blow up of solutions of some nonlinear hyperbolic equations*, Manuscripta Math., **28** (1980), 235–268.

[26] H. LEVINE AND G. TODOROVA, *Blow-up of solutions of the cauchy problem for a wave equations with nonlinear damping and source terms and positive initial energy*, in Proceedings of the American Mathematical Society, **129** (2000), 793–805.

[27] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[28] S. A. MESSAOUDI, *Blow up in a nonlinearly damped wave equation*, Math. Nachr., **231** (2001), 105–111.

[29] S. A. MESSAOUDI, *Blow up in the Cauchy problem for a nonlinearly damped wave equation*, Commun. Appl. Anal., **7** (2003), 379–386.

[30] S. A. MESSAOUDI, *On the decay of solutions for a class of quasilinear hyperbolic equations with non -linear damping and source terms*, Mathematical Methods in the Applied Sciences, **28** (2005), 1819–1828.

[31] S. A. MESSAOUDI AND B. SAID HOUARI, *Global non-existence of solutions of a class of wave equations with non-linear damping and source terms*, Math. Methods Appl. Sci., **27** (2004), 1687–1696.

[32] C. MU, R. ZENG, AND B. CHEN, *Blow-up phenomena for a doubly degenerate equation with positive initial energy*, Nonlinear Anal., **72** (2010), 782–793.

[33] M. NAKAO AND Y. ZHIJIAN, *Global attractors for some quasi-linear wave equations with a strong dissipation*, Adv. Math. Sci. Appl., **17** (2007), 89–105.

[34] J. P. PINASCO, *Blow-up for parabolic and hyperbolic problems with variable exponents*, Nonlinear Anal., **71** (2009), 1094–1099.

[35] K. RAJAGOPAL AND M. RÚŽIČKA, *Mathematical modelling of electro-rheological fluids*, Cont. Mech. Therm., **13** (2001), 59–78.

[36] M. RÚŽIČKA, *Electrorheological fluids: modeling and mathematical theory*, Springer, Berlin, 2000. Lecture Notes in Mathematics, 1748.

[37] A. A. SAMARSKII, V. A. GALAKTIONOV, S. P. KURDYUMOV, AND A. P. MIKHAILOV, *Blow-up in quasilinear parabolic equations*, Walter de Gruyter & Co., Berlin, 1995. Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors.

[38] S. G. SAMKO, *Density $C_0^\infty(\mathbb{R}^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^n)$*, Dokl. Akad. Nauk, **369** (1999), 451–454.

[39] J. SIMON, *Compact sets in the space $l^p(0,t;b)$*, Ann. Mat. Pura Appl., IV. Ser., **146** (1952), 65–96.

[40] G. TODOROVA AND E. VITILLARIO, *Blow-up for nonlinear dissipative wave equations in $\mathbb{R}^n$*, J. Math. Anal. Appl., (2005), 242–257.

[41] Z. WILSTEIN, *Global well-posedness for a nonlinear wave equation with p-Laplacian damping*, Dissertation, University of Nebraska-Lincoln, downloaable at :http://digitalcommons.unl.edu/mathstudent/24, (2011), 1–116.

[42] Z. YANG, *Cauchy problem for quasi-linear wave equations with viscous damping*, J. Math. Anal. Appl., **320** (2006), 859–881.

[43] Z. YANG AND G. CHEN, *Global existence of solutions for quasi-linear wave equations with viscous damping*, J. Math. Anal. Appl., **285** (2003), 604–618.

[44] Y. ZHIJIAN, *Existence and asymptotic behaviour of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms*, Mathematical Methods in the Applied Sciences, **25** (2002), 795–814.

[45] Y. ZHIJIAN, *Global existence, asymptotic behavior and blowup of solutions for a class of nonlinear wave equations with dissipative term*, J. Differential Equations, **187** (2003), 520–540.

[46] Y. ZHIJIAN, *Initial boundary value problem for a class of non-linear strongly damped wave equations*, Math. Methods Appl. Sci., **26** (2003), 1047–1066.

[47] Y. ZHIJIAN, *Cauchy problem for a class of nonlinear dispersive wave equations arising in elasto-plastic flow*, J. Math. Anal. Appl., **313** (2006), 197–217.

[48] V. V. ZHIKOV, *On the density of smooth functions in Sobolev-Orlich spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), **310** (2004), 1–14.

(Received August 31, 2011)

S. Antontsev
CMAF
University of Lisbon
Portugal
e-mail: anton@ptmat.fc.ul.pt